

Note on the surface wave due to the prescribed elevation

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Abstract. In the present paper, a study of the deep-sea water wave caused by an oscillatory wind stress due to the atmospheric depression, resulting in spiral cyclonic wind pressure on the surface of the sea is made. It has been observed that the motion of the water wave in the case of wind stress exhibits a greater elevation on the sea surface as g the acceleration due to gravity decreases and maintains the oscillatory nature with the increase of time. For the case of spiral cyclonic motion for which the sea surface experiences the elliptical pressure on the surface, the motion diminishes as g diminishes and oscillates with the variation of time. The motion also diminishes asymptotically as the radius vector of the elliptical pressure approaches unity.

Keywords. Surface wave; irrotational; wind stress; velocity potential.

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1. Introduction

The study of deep-sea waves has long been regarded as a subject of study in oceanography as well as in theoretical physics.

Different mechanisms are available in nature for the generation of the wave motions in sea and ocean. Among these, the most common mechanism is the wind stress distribution due to the atmospheric depression, resulting in spiral cyclonic wind pressure on the surface of the sea.

Regarding the wind stress of oscillatory motion, it exhibits a greater elevation on the surface of the sea as g the acceleration due to gravity decreases and the oscillatory motion with the increase of time, leaving behind the highest elevation initially.

On the other hand in case of spiral cyclonic motion for which the sea surface experiences the elliptical pressure distribution, the motion diminishes as g the acceleration due to gravity diminishes and oscillates with the variation of time. It also diminishes asymptotically as the radius vector of the elliptical distribution of the pressure approaches unity.

The formation of waves in deep sea water by local disturbances on the surface was investigated in classical treatises by Lamb [1] and Stoker [2]. Kalisch *et al* [3] in their recent work, studied the formation of wave in deep water. Zakharov [4] has done some significant work on the theory of gravity and capillary waves on the surface of the fluid.

Two forms of the problem may arise: we may start with an initial elevation of the free surface without initial velocity or we may start the surface undisturbed, prescribe an initial distribution of surface impulse.

The motion is generated originally from rest by the action of forces or an impulse admits of a velocity potential.

Asymptotic expressions for the surface elevations are obtained by the applications of Kelvin's method of stationary phase.

Snedden [5] has considered the propagation of the surface waves in two dimensions by initial elevation

$$\zeta = f(r)$$

when (r, z) are cylindrical coordinates with origin in the free surface, for the case of symmetry about z -axis.

In the present context, a study has been made on the wave motion produced by oscillatory wind stress in the form [6]

$$f(r) = \frac{1}{\varepsilon \sqrt{\pi}} e^{-r^2/\varepsilon^2}.$$

Our second consideration is the function of the form [7]

$$\begin{aligned} f(r) &= \varepsilon (1 - r^2/a^2)^n; \quad 0 < r < a \quad \text{when } n > -1, \\ &= 0; \quad r \geq a. \end{aligned}$$

A symmetrical function confined over elliptical region, is represented in the case of spiral cyclonic motion for which the sea surface experiences the elliptical distribution of pressure.

The elevation of the surface and velocity potential are obtained in explicit forms by the applications of zero order Hankel transform.

2. Formalism

The equation of motion in cylindrical polar coordinates (r, z) in the case of symmetry about z -axis is given by

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad (1)$$

where ϕ is the velocity potential and ζ denotes the elevation of the free surface at time t .

$$\zeta = \frac{1}{g} \left[\frac{\partial \phi}{\partial t} \right]_{z=0}. \quad (2)$$

The boundary and surface conditions are

$$\phi \rightarrow 0 \quad \text{as} \quad z \rightarrow -\infty, \quad (3)$$

$$\partial \zeta / \partial t = -[\partial \phi / \partial z]_{z=0}. \quad (4)$$

Hence, for $z = 0$, we must have

$$\partial^2 \phi / \partial t^2 + g[\partial \phi / \partial z] = 0. \quad (5)$$

