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$U(1)$ Gauge theory as quantum hydrodynamics

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Abstract. It is shown that gauge theories are most naturally studied via a polar decomposition of the field variable. Gauge transformations may be viewed as those that leave the density invariant but change the phase variable by additive amounts. The path integral approach is used to compute the partition function. When gauge fields are included, the constraint brought about by gauge invariance simply means an appropriate linear combination of the gradients of the phase variable and the gauge field is invariant. No gauge fixing is needed in this approach that is closest to the spirit of the gauge principle. We derive an exact formula for the condensate fraction and in case it is zero, an exact formula for the anomalous exponent. We also derive a formula for the vortex strength which involves computing radiation corrections.

Keywords. Bosons; hydrodynamics; bosonization.

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1. Introduction

The density phase transformation for bosons is quite well-known to those who work in the fields of superfluidity [1,2] and other related areas such as superconductivity and possibly even quantum optics. Jackiw and collaborators [3] have recently introduced some of these ideas independently in the context of relativisitic quarks. Since the subject of bosons is vast, we refer the reader to the review by Ceperley [4] for further references. However, a few basic works do deserve mention. First there is the work by Penrose and Onsager [5] on estimating the superfluid fraction in He^4. Then there is the Hohenberg–Mermin–Wagner theorem [6] that precludes long-range order with a broken continuous symmetry in two dimensions. Finally, there is work by Ceperley [4] using quantum Monte Carlo.

The main point of this article is to highlight the usefullness of the hydrodynamical approach in tackling gauge theories. We argue that this approach is the natural way in which to study gauge theories. Although by itself none of the results in this article are particularly new, the novel aspect is the manner in which the theory is formulated that renders it susceptible to generalization in future publications to fermions coupled to gauge fields and later on to relativistic gauge theories and finally to nonabelian relativistic gauge theories. Much of this agenda has already been completed by the high energy community recently, specifically, the work of Jackiw and Polychronakos [3] in introducing anticommuting Grassmann variables into a fluid dynamical description will be very relevent to our future work. But our thrust is to compute microscopic correlation functions rather than just develop formalism.

One concrete result in this article is an exact formula for the condensate fraction. Since the condensate fraction depends only on the asymptotic properties of the one-particle Green function, and this is given exactly via bosonization, we may conclude that the derived result for this quantity is therefore, exact. The formula for the condensate fraction may also be derived using other methods, most recently Liu [7] has used the eigenfunctional theory to derive an expression for the same quantity that agrees with the expression derived in this article.

We apply this method to compute the velocity–velocity correlation function and demonstrate the power of this method in accounting for vortices in a gauge invariant manner.

2. Density phase transformation

Consider the action of non-relativistic spinless bosons. We use units such that $\hbar = 2m = 1$.

$$S_{\text{free}} = \int_0^{-i\beta} \mathrm{d}t \int \mathrm{d}^d x \psi^\dagger(\mathbf{x}) \left(i\partial_t + \nabla^2\right) \psi(\mathbf{x}). \tag{1}$$

In the path integral approach, $\psi(\mathbf{x})$ is just a complex number that is defined at each point $\mathbf{x}$. Every complex number can be decomposed into a magnitude and a phase. Using the ideas explained in our earlier work [8]

$$\psi(\mathbf{x}) = \mathrm{e}^{-i\Pi(\mathbf{x})}\sqrt{\rho(\mathbf{x})}. \tag{2}$$

From this we may write down the current,

$$\mathbf{J}(\mathbf{x}) = -\rho(\mathbf{x})\nabla\Pi(\mathbf{x}) \tag{3}$$

$$S_{\text{free}} = \int_0^{-i\beta} \mathrm{d}t \int \mathrm{d}^d x \left[\rho\partial_t\Pi - \rho(\nabla\Pi)^2 - \frac{(\nabla\rho)^2}{4\rho}\right]. \tag{4}$$

In deriving this action we have already used some boundary conditions. From the text by Kadanoff and Baym [9] we learn that the Green functions of bosons obey the KMS (Kubo–Martin–Schwinger) boundary conditions. These boundary conditions translate in the path integral representation to

$$\psi(\mathbf{x}, t - i\beta) = \mathrm{e}^{\beta\mu}\psi(\mathbf{x}, t). \tag{5}$$

This in turn means that the number conserving product is invariant under this discrete time translation. To see this we examine, $\psi^\dagger(\mathbf{x}, t-i\beta)\psi(\mathbf{x}', t-i\beta)$. Observe that $\psi^\dagger(\mathbf{x}, t - i\beta) = [\psi(\mathbf{x}, t + i\beta)]^\dagger = \mathrm{e}^{-\beta\mu}\psi^\dagger(\mathbf{x}, t)$ and $\psi(\mathbf{x}', t - i\beta) = \mathrm{e}^{\beta\mu}\psi(\mathbf{x}', t)$.

Thus, $\psi^\dagger(\mathbf{x}, t-i\beta)\psi(\mathbf{x}', t-i\beta) = \psi^\dagger(\mathbf{x}, t)\psi(\mathbf{x}', t)$. As pointed out in an earlier work [8] the phase variable may be written as a sum of two terms, a position independent term which is the conjugate to the total number and a position dependent term that is related to currents and densities. Thus, $\Pi(\mathbf{x}) = X_0 + \tilde{\Pi}(\mathbf{x})$. We have just argued that $\tilde{\Pi}(\mathbf{x})$ and $\rho(\mathbf{x})$ are invariant under the discrete time translation. Thus in order to preserve the KMS boundary condition we must impose $X_0(t-i\beta) = X_0(t) + i\beta\mu$. Therefore, we find that the conjugate to the total number makes its presence felt in a very non-trivial manner. The boundary condition that has been used in deriving the action is $N(-i\beta) = N(0)$ where N is the total number of particles.

