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Painlevé analysis and integrability of two-coupled non-linear oscillators

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Abstract. Integrability of a linearly damped two-coupled non-linear oscillators equation

$$\ddot{x} = -d\dot{x} - \alpha x - \delta_1 \left(x^2 + y^2\right) - 2\delta_2 xy ,$$
$$\ddot{y} = -d\dot{y} - \beta y - \delta_2 \left(x^2 + y^2\right) - 2\delta_1 xy$$

is investigated by employing the Painlevé analysis. The following two integrable cases are identified: (i) $d = 0$, $\alpha = \beta$, δ_1 and δ_2 are arbitrary. (ii) $d^2 = 25\alpha/6$, $\alpha = \beta$, δ_1 and δ_2 are arbitrary. Exact analytical solution is constructed for the integrable choices.

Keywords. Two-coupled non-linear oscillators; Painlevé analysis; exact analytical solution.

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1. Introduction

For non-linear systems integrating the equations of motion completely, obtaining analytical solutions and finding acceptable constants of motions seem to be rare. From a qualitative point of view, integrability can be considered as a mathematical property that can be successfully used to obtain more predictive power and quantitative informations to understand the dynamics of the system locally and globally [1–8]. Integrability nature of dynamical systems can be methodologically investigated employing Painlevé analysis [1–3]. If the solution of a non-linear differential equation is free from movable critical singular points, it is said to possess the Painlevé property. In this case the system is expected to be integrable.

In recent years more attention is paid to the study of coupled non-linear oscillators. Coupled non-linear oscillators describe a variety of self-organization phenomena. Examples include multi-rhythmicity of heart beating [9,10], wave in ensembles of intestinal cells [11], oscillations in chemical reactions [12,13], transition between two oscillation modes [14], wave fronts in coupled Lorentz oscillators [15] and coupled MLC circuits [3,16], in-phase and out-of-phase solutions [17] and self-organized criticality in pulse coupled relaxation oscillators [18,19].

We consider a two-coupled non-linear oscillator with the potential

$$V(x,y) = \frac{\alpha}{2}x^2 + \frac{\beta}{2}y^2 + \frac{\delta_1}{3}x^3 + \frac{\delta_2}{3}y^3 + \delta_1 xy^2 + \delta_2 x^2 y. \tag{1}$$

The equation of motion of the system with the addition of linear damping is written as

$$\ddot{x} = -d\dot{x} - \alpha x - \delta_1\left(x^2 + y^2\right) - 2\delta_2 xy, \tag{2a}$$

$$\ddot{y} = -d\dot{y} - \beta y - \delta_2\left(x^2 + y^2\right) - 2\delta_1 xy. \tag{2b}$$

$V(x,y)$ given by eq. (1) is the potential of the two harmonic oscillators coupled by conservative coupling. Further, the type of non-linear coupling in the potential (1) is present in a variety of plasma waves [20,21], for example, Rossby waves in atmosphere and gravity-capillary waves in deep water. Analog simulation of eq. (2) is also possible. The system (2) is considered as the simplest model for a hard in amplitude transition to chaos [22,23]. Sudden transition to chaos was found in an analog simulation of the system (2) [24]. The effect of noise has also been studied in eq. (2) [25].

In the present work, we wish to investigate integrability of eq. (2) by applying the Painlevé analysis. To be self-contained, in §2 we briefly outline the salient features of the Painlevé analysis. In §3 we perform the Painlevé test to eq. (2). We identify two integrable limits. Then, in §4 we obtain the analytical solution for the two integrable cases. Finally, §5 contains conclusions.