For the solution of (1) we introduce zero-order Hankel transform

$$\bar{\phi}(\xi, z; t) = \int_0^\infty r \phi(r, z; t) J_0(\xi r) dr \quad (6)$$

and get

$$d^2 \bar{\phi} / dz^2 - \xi^2 \bar{\phi} = 0. \quad (7)$$

Therefore

$$\bar{\phi} = A(\xi, t) e^{\xi z}. \quad (8)$$

From (5),

$$d^2 A / dt^2 + g \xi A = 0. \quad (9)$$

Let the initial conditions be given by

$$\zeta = f(r) \quad \text{when} \quad t = 0, \quad z = 0, \quad (10)$$

$$\phi = 0 \quad \text{when} \quad t = 0. \quad (11)$$

From (8) we have, $A = 0$ when $t = 0$. From (2),

$$g f(r) = \left(\frac{\partial \phi}{\partial t} \right)_{z=0}. \quad (12)$$

Therefore

$$g \bar{f}(\xi) = \left(\frac{\partial \phi}{\partial t} \right)_{z=0} = dA/dt \quad \text{when} \quad t = 0. \quad (13)$$

From (9) we get

$$A = \alpha(\xi) \cos \sqrt{g\xi} t + \beta(\xi) \sin \sqrt{g\xi} t.$$

Since $A = 0$ when $t = 0$, $\alpha(\xi) = 0$.

Therefore

$$A = \beta(\xi) \sin \sqrt{g\xi} t. \quad (14)$$

From (13) and (14)

$$\beta(\xi) = (g/\xi)^{1/2} \bar{f}(\xi). \quad (15)$$

Therefore

$$\bar{\phi} = (g/\xi)^{1/2} \bar{f}(\xi) \sin \sqrt{g\xi t} e^{\xi z}. \quad (16)$$

Hence, by Hankel inversion formula we get

$$\phi = \int_0^\infty (g\xi)^{1/2} \bar{f}(\xi) \sin \sqrt{g\xi t} J_0(\xi r) e^{\xi z} d\xi. \quad (17)$$

And from (2),

$$\zeta = \int_0^\infty \xi \bar{f}(\xi) J_0(\xi r) \cos \sqrt{g\xi t} d\xi.$$

Let

$$f(r) = \frac{1}{\varepsilon \sqrt{\pi}} e^{-r^2/\varepsilon^2}.$$

Then [8]

$$\begin{aligned} (\phi)_{z=0} &= (gt)/\varepsilon \sqrt{\pi} \sum_{n=0}^{\alpha} (-1)^n (gt^2)^n / (2n+1)! (2/\varepsilon)^n \\ &\quad \times \Gamma(n/2+1)_1 F_1(n/2+1; 1; r^2/\varepsilon^2) \end{aligned} \quad (18)$$

and [9]

$$\zeta = 1/\varepsilon \sqrt{\pi} \cos \sqrt{2gt/\varepsilon} \cdot t e^{-r^2/\varepsilon^2} \sum_{n=0}^{\alpha} \Gamma(n/2+1)_1 F_1(-n/2; 1; r^2/\varepsilon^2). \quad (19)$$

Equation (19) represents the explicit form of the surface elevation, consisting of the transient and the steady states.

Let $\zeta = f_1(t) x f_1(r)$ when

$$f_1(t) = 1/\varepsilon \sqrt{\pi} \cos \sqrt{2g/\varepsilon} \cdot t$$

and

$$f_1(r) = e^{-r^2/\varepsilon^2} \sum_{n=0}^{\alpha} (-1)^n \Gamma(n/2+1)_1 F_1(-n/2; 1; r^2/\varepsilon^2).$$

The motion has been studied under the circumstances of variation of t , variation of r/ε and $t, r/\varepsilon$ simultaneously (figures 1–3).

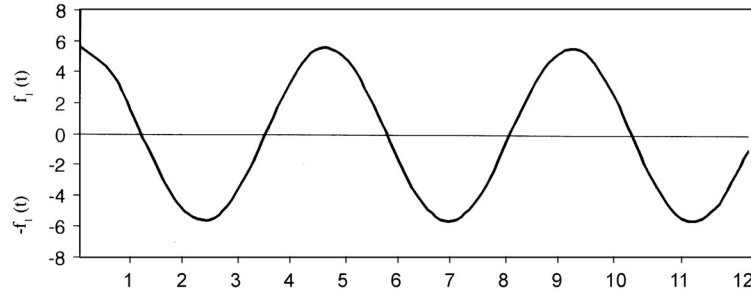


Figure 1.