3. Gauge transformations

Here we examine what sorts of changes are brought about by the imposition of local gauge invariance. Gauge transformations in the usual language is given by

$$\psi'(\mathbf{x}) = \mathrm{e}^{ie\Lambda}\psi(\mathbf{x}), \tag{6}$$

$$A_0' = A_0 + \partial_t\Lambda, \tag{7}$$

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda. \tag{8}$$

In order to find an action invariant under these transformations, we replace derivatives by covariant derivatives (minimal coupling). Thus we have $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$. In the usual language,

$$S = \int \psi^\dagger \left(i\partial_t + eA_0 + (\nabla - ie\mathbf{A})^2\right)\psi. \tag{9}$$

Here and henceforth by $\int$ we mean $\int_0^{-i\beta}\mathrm{d}t\int \mathrm{d}^d x$. Now we rewrite this as a sum of two parts, the free term plus the term that couples to the gauge fields and finally the term involving only gauge fields namely, the curvature term.

$$S = S_{\text{free}} + S_{\text{int}} + S_{\text{gauge}}, \tag{10}$$

$$S_{\text{free}} = \int \left[\rho\partial_t\Pi - \rho(\nabla\Pi)^2 - \frac{(\nabla\rho)^2}{4\rho}\right], \tag{11}$$

$$S_{\text{int}} = \int \rho e A_0 - e^2 \int \rho\mathbf{A}^2 - 2e\int \rho\left(\mathbf{A}\cdot\nabla\Pi\right), \tag{12}$$

$$S_{\text{gauge}} = -\frac{1}{4}\int F^2. \tag{13}$$

This action is gauge invariant provided we have

$$\Pi' = \Pi - e\Lambda; \quad \rho' = \rho. \tag{14}$$

4. The Jacobian determinant

In this section we attempt to rewrite the partition function in terms of the density and phase variables. Consider the partition function in the original Bose language.

$$Z = \int D[\psi]D[\psi^{\dagger}]\mathrm{e}^{iS[\psi,\psi^{\dagger}]}. \tag{15}$$

In the density phase variable language we have,

$$Z = \int D[\rho]D[\Pi]J(\Pi,\rho)\mathrm{e}^{iS[\Pi,\rho]}. \tag{16}$$

Here $J(\Pi,\rho)$ is the appropriate Jacobian determinant that tells us how the measure transforms. Fortunately, this Jacobian is a constant. To see this we write the definition of J as

$$J(\Pi,\rho) = \begin{vmatrix} \frac{\delta\psi}{\delta\Pi} & \frac{\delta\psi}{\delta\rho} \\ \frac{\delta\psi^{\dagger}}{\delta\Pi} & \frac{\delta\psi^{\dagger}}{\delta\rho} \end{vmatrix} \quad = \quad -i. \tag{17}$$

5. Propagator with two-body forces

Here we compute the free propagator using path integrals in the density phase variable language. This is done to convince ourselves of the basic soundness of the approach. Also we operate in the limit where the mean density is constant so that we may ignore the density fluctuations in the long-wavelength limit. This will be made precise soon. The main advantage of this approach is the ease with which we may treat interactions. Just to highlight this fact we also include density–density interactions.

$$S_{\text{free}} \approx \int \left[\rho\partial_t\Pi - \rho_0(\nabla\Pi)^2 - \frac{(\nabla\rho)^2}{4\rho_0}\right]. \tag{18}$$

Using the boundary conditions we may write,

$$\rho(\mathbf{x},t) = \frac{1}{V}\sum_{\mathbf{q},n}\mathrm{e}^{-i\mathbf{q}\cdot\mathbf{x}}\rho_{\mathbf{q}n}\mathrm{e}^{-z_n t}, \tag{19}$$

$$\Pi(\mathbf{x},t) = -\mu t + \sum_{\mathbf{q},n}\mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}}X_{\mathbf{q}n}\mathrm{e}^{z_n t}. \tag{20}$$

The action then may be written as follows: We have added a two-body potential since we may do so without additional effort.

$$\begin{aligned} S_{\text{full}} = {} & i\beta\mu\rho_{\mathbf{0},0} + \sum_{\mathbf{q},n}(-i\beta z_n)\rho_{\mathbf{q}n}X_{\mathbf{q}n} \\ & + i\beta N^0\sum_{\mathbf{q},n}\mathbf{q}^2X_{\mathbf{q},n}X_{-\mathbf{q},-n} + \frac{i\beta}{4N^0}\sum_{\mathbf{q},n}\mathbf{q}^2\rho_{\mathbf{q},n}\rho_{-\mathbf{q},-n} \\ & + i\beta\sum_{\mathbf{q},n}\frac{v_{\mathbf{q}}}{2V}\rho_{\mathbf{q},n}\rho_{-\mathbf{q},-n}. \end{aligned} \tag{21}$$

Here $z_n = 2\pi n/\beta$. From this we may write down a formal expression for the propagator. We set $\rho \approx \rho_0$ in the argument of the square roots.