2. Painlevé analysis

Consider an nth order ordinary differential equation of the form

$$\dot{x}_i = F_i\left(x_1, \ldots, x_n; t\right), \quad i = 1, 2, \ldots, n, \tag{3}$$

where F_i are rational in $x_1, x_2, \ldots, x_n$ and analytic in t. The ordinary differential eq. (3) is said to have the Painlevé property if all movable singularities of the solutions are poles. The term 'strong Painlevé' is used when the solution in the neighbourhood of an arbitrary singularity t^* can be expressed as $\tau = (t - t^*)^{-p}$, where p is an integer determined from the leading order, so that the movable algebraic or logarithmic branch points as well as essential singularities are excluded. A necessary condition for eq. (3) to have the Painlevé property is that there is a Laurent series expansion with $(n-1)$ arbitrary expansion coefficients. The analysis consists of three steps, dealing with the dominant behaviours, the resonances and the constants of integrations, respectively [1–3,26,27].

2.1 *Dominant behaviours*

The first step is the determination of the leading order behaviours of x_i in the neighbourhood of a movable singularity t^* in the form $x_i \approx a_{i0}\,(t - t^*)^{p_i}$, as

$t \rightarrow t^*$, a_{i0} = constant. If all the allowed p_is are negative integers, the solution may correspond to the strong Painlevé property. On the other hand, if any of the p_is is a rational fraction, the solution may be associated with the weak P-property [28]. In either case, the solution is written in the form of Laurent series as

$$x_i(t) = \tau^{p_i} \sum_{k=0}^{\infty} a_{ik} \tau^k \,, \tag{4}$$

where $\tau = t - t^*$.

2.2 *Resonances*

The next step is to find the resonances, that is, the values of the order at which arbitrary constants will enter in the expansion of the solution near the singularity at $t = t^*$. Apart from t^*, there are $(n-1)$ other arbitrary constants for eq. (3). To determine the resonances, the solution

$$x_i \approx a_{i0}\tau^{p_i} + \Omega_i \tau^{p_i + r}, \quad r > 0, \quad i = 1, 2, \ldots, n \tag{5}$$

is substituted in eq. (3) and the leading order terms in Ω_i are retained. The reduced equation will be of the form

$$Q(r) \cdot \Omega = 0, \quad \Omega = (\Omega_1, \ldots, \Omega_n)\,, \tag{6}$$

where $Q(r)$ is a square matrix of order n with r appearing only in its diagonal elements. The roots of equation $\det Q(r) = 0$ are the resonances.

2.3 *The constants of integration*

The final step is to verify the existence of sufficient number of arbitrary constants at the resonances without the introduction of logarithmic branch points. For this purpose, the truncated expansion

$$x_i = a_{i0}\tau^{p_i} + \sum_{k=1}^{r_l} a_{ik} \tau^{p_i + k} \,, \tag{7}$$

where r_l the largest resonance value which is substituted in eq. (3). At the resonances, one usually finds some conditions termed ‘compatibility conditions’ that have to be satisfied in order to secure arbitrariness of the coefficients.

3. The Painlevé property of the two-coupled non-linear oscillators equation (2)

We apply the P-analysis to the two-coupled anharmonic oscillators eq. (2). The analysis starts with the leading order behaviours.

3.1 *Leading order analysis*

We assume the leading orders to be

$$x = a_0 \tau^p, \quad y = b_0 \tau^q, \quad \tau = (t - t^*) \to 0. \tag{8}$$

To determine p, q, a_0 and b_0 we substitute eq. (8) in eq. (2). We obtain the following pairs of leading order equations:

$$\begin{aligned} a_0 p(p-1)\tau^{p-2} &= -d a_0 p \tau^{p-1} - \alpha a_0 \tau^p - \delta_1 a_0^2 \tau^{2p} - \delta_1 b_0^2 \tau^{2q} \\ &\quad -2\delta_2 a_0 b_0 \tau^{p+q}, \end{aligned} \tag{9a}$$

$$\begin{aligned} b_0 q(q-1)\tau^{q-2} &= -d b_0 q \tau^{q-1} - \beta b_0 \tau^q - \delta_2 b_0^2 \tau^{2q} - \delta_2 a_0^2 \tau^{2p} \\ &\quad -2\delta_1 a_0 b_0 \tau^{p+q}. \end{aligned} \tag{9b}$$