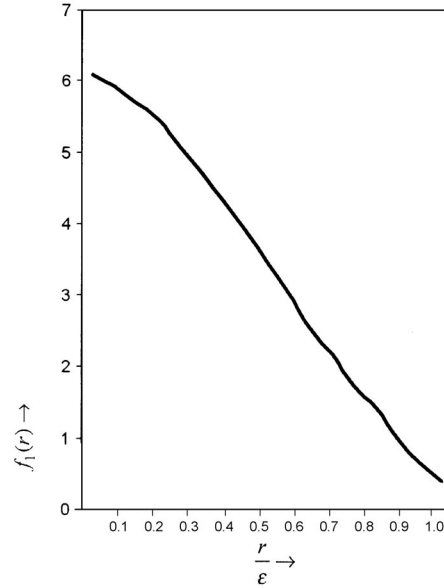


Figure 2.

3. Discussion of the result

Equation (19) which is a solution of the problem is characteristic with the two independent variables t and r . It also experiences with the variation of g the acceleration due to gravity. The motion exhibits greater elevation as g decreases and oscillates with the increase of time, keeping the highest elevation, initially at $t = 0$.

It also reveals the fact that when the sea surface experiences the oscillatory wind stress, the surface does not exhibit the same elevation throughout the sea surface. It experiences greater elevation on places where the value of g is comparatively less than where g has the maximum value of 980 cm/s^2 .

It has also been observed that the surface elevation diminishes asymptotically as $r/\epsilon \rightarrow 1$ and oscillates with the variation of $t, r/\epsilon$ simultaneously.

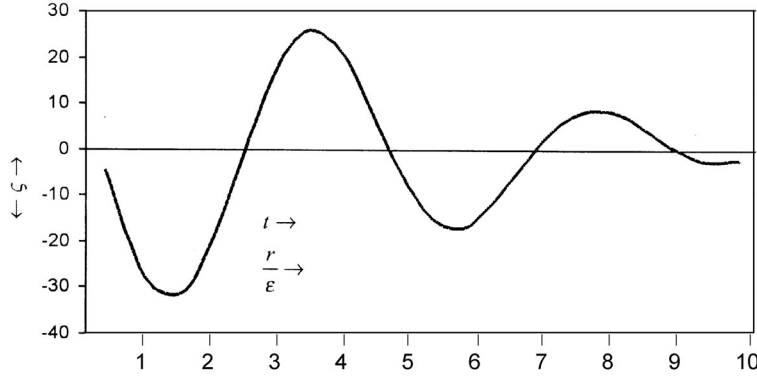


Figure 3.

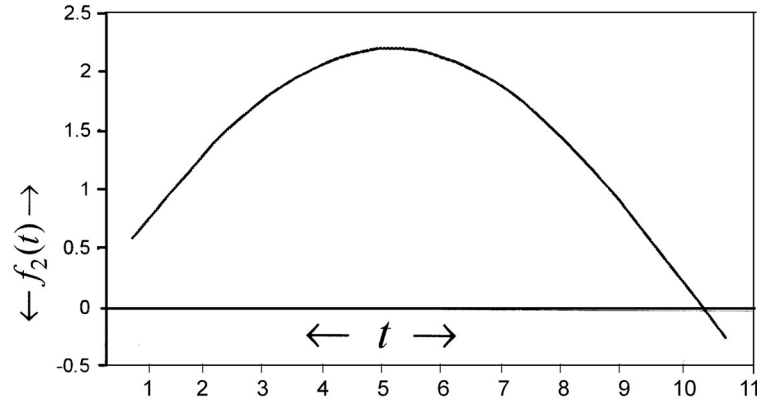


Figure 4.

Our second consideration is the function of the type [7]

$$\begin{aligned} f(r) &= \varepsilon(1 - r^2/a^2)^n, & 0 < r < a & \text{ when } n > -1, \\ &= 0; & r \geq a, \end{aligned} \quad (20)$$

which follows the elliptical distribution of perturbation due to spiral cyclonic wind stress and also quickly assumes a constant pressure at $0 < r < a$ when $n = 0$.