$$G(\mathbf{x},t) = -i\rho_0 \mathrm{e}^{i\mu t} \exp\left[\frac{1}{4}\sum_{\mathbf{q},n} \frac{(2 - \mathrm{e}^{i\mathbf{q}\cdot\mathbf{x}}\mathrm{e}^{z_n t} - \mathrm{e}^{-i\mathbf{q}\cdot\mathbf{x}}\mathrm{e}^{-z_n t})}{\beta N^0 \mathbf{q}^2 + \frac{1}{2}\frac{\beta^2 z_n^2}{\frac{\beta \mathbf{q}^2}{2N^0} + \frac{\beta v_{\mathbf{q}}}{V}}}\right]. \tag{22}$$

Here $\omega_{\mathbf{q}}^2 = \mathbf{q}^2(\mathbf{q}^2 + 2\rho_0 v_{\mathbf{q}})$ is the Bogoliubov dispersion. The Matsubara sums may be performed quite easily and we may derive an expression for the equal-time version of the time-ordered propagator,

$$\gamma_0 + S^<(\mathbf{x},0) = -\frac{1}{4N^0}\sum_{\mathbf{q}}\left(\frac{(\mathbf{q}^2 + 2\rho_0 v_{\mathbf{q}})^{1/2}}{|\mathbf{q}|} - 1\right)(1 - \cos(\mathbf{q}\cdot\mathbf{x})), \tag{23}$$

where

$$G^<(\mathbf{x},0) = \mathrm{e}^{\gamma_0 + S^<(\mathbf{x},0)} G_0^<(\mathbf{x},0). \tag{24}$$

The main reason why the one-particle propagator is interesting is that, while the Gaussian approximation leads to just Bogoliubov's theory as far as the computation of density–density correlation functions are concerned, the one-particle properties are singular in one dimension, just as in the case of fermions in one dimension.

5.1 *The condensate fraction at zero momentum*

The analysis in the preceeding sections employs a Gaussian approximation which is valid if the density fluctuations are small compared to the mean density. To see this more clearly we write the condition as follows:

$$\sqrt{\langle \rho_{\mathbf{q}} \rho_{-\mathbf{q}} \rangle} \ll N^0, \tag{25}$$

where N^0 is the total number of particles. The density–density correlation is given by $\langle \rho_{\mathbf{q}} \rho_{-\mathbf{q}} \rangle = N^0 S(\mathbf{q})$. Using the same Gaussian approximation we may deduce that

$$S(\mathbf{q}) = \frac{|\mathbf{q}|}{(\mathbf{q}^2 + 2\rho_0 v_{\mathbf{q}})^{1/2}}. \tag{26}$$

Thus this scheme is self-consistent (as opposed to self-contradictory) only if,

$$\frac{|\mathbf{q}|}{(\mathbf{q}^2 + 2\rho_0 v_{\mathbf{q}})^{1/2}} \ll N^0. \tag{27}$$

If we assume that $v_{\mathbf{q}} = \lambda|\mathbf{q}|^{m+2}$ we have, the condition (for $N^0 \gg 1$),

$$(2\rho_0\lambda)|\mathbf{q}|^m \gg -1. \tag{28}$$

It would appear that so long as $\lambda > 0$ and $m \leq -2$, this holds for small $|\mathbf{q}|$. Thus the delta-function potential ($m = -2$), the Coulomb potential in 2d ($m = -3$) and 3d ($m = -4$) all obey the inequality for small $|\mathbf{q}|$. For large enough $|\mathbf{q}|$ the left-hand side is zero but $0 \gg -1$ is not acceptable, and hence the Gaussian approximation breaks down for large $|\mathbf{q}|$. Thus it would appear that the results for the correlation functions are exact in the asymptotic limit. In real space, this is the $|\mathbf{x}| \to \infty$ limit. Let us examine this limit of the one-particle Green function. To this end let us write a formal expression for the momentum distribution as

$$\bar{n}_{\mathbf{k}} = f_0 N^0 \delta_{\mathbf{k},0} + f_{\mathbf{k}}. \tag{29}$$

Here $f_{\mathbf{k}}$ is a continuous function of $|\mathbf{k}|$ and is of order unity. $0 \leq f_0 \leq 1$ is the condensate fraction and N^0 is the total number of particles. The propagator in real space is then given by

$$\langle \psi^{\dagger}(\mathbf{x},0)\psi(\mathbf{0},0)\rangle = f_0\rho_0 + \frac{1}{V}\sum_{\mathbf{k}}\cos(\mathbf{k}\cdot\mathbf{x})f_{\mathbf{k}}. \tag{30}$$

In the ultra-asymptotic limit $|\mathbf{x}| \to \infty$, the cosine may be set equal to zero and we have just the first term. Thus it would appear that the Gaussian approximation being exact in this limit, yields the exact condensate fraction but not the exact $f_{\mathbf{k}}$. This is analogous to the assertion that in the Fermi case, the analogous method yields the exact quasi-particle residue but not the full momentum distribution and in case the quasi-particle residue is zero it yields the exact anomalous exponent. Here too when the condensate fraction is zero, we have to instead compute the anomalous exponent. Thus the exact condensate fraction is given by

$$f_0 = \exp\left[-\frac{1}{4N^0}\sum_{\mathbf{q}}\left(\frac{(\mathbf{q}^2 + 2\rho_0 v_{\mathbf{q}})^{1/2}}{|\mathbf{q}|} - 1\right)\right]. \tag{31}$$

For the delta-function interaction in 1d, the condensate fraction is zero. This could not have been guessed from Bogoliubov's theory that predicts (rather assumes) that a condensate always exists. In two dimensions, for the gauge potential $v_q = \text{const.}/q^2$, the condensate fraction is zero. We could now evaluate the total energy of the system (per particle) and compare with the results of Lieb and Liniger [10] who solve their model in 1d or with the results of Schick [11] which is valid in 2d but we do not expect the comparisons to be favorable since for the total energy to come out right we need the exact $f_{\mathbf{k}}$ rather than f_0 which does not contribute at all. Since the Gaussian approximation does not give this we shall not bother performing this calculation.