From eq. (9) the leading order behaviour is obtained as $p = q = -2$. Then, from the coefficient of (τ^{-4}, τ^{-4}) in eq. (9) we have

$$6a_0 = -\delta_1 a_0^2 - \delta_1 b_0^2 - 2\delta_2 a_0 b_0, \tag{10a}$$

$$6b_0 = -\delta_2 b_0^2 - \delta_2 a_0^2 - 2\delta_1 a_0 b_0. \tag{10b}$$

Multiplying eq. (10a) by b_0 and eq. (10b) by a_0 and subtracting one from the other we get

$$\left(a_0^2 - b_0^2\right)(\delta_2 a_0 + \delta_1 b_0) = 0. \tag{11}$$

Equation (11) is satisfied if $a_0 = b_0$ or $a_0 = -b_0$ or $\delta_2 a_0 + \delta_1 b_0 = 0$. For each of these choices a_0 and b_0 can be determined from eqs (10). We have the following three leading orders:

Case 1

$$p = q = -2, \quad a_0 = b_0 = -\frac{3}{\delta_1 + \delta_2}. \tag{12a}$$

Case 2

$$p = q = -2, \quad a_0 = -b_0 = \frac{3}{\delta_2 - \delta_1}, \quad a_0 - b_0 = \frac{6}{\delta_2 - \delta_1}. \tag{12b}$$

Case 3

$$p = q = -2, \quad \delta_2 a_0 + \delta_1 b_0 = 0, \quad a_0 = \frac{6\delta_1}{\delta_2^2 - \delta_1^2}, \quad b_0 = -\frac{6\delta_2}{\delta_2^2 - \delta_1^2}. \tag{12c}$$

The next step is the resonance analysis for each of the three leading orders.

3.2 *Resonances*

To identify the resonances, the powers of eq. (5) at which arbitrary constants will enter, we write

$$x \approx a_0\tau^p + \Omega_1\tau^{p+r}, \quad y \approx b_0\tau^q + \Omega_2\tau^{q+r}. \tag{13}$$

Equation (13) is substituted into eq. (2). From the coefficients of (τ^{r-4}, τ^{r-4}) we obtain

$$\Omega_1\left[(r-2)(r-3) + 2\delta_1 a_0 + 2\delta_2 b_0\right] + \Omega_2\left[2\delta_2 a_0 + 2\delta_1 b_0\right] = 0, \tag{14a}$$

$$\Omega_1\left[2\delta_2 a_0 + 2\delta_1 b_0\right] + \Omega_2\left[(r-2)(r-3) + 2\delta_1 a_0 + 2\delta_2 b_0\right] = 0. \tag{14b}$$

The above system of linear equations is rewritten in the form of (6) as

$$Q(r)\cdot\Omega = \begin{pmatrix} (r-2)(r-3) + 2\delta_1 a_0 + 2\delta_2 b_0 & 2\delta_2 a_0 + 2\delta_1 b_0 \\ 2\delta_2 a_0 + 2\delta_1 b_0 & (r-2)(r-3) + 2\delta_2 b_0 + 2\delta_1 a_0 \end{pmatrix} \times \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = 0. \tag{15}$$

The condition for non-trivial solutions (Ω_1, Ω_2) is the determinant of $Q(r)$ equal to zero.

Case 1

For eq. (12a) the matrix Q is

$$Q(r) = \begin{pmatrix} r^2 - 5r & -6 \\ -6 & r^2 - 5r \end{pmatrix}. \tag{16}$$

The roots of $\det Q(r) = 0$ are $r = -1, 2, 3, 6$. The root -1 corresponds to the arbitrariness of t^* in eq. (8).

Case 2

For eq. (12b), the values of r are found to be $r = -1, 2, 3, 6$ which are the same as that of Case 1.

Case 3

For eq. (12c) the determinant of $Q(r)$ leads to the equation $(r+1)^2(r-6)^2 = 0$. The resonances are thus $r = -1, -1, 6, 6$.