Here $\bar{f}(\xi) = \int_0^\infty r(1 - r^2/a^2)^n J_0(\varepsilon r) dr$. Putting $r/a = \sin \theta$, $r \rightarrow 0$, $\theta \rightarrow 0$, $r \rightarrow a$, $\theta \rightarrow \frac{\pi}{2}$. Therefore $\bar{f}(\xi) = \varepsilon a^2 2^n \Gamma(n+1) (\xi a)^{-n-1} J_{n+1}(\xi a)$. From (17) we get

$$(\phi)_{z=0} = \varepsilon g t \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+1)}{(2n+1)!} 2^n / a^{n-1} (g t^2)^n \int_0^\alpha J_{n+1}(\xi a) J_0(\xi r) d\xi \quad (21)$$

$$= \varepsilon \sqrt{g/2} \sin(\sqrt{2} g t) \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{a^n} 2 \cdot F_1(n/2 + 1; -n/2; 1; r^2/a^2) \quad (22)$$

and

$$\zeta = \varepsilon \cos(\sqrt{2gt}) \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{a^n} {}_2F_1((n/2) + 1; -n/2; 1; r^2/a^2). \quad (23)$$

Equation (22) represents the explicit form of the velocity potential, which also has the transient and steady states. As the problem has been considered of forced gravity wave, the motion varies with the variation of g , the acceleration due to gravity. It diminishes as g diminishes, that is, it reveals the fact that when the sea surface experiences the elliptical distribution of force due to spiral cyclonic wind pressure of the atmospheric depression, the sea surface does not experience the same wave motion throughout the surface. It experiences greater wave motion in places where the acceleration due to gravity is maximum say 980 cm/s^2 . The motion oscillates with the variation of time. It has also been observed that the motion diminishes asymptotically as $r \rightarrow 1$ when $a = 1$, in its steady state. The motion

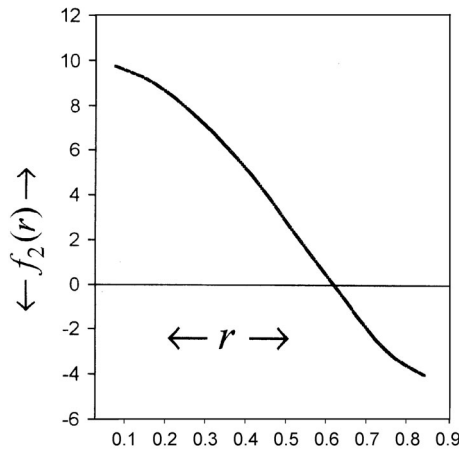


Figure 5.

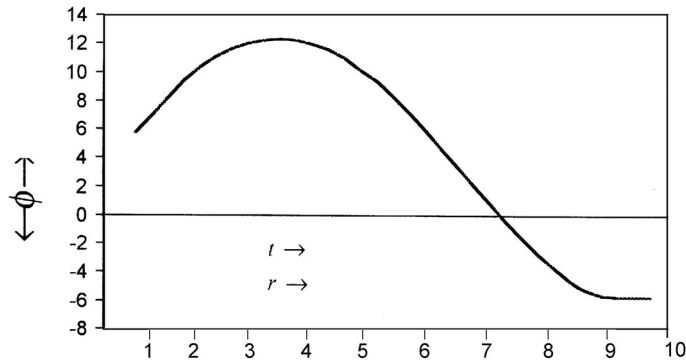


Figure 6.

has also been studied when both r and t vary simultaneously, when it exhibits the oscillatory nature (figures 4–6).

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References

- [1] H Lamb, *Hydrodynamics*, 6th edition (Cambridge University Press, 1932) p. 429
- [2] J J Stoker, *Water waves* (Inter Science Publishers Inc., New York, 1957) vol. 64, p. 156
- [3] Kalesch, Henrik and Bona L Jerry, *Discrete Contin Dynam System* **6**, 1 (2000)
- [4] Zakharov and E Vladimir, *Moscow Eur. J. Mech. B. Fluids* **18**(3), 327 (1999)
- [5] I N Snedden, *Fourier transform* (Mc Graw Hill Book Co. Inc., New York, 1951) vol. 326, p. 290
- [6] Carslaw and Jargar, *Operational methods in applied mathematics*, 2nd edition (Dover Publications, New York, 1947) p. 234
- [7] A R Sen, *Deep water surface waves due to arbitrary periodic pressure*, *Proc. NISI* **28**, 612 (1962)
- [8] Erdelyi *et al*, *Higher transcendental functions*, *Batman manuscript project* (McGraw Hill Book Co. Inc., 1963)
- [9] B B Sen, *Kummer's results - A treatise on special functions* (Allied Publications Pvt. Ltd., 1967) p. 71
- [10] F Magnus Want, *Ober Hettinger, Special Functions of Mathematical Physics* (Chelsae, New York, 1949)