5.2 *Anomalous exponents*

In case the condensate fraction is zero, we have the bosonic analog of the Luttinger liquid or the Lieb–Liniger liquid. Here we have to instead compute the anomalous exponents which are also given exactly in the Gaussian approximation. Interestingly we may address the question whether the condensate is destroyed in more than one dimension also. In passing we note that the main purpose of this article is to hint at the usefullness of this approach in studying *fermions* coupled to gauge fields. There we have to polar decompose Grassmann variables – an exercise which is still in its infancy, although considerable inroads have been made by the author. For a delta-function interaction in 1d it is well-known that there is no condensation and instead we have to compute the anomalous exponent of the propagator. It is clear from the preceeding sections that we may write,

$$\gamma_0 + S^{<}(x,0) \approx -\frac{1}{4\pi\rho_0}\int_0^{\infty} \mathrm{d}q \frac{(2v_0\rho_0)^{1/2}}{|q|}(1-\cos(q\cdot x)) \sim -\gamma \mathrm{Ln}(|x|), \tag{32}$$

where $\gamma = (2v_0\rho_0)^{1/2}/(4\pi\rho_0)$ is the anomalous exponent. The interesting question is whether we can have the destruction of the condensate in two or three dimensions for realistic potentials? This would be the bosonic analog of the question 'Does Fermi liquid theory break down in two or three dimensions?' In two dimensions, for the interaction $v_q = \mathrm{const.}/q^2$ (gauge potential) the condensate fraction is zero. It would appear that for realistic Fourier transformable potentials in three dimensions we have Bose condensation but not in less than three dimensions. We have derived an exact formula for the condensate fraction that is valid for functional forms of the interaction that are Fourier transformable and are subject to the constraint mentioned previously, namely, that the long-wavelength limit is exactly given by the Gaussian approximation.

6. Correlation functions with gauge fields

When one is performing the path integral with gauge fields, one must be careful about preserving gauge symmetry. This is the crucial aspect that may be elegantly treated in the present approach. The usual method for treating such constraints in the context of path integrals is the well-known Faddeev–Popov method [12]. However, we are able to treat gauge symmetry in all its generality without ever having to fix the gauge at any time. Thus the path integral is to be performed such that the following

$$eA_0' + \partial_t\Pi' = eA_0 + \partial_t\Pi \tag{33}$$

$$e\mathbf{A}' + \nabla\Pi' = e\mathbf{A} + \nabla\Pi \tag{34}$$

constraints are obeyed. We may therefore simply solve for the gauge fields straight away in terms of the conjugate variables. Define an arbitrary gauge constant C. This is constant in the sense that changes in the phase of the field cancel out the changes in the vector potential. However, in the end we will have to integrate over this variable as well. It will be shown that when $e = 0$ we recover the non-interacting propagator.

$$C_0 = eA_0 + \partial_t\Pi, \tag{35}$$

$$\mathbf{C} = e\mathbf{A} + \nabla\Pi. \tag{36}$$

In the non-interacting limit ($e \equiv 0$), Π' and Π differ at most by a trivial constant and hence the vector $\mathbf{C}$ is irrotational. This observation will be made use of later. The action now may be recast in terms of the gauge constant and the density and phase variables. The field tensor is then simply given by, $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. Thus the total action simply reads,

$$S = -\int \frac{(\nabla\rho)^2}{4\rho} + \int \rho C_0 - \int \rho \mathbf{C}^2 - \frac{1}{4e^2}\int F^2. \tag{37}$$

We may see by examining eq. (37) that in the limit $e \to 0$, the field tensor has to be zero for the partition function to be non-vanishing. Thus in this limit we must have $C_\mu = \partial_\mu \Pi$ for some scalar Π so that $F_{\mu\nu} \equiv 0$. This then gives us the action functional of the free theory as it should.

The next question that is worth answering is the following. How should the current operator be defined when there are gauge fields? In particular, is the current operator gauge invariant? The density surely is gauge invariant. If we would like to treat currents and densities as part of a canonical set of variables then we better have a definition that is gauge invariant. The definition $\mathbf{J} = -\rho\nabla\Pi$ is not gauge invariant. Indeed, it changes by additive amounts everytime a gauge transformation is performed. In order to remedy this, we redefine the current to be $\mathbf{J} = -\rho\mathbf{C}$. As we have seen, in the non-interacting limit, this reduces to the form already shown, since there $\mathbf{C} = \nabla\Pi$. In general however, it does not. Now we would like an expression for the field operator also in terms of C. Using the DPVA [8] in terms of currents and densities we may write

$$\psi(\mathbf{x}) = \mathrm{e}^{-iX_0 + i\int^{\mathbf{x}} d\mathbf{l}\cdot\frac{\mathbf{J}}{\rho}}\sqrt{\rho(\mathbf{x})}. \tag{38}$$