Thus, for eq. (2) we identified three sets of full resonances. The resonance analysis tells us which coefficients should be arbitrary. In Cases 1 and 2 apart from the arbitrariness of t^* the coefficients a_2 or b_2, a_3 or b_3 and a_6 or b_6 must be arbitrary to satisfy P-property. Then the Laurent series will have four arbitrary constants. Such a branch of solutions $x(t)$ and $y(t)$ with sufficient number of arbitrary constants is termed as main branch [2,3]. In Case 3 the resonance values are $r = -1, -1, 6, 6$. $r = -1$ and 6 appear twice. One of the $r = -1$ values corresponds to arbitrariness of t^*. For the solution to be free from movable critical points, a_6 and b_6 must be arbitrary. In this case the Laurent series will have only three arbitrary constants. Such a branch of solution with less number of arbitrary constants is known as subsidiary branch [2,3]. Next, we check the arbitrariness of the coefficients indicated by the resonance analysis.

3.3 *Identification of arbitrary constants of integration*

To verify the occurrence of sufficient number of arbitrary constants, let us consider the series expansion

$$x = a_0\tau^p + \sum_{k=1}^{r_l} a_k\tau^{p+k}, \quad y = b_0\tau^q + \sum_{k=1}^{r_l} b_k\tau^{q+k}. \tag{17}$$

We substitute eq. (17) in eq. (2) and equate the coefficients of like powers of τ on both sides of the expansion.

Case 1

The resonance values $r = -1, 2, 3, 6$ imply that in addition to t^*, three more arbitrary constants exist. Thus, for eq. (2) to satisfy P-property, a_2 (or b_2), a_3 (or b_3) and a_6 (or b_6) must be arbitrary.

From the coefficients of (τ^{-3}, τ^{-3}) we obtain

$$3a_1 + 2b_1 = -da_0, \tag{18a}$$

$$3b_1 + 2a_1 = -da_0, \tag{18b}$$

which imply $a_1 = b_1$. Substitution of $a_1 = b_1$ in (18) gives

$$a_1 = b_1 = -\frac{d}{5}a_0. \tag{19}$$

From the coefficients of (τ^{-2}, τ^{-2}) we have

$$a_2 + b_2 = \frac{(25\alpha - d^2)}{150}\, a_0, \tag{20a}$$

$$a_2 + b_2 = \frac{(25\beta - d^2)}{150}\, a_0. \tag{20b}$$

From (20) we get

$$0 \cdot a_2 + 0 \cdot b_2 = \frac{(\alpha - \beta)}{6}\, a_0. \tag{21}$$

Thus, a_2 or b_2 is arbitrary provided $\alpha = \beta$.

The coefficients of (τ^{-1}, τ^{-1}) in (2) give

$$a_3 + b_3 = -\frac{d^3}{750}\, a_0, \tag{22a}$$

$$a_3 + b_3 = -\frac{d^3}{750}\, a_0, \tag{22b}$$

from which we get

$$0 \cdot a_3 + 0 \cdot b_3 = 0. \tag{23}$$

a_3 or b_3 is arbitrary without any further restriction on the parameters. Proceeding further, from the coefficients of (τ^0, τ^0) and (τ, τ) we have the relations

$$a_4 + b_4 = \left(\frac{125\alpha^2 - 7d^4}{15000}\right) a_0, \tag{24a}$$

$$a_5 + b_5 = \left(\frac{1375d\alpha^2 - 79d^5}{225000}\right) a_0. \tag{24b}$$

Finally, from the coefficients of (τ^2, τ^2) we obtain

$$\begin{aligned} 0 \cdot a_6 + 0 \cdot b_6 = {} & \frac{27}{5} d\left(a_5 + b_5\right) + \alpha\left(a_4 + b_4\right) + \left(\delta_1 + \delta_2\right)\left(a_3 + b_3\right)^2 \\ & + 2\left(\delta_1 + \delta_2\right)\left(a_2 + b_2\right)\left(a_4 + b_4\right). \end{aligned} \tag{25}$$