Using the formula for the current in terms of the gauge constant we may write

$$\psi(\mathbf{x}, t) = \mathrm{e}^{-iX_0(t) - i\int^{\mathbf{x}} \mathrm{d}\mathbf{l}\cdot\mathbf{C}(\mathbf{x}', t)}\sqrt{\rho(\mathbf{x}, t)}. \tag{39}$$

Computing the propagator of the interacting theory then means evaluating the following path integral:

$$G(\mathbf{x}, t) = \frac{-i}{Z}\int D[C]D[\rho]\mathrm{e}^{iS}\psi(\mathbf{x}, t)\psi^\dagger(\mathbf{0}, 0) \tag{40}$$

and

$$Z = \int D[C]D[\rho]\mathrm{e}^{iS}, \tag{41}$$

where S is given as in eq. (37). The integration over the gauge constant C may be justified as follows: In the original path integral we had to integrate over ψ, $\psi^\dagger$ and the four A^μs. Thus we had six variables in all, except that the gauge constraint meant that one of the variables, say the phase of the field, depended on the gauge fields thus the number of independent variables reduced to five. In the definition eq. (41) we have a similar situation – integrating over the four components of the gauge constant plus the density ρ makes five variables as it should. The action of the free gauge field is given by

$$S_{\rm guage} = \frac{1}{2e^2}\int (\partial_0 \mathbf{C} - \nabla C_0)^2 + \frac{v_{\rm l}^2}{2e^2}\int \mathbf{C}\nabla^2\mathbf{C} + \frac{v_{\rm l}^2}{2e^2}\int (\nabla\cdot\mathbf{C})^2. \quad (42)$$

A word regarding units is appropriate. We have set $\hbar = 2m = 1$. The presence of gauge fields implies that there is a further dimensionful parameter namely the speed of light which we denote by $v_{\rm l}$. If we set $v_{\rm l} = 1$ then all quantities are dimensionless. This is undesirable. Hence we retain the speed of light as it is. The various quantities in this work now have the following dimensions. $[\rho_{\mathbf{k}n}] = 1$ but $[\rho(\mathbf{x},t)] = [V]^{-1}$; $[\mathbf{C}(\mathbf{x},t)] = [\mathbf{C}_{\mathbf{k}n}] = [\mathbf{k}] = [\nabla] = [v_{\rm l}]$; $\mathbf{k}^2 = [\text{Energy}] = [\partial_0]$; $\beta = [\text{Energy}]^{-1}$ and finally $[e^2/V] = [\text{Energy}]^2$. Using the decomposition into the various modes we may write

$$\begin{aligned}
S = {} & \frac{i\beta}{4N^0}\sum_{\mathbf{k}}\mathbf{k}^2\rho_{\mathbf{k},n}\rho_{-\mathbf{k},-n} + (-i\beta)\sum_{\mathbf{k},n}\rho_{\mathbf{k},n}C^0_{\mathbf{k},n} \\
& +(i\beta N^0)\sum_{\mathbf{k},n}\mathbf{C}_{\mathbf{k},n}\cdot\mathbf{C}_{-\mathbf{k},-n} + (i\beta)\frac{V}{2e^2}\sum_{\mathbf{k},n}z_n^2\mathbf{C}_{\mathbf{k},n}\cdot\mathbf{C}_{-\mathbf{k},-n} \\
& -(i\beta)\frac{V}{2e^2}\sum_{\mathbf{k},n}(2iz_n\mathbf{k})\mathbf{C}_{\mathbf{k},n}C^0_{-\mathbf{k},-n} - (i\beta)\frac{V}{2e^2}\sum_{\mathbf{k}}\mathbf{k}^2C^0_{\mathbf{k},n}C^0_{-\mathbf{k},-n} \\
& +(i\beta)v_{\rm l}^2\ \frac{V}{2e^2}\sum_{\mathbf{k},n}\mathbf{k}^2\ \mathbf{C}_{\mathbf{k},n}\cdot\mathbf{C}_{-\mathbf{k},-n} \\
& -(i\beta)v_{\rm l}^2\frac{V}{2e^2}\sum_{\mathbf{k},n}(\mathbf{k}\cdot\mathbf{C}_{\mathbf{k},n})(\mathbf{k}\cdot\mathbf{C}_{-\mathbf{k},-n}). \qquad (43)
\end{aligned}$$

After integrating out C^0 we have,

$$\begin{aligned}
S = {} & \frac{i\beta}{4N^0}\sum_{\mathbf{k}n}\left(\mathbf{k}^2 + \frac{2\rho_0 e^2}{\mathbf{k}^2}\right)\rho_{\mathbf{k},n}\rho_{-\mathbf{k},-n} + (i\beta)\sum_{\mathbf{k},n}\frac{iz_n}{\mathbf{k}^2}\rho_{\mathbf{k}n}(\mathbf{k}\cdot\mathbf{C}_{\mathbf{k}n}) \\
& +(i\beta)\sum_{\mathbf{k}n}\left(N^0 + v_{\rm l}^2\frac{V}{2e^2}\mathbf{k}^2\right)\mathbf{C}_{\mathbf{k},n}\cdot\mathbf{C}_{-\mathbf{k},-n} \\
& -(i\beta)v_{\rm l}^2\frac{V}{2e^2}\sum_{\mathbf{k}n}(\mathbf{k}\cdot\mathbf{C}_{\mathbf{k}n})(\mathbf{k}\cdot\mathbf{C}_{-\mathbf{k},-n}) \\
& +(i\beta)\frac{V}{2e^2}\sum_{\mathbf{k}n}z_n^2\mathbf{C}_{\mathbf{k}n}\cdot\mathbf{C}_{-\mathbf{k},-n} \\
& -(i\beta)\frac{V}{2e^2}\sum_{\mathbf{k}n}\frac{z_n^2}{\mathbf{k}^2}(\mathbf{k}\cdot\mathbf{C}_{\mathbf{k}n})(\mathbf{k}\cdot\mathbf{C}_{-\mathbf{k},-n}). \qquad (44)
\end{aligned}$$

Observe that in the limit $e \to 0$, we must have $\mathbf{C}_{\mathbf{k},n} = \mathbf{k}X_{\mathbf{k},n}$ for the partition function to be non-vanishing. Thus we recover the non-interacting theory in this limit. Now we would like to compute two quantities. One is the dynamical density–density correlation function. Here we would like to see radiation corrections to corresponding spectral function. However, it is not present at the Gaussian level at which we are presently operating. Thus we shall be content at reproducing results equivalent to Bogoliubov theory. This is a novel way of recovering the Bogoliubov spectrum by introducing gauge fields and using the path integral formalism. The other quantity of interest is the circulation of the gauge constant. This quantity is zero in the non-intearcting case since the vector $\mathbf{C}$ may be expressed as the gradient of a scalar. In fact we may see from the expression of the current that the velocity of the fluid is simply given by $\mathbf{v} = -\mathbf{C}$. This is irrotational in the absence of gauge fields. In general however this is not the case. The circulation of the velocity around a closed loop P is given by, $V(P,t) = -\oint_P \mathrm{d}\mathbf{l} \cdot \mathbf{C}(\mathbf{x},t)$. Physically, the quantity $V(P,t)$ is a measure of the strength of vortices in the system. The average $\langle V(P,t)\rangle = 0$ is trivially zero. Thus we have to examine the fluctuation $\langle V^2(P,t)\rangle^{1/2}$. First the density–density correlation function: In order to evaluate this we must first perform the integral: $\int D[\mathbf{C}]\mathrm{e}^{iS}$. Again, here we may use the usual procedure of translating the integration variable by a constant amount and we obtain the following effective action.

$$S = \frac{i\beta}{4N^0}\sum_{\mathbf{k},n}\left(\mathbf{k}^2 + \frac{z_n^2 + 2e^2\rho_0}{\mathbf{k}^2}\right)\rho_{\mathbf{k},n}\rho_{-\mathbf{k},-n}. \tag{45}$$

From this we may immediately deduce the static structure factor as

$$S(\mathbf{k}) = \frac{\mathbf{k}^2}{\omega_{\mathbf{k}}}, \tag{46}$$

where the Bogoliubov dispersion $\omega_{\mathbf{k}}$ is given by the positive real solution to

$$\mathbf{k}^2 + \frac{(iz)^2 + 2e^2\rho_0}{\mathbf{k}^2} = 0. \tag{47}$$

In other words,

$$\omega_{\mathbf{k}} = |\mathbf{k}|\sqrt{\mathbf{k}^2 + 2\rho_0 v_{\mathbf{k}}}, \tag{48}$$

where $v_{\mathbf{k}} = e^2/\mathbf{k}^2$. Therefore as is well-known, independent of the dimensionality of space, the interaction in the presence of gauge fields is forced to be of the form $\sim 1/\mathbf{k}^2$. In three space dimensions, this corresponds to Coulomb interaction. This is a necessary consistency check.

Some may want to evaluate the propagator. For this we have to evaluate the correlation function $\langle \mathbf{C}^i(\mathbf{x},t)\mathbf{C}^j(\mathbf{0},0)\rangle$. There is a more interesting reason to study this, namely to compute the vortex strength. We have found that even though at the Gaussian level the density–density correlation is unremarkable, the velocity–velocity correlation function does exhibit some new physics. To see this we note that integrating out the velocity variable means replacing $\mathbf{C}_{\mathbf{k}n} \to \mathbf{C}_{\mathbf{k}n} + \mathbf{k}\Lambda_{\mathbf{k}n}$ for an appropriate Λ. This makes all the terms involving the speed of light and most other terms drop out and we are led to eq. (45). However, if we integrate out the density first and retain the velocity as it is then we find that the final action is no longer so simple. In particular, it will involve both longitudinal and transverse terms, the latter being responsible for vortices as we shall see. If we integrate out ρ first we are led to the following effective action. Here we have also introduced a source for velocity.

$$
\begin{aligned}
S_0 = & -i\beta N^0 \sum_{\mathbf{k}n} \frac{(iz_n/\mathbf{k}^2)^2}{(\mathbf{k}^2 + 2\rho^0 e^2/\mathbf{k}^2)} (\mathbf{k}\cdot\mathbf{C}_{\mathbf{k}n})(\mathbf{k}\cdot\mathbf{C}_{-\mathbf{k},-n}) \\
& + \sum_{\mathbf{k}n} \mathbf{A}_{\mathbf{k}n}\cdot\mathbf{C}_{\mathbf{k}n} + (i\beta)\sum_{\mathbf{k}n}\left(N^0 + v_1^2 \frac{V}{2e^2}\mathbf{k}^2\right)\mathbf{C}_{\mathbf{k},n}\cdot\mathbf{C}_{-\mathbf{k},-n} \\
& -(i\beta)v_1^2\frac{V}{2e^2}\sum_{\mathbf{k}n}(\mathbf{k}\cdot\mathbf{C}_{\mathbf{k}n})(\mathbf{k}\cdot\mathbf{C}_{-\mathbf{k},-n}) \\
& +(i\beta)\frac{V}{2e^2}\sum_{\mathbf{k}n} z_n^2 \; \mathbf{C}_{\mathbf{k}n}\cdot\mathbf{C}_{-\mathbf{k},-n} \\
& -(i\beta)\frac{V}{2e^2}\sum_{\mathbf{k}n}\frac{z_n^2}{\mathbf{k}^2}(\mathbf{k}\cdot\mathbf{C}_{\mathbf{k}n})(\mathbf{k}\cdot\mathbf{C}_{-\mathbf{k},-n}).
\end{aligned} \tag{49}
$$

It can be seen from eq. (49) that in the limit $v_1 \to \infty$, the partition function is non-vanishing only if $\mathbf{k}^2\mathbf{C}_{\mathbf{k}n}\cdot\mathbf{C}_{-\mathbf{k},-n} = (\mathbf{k}\cdot\mathbf{C}_{\mathbf{k}n})(\mathbf{k}\cdot\mathbf{C}_{-\mathbf{k},-n})$. In other words, only if $\mathbf{C}_{\mathbf{k}n} = \mathbf{k}\Lambda_{\mathbf{k}n}$. This is the same as saying that the velocity is the gradient of some scalar $\mathbf{v} = -\nabla\Lambda$. Thus in this limit there are no vortices. Vortices arise as a result of a finite speed of light, in other words due to radiation corrections. Now we evaluate the partition function $\tilde{Z}([\mathbf{A}]) = Z([\mathbf{A}])/Z([\mathbf{0}])$ where $Z([\mathbf{A}]) = \int D[\mathbf{C}]\mathrm{e}^{iS_0}$. After some tedious algebra we arrive at,

$$
\begin{aligned}
\tilde{Z}(\mathbf{A}) = \exp & \left[-\frac{1}{2}\sum_{\mathbf{k}n} f(\mathbf{k},n)(\mathbf{k}\cdot\mathbf{A}_{\mathbf{k},n})(\mathbf{k}\cdot\mathbf{A}_{-\mathbf{k},-n}) \right. \\
& \left. -\frac{1}{2}\sum_{\mathbf{k}n} g(\mathbf{k},n)\mathbf{A}_{\mathbf{k},n}\cdot\mathbf{A}_{-\mathbf{k},-n} \right],
\end{aligned} \tag{50}
$$

where

$$
\begin{aligned}
f(\mathbf{k},n) = & (2\beta N^0)^{-1}\left[\frac{(iz_n/\mathbf{k}^2)^2}{\mathbf{k}^2 + 2\rho^0 e^2/\mathbf{k}^2} + \frac{1}{2\rho^0 e^2}\left(v_1^2 + \frac{z_n^2}{\mathbf{k}^2}\right)\right] \\
& \times \left(1 + \frac{1}{2\rho^0 e^2}(v_1^2\mathbf{k}^2 + z_n^2)\right)^{-1}\left(1 + \frac{z_n^2/\mathbf{k}^2}{\mathbf{k}^2 + 2\rho^0 e^2/\mathbf{k}^2}\right)^{-1}
\end{aligned} \tag{51}
$$

$$
g(\mathbf{k},n) = (2\beta N^0)^{-1}\left(1 + \frac{1}{2\rho^0 e^2}(v_1^2\mathbf{k}^2 + z_n^2)\right)^{-1}. \tag{52}
$$

We may deduce a formula for the time ordered velocity–velocity correlation function,

$$
\left\langle T\mathbf{C}^i(\mathbf{x}',t')\mathbf{C}^j(\mathbf{x},t)\right\rangle = \left[\nabla^i_{\mathbf{x}'}\nabla^j_{\mathbf{x}}F(\mathbf{x}'-\mathbf{x},t'-t) + H(\mathbf{x}'-\mathbf{x},t'-t)\delta_{i,j}\right], \tag{53}
$$

$$
F(\mathbf{x}',t') = \sum_{\mathbf{k},n} f(\mathbf{k},n)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}'}\mathrm{e}^{-z_n t'}; H(\mathbf{x}',t') = \sum_{\mathbf{k},n} g(\mathbf{k},n)\mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}'}\mathrm{e}^{-z_n t'}. \tag{54}
$$

The Σ and σ below are defined in the appendix.

$$\Sigma(t-t';P,P') = -i\oint_{\mathbf{l}'\in P'}\oint_{\mathbf{l}\in P} d\mathbf{l}'\cdot \mathrm{d}\mathbf{l}H(\mathbf{x}'-\mathbf{x},t'-t). \tag{55}$$

Also we may write for the propensity to create vortices (defined in the appendix),

$$\sigma(P,P') = -\frac{e^2}{4\pi v_{\mathrm{l}}^2}\oint_{\mathbf{l}'\in P'}\oint_{\mathbf{l}\in P} \mathrm{d}\mathbf{l}'\cdot \mathrm{d}\mathbf{l}\frac{\mathrm{e}^{-\lambda|\mathbf{x}'-\mathbf{x}|}}{|\mathbf{x}'-\mathbf{x}|}, \tag{56}$$

where $\lambda = (2\rho^0 e^2/v_{\mathrm{l}}^2)^{1/2}$. This quantity looks finite. Hence there is no vortex instability in the Gaussian approximation. Let us now compute the vortex strength as defined earlier.

$$\langle V^2(P,t)\rangle = \oint_{\mathbf{l}'\in P}\oint_{\mathbf{l}\in P} \mathrm{d}\mathbf{l}'\cdot \mathrm{d}\mathbf{l}H(\mathbf{x}'-\mathbf{x},0), \tag{57}$$

$$H(\mathbf{x}'-\mathbf{x},0) = \frac{e^2}{2V}\sum_{\mathbf{k}}\frac{\mathrm{e}^{i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})}}{\omega_{\mathrm{l}}(\mathbf{k})}\coth\left(\frac{\beta\omega_{\mathrm{l}}(\mathbf{k})}{2}\right), \tag{58}$$

where $\omega_{\mathrm{l}}(\mathbf{k}) = (2\rho^0 e^2 + v_{\mathrm{l}}^2\mathbf{k}^2)^{1/2}$. This dispersion shows that the photons have acquired mass by coupling to matter. An explicit evaluation of this and further analysis is possible and will not be done here since our interest is merely to highlight the usefulness of these ideas. However, some intuitive justification of the above formulas is in order. First the vortex strength and susceptibility are zero for a non-interacting system. This is hardly surprising since the velocity operator of free bosons is irrotational in our formalism. The author has tried to prove this independently using the second quantization formalism but has been unsuccessful in proving it rigorously. It appears that the only obstruction to this conclusion at least at the formal mathematical level, is the lack of self-adjointness of the canonical conjugate to the total number of bosons [8,13]. In the high density limit this is not a problem. Thus, as far as the asymptotics are concerned, our approach provides reliable answers. In other words, the expression for the velocity–velocity correlation function is valid in the large separation $|\mathbf{x}-\mathbf{x}'|\to\infty$ limit. However, the main message is that the vortices are due to the finiteness of the speed of light. That is, even in an interacting system, there are no vortices if we include only the longitudinal density–density interactions. Coupling to transverse radiation fields is needed to generate vortices. This may be understood by realizing that in order to create vortices we have to supply angular momentum. Photons being spin-1 particles, possess the angular momentum that the charged bosons use to generate vortices.

7. Conclusions

We have written down formulas for the one-particle Green function of a homogeneous Bose system that is exact in the asymptotic limit. From the exact asymptotic form we may extract an exact formula for the condensate fraction and if it is zero an exact formula for the anomalous exponent. We have also computed the vortex strength and shown that radiation corrections are responsible for the vortices. No other quantity is given exactly in the formalism outlined. The total energy per particle, roton minimum and other important physical attributes will have to involve going beyond the Gaussian approximation. Nevertheless, the agenda in the immediate future is to apply these ideas to fermions and then write down a theory of neutral matter with nuclei and electrons treated on an equal footing.

Appendix

In this section, we derive a Kubo-like formula that relates the propensity for vortices emerging (an analog of d.c. conductivity) with microscopic velocity–velocity correlation functions. Let us write down the following formula for the vortex strength with a source for vortices in the interaction representation. As defined in the main text, the vorex strength may be written as follows:

$$V(P,t) = -\oint_P \mathrm{d}\mathbf{l} \cdot \mathbf{C}(\mathbf{x},t) \tag{59}$$

$$\langle V(P,t)\rangle = \frac{\left\langle TS\hat{V}(P,t)\right\rangle}{\langle TS\rangle} \tag{60}$$

$$S = \exp\left[-i\int_0^{-i\beta} \mathrm{d}t' \sum_{P'} \hat{V}(P',t')W(P',t')\right]. \tag{61}$$

Here $W(P,t)$ is a whirlpool source for the vortex $V(P,t)$. From linear response theory, we may expect that for weak sources the vortex strength is proportional to the convolution of the source with a linear response coefficient.

$$\langle V(P,t)\rangle = \sum_{P'} \int_0^{-i\beta} \mathrm{d}t' \Sigma(t-t';P,P')W(P',t'). \tag{62}$$

From this we see that

$$\Sigma(t-t';P,P') = \left(\frac{\delta}{\delta W(P',t')}\langle V(P,t)\rangle\right)_{W\equiv 0} \tag{63}$$

$$\Sigma(t-t';P,P') = -i\left\langle T\hat{V}(P',t')\hat{V}(P,t)\right\rangle. \tag{64}$$

If the whirlpool source is time-independent, we may write

$$\langle V(P)\rangle = \sum_{P'} \sigma(P,P')W(P') \tag{65}$$

$$\sigma(P,P') = \int_0^{-i\beta} \mathrm{d}t' \Sigma(t-t';P,P'). \tag{66}$$

If $\sigma(P,P') = \infty$ then it signals instability to vortices, namely arbitrarily weak perturbations can lead to vortices. Thus we have,

$$\Sigma(t-t';P,P') = -i\oint_{P'} \mathrm{d}\mathbf{l}'_i \oint_P \mathrm{d}\mathbf{l}_j \left\langle T\mathbf{C}^i(\mathbf{x}',t')\mathbf{C}^j(\mathbf{x},t)\right\rangle. \tag{67}$$

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