Substituting the expressions obtained above for $a_i + b_i$, $i = 2, 3, 4, 5$ and a_0 from eq. (12a) in eq. (25) we get

$$0 \cdot a_6 + 0 \cdot b_6 = \frac{d^2}{4}\left[\frac{36}{625} d^4 - \alpha^2\right]. \tag{26}$$

Thus, a_6 or b_6 is arbitrary if $d = 0$ or $d^2 = 25\alpha/6$. Therefore, eq. (2) satisfies P-property for

$$d = 0, \quad \alpha = \beta, \quad \delta_1 \text{ and } \delta_2 \text{ are arbitrary.} \tag{27}$$

$$d^2 = \frac{25\alpha}{6}, \quad \alpha = \beta, \quad \delta_1 \text{ and } \delta_2 \text{ are arbitrary.} \tag{28}$$

Case 2

For Case 2 the resonance values are $r = -1, 2, 3, 6$. We obtain

$$a_1 = -b_1 = -\frac{d}{5}\, a_0. \tag{29a}$$

From the coefficients of (τ^{-2}, τ^{-2}) we obtain

$$a_2 - b_2 = \frac{\left(25\alpha - d^2\right)}{150}\, a_0, \tag{29b}$$

$$b_2 - a_2 = \frac{\left(d^2 - 25\beta\right)}{150}\, a_0. \tag{29c}$$

From (29b) and (29c) the condition for a_2 or b_2 to become arbitrary is $\alpha = \beta$. a_3 or b_3 is arbitrary without any further restriction on the parameters and

$$a_3 - b_3 = -\frac{d^3}{750}\, a_0. \tag{29d}$$

The values of a_4, b_4, a_5 and b_5 are fixed and we have

$$a_4 - b_4 = \left(\frac{125\alpha^2 - 7d^4}{15000}\right) a_0, \tag{29e}$$

$$a_5 - b_5 = \left(\frac{1375d\alpha^2 - 79d^5}{225000}\right) a_0. \tag{29f}$$

From the coefficients of (τ^2, τ^2) we get

$$\begin{aligned} 0 \cdot a_6 + 0 \cdot b_6 &= \frac{27d}{5}(a_5 - b_5) + \alpha (a_4 - b_4) + (\delta_1 - \delta_2)(a_3 - b_3)^2 \\ &\quad + 2(\delta_1 - \delta_2)(a_2 - b_2)(a_4 - b_4) \\ &= \frac{d^2}{4}\left(\frac{36d^4}{625} - \alpha^2\right). \end{aligned} \tag{30}$$

For a_6 or b_6 to be arbitrary we require $d = 0$ or $d^2 = 25\alpha/6$. Thus, the system (2) possesses P-property for the parametric conditions given by eqs (27) and (28).

Case 3

The resonance values are $r = -1, -1, 6, 6$ with the parametric condition given by eq. (12c). From the coefficients of (τ^i, τ^i), $i = -3, -2, -1, 0, 1$ the constants a_i, b_i, $i = 1, 2, ..., 5$ are determined as

$$a_1 = -\frac{d}{5} a_0, \quad b_1 = -\frac{d}{5} b_0, \tag{31a}$$

$$a_2 = \frac{(25\alpha - d^2)}{300} a_0, \quad b_2 = \frac{(25\beta - d^2)}{300} b_0, \tag{31b}$$

$$a_3 = -\frac{d^3}{1500} a_0, \quad b_3 = -\frac{d^3}{1500} b_0, \tag{31c}$$

$$a_4 = \left(\frac{250\alpha^2 - 7d^4}{30000}\right) a_0 + \frac{1}{1440}\left(\delta_1 \alpha^2 a_0^2 + \delta_1 \beta^2 b_0^2 + 2\alpha\beta\delta_2 a_0 b_0\right), \tag{31d}$$

$$b_4 = \left(\frac{250\beta^2 - 7d^4}{30000}\right) b_0 + \frac{1}{1440}\left(\delta_2 \beta^2 b_0^2 + \delta_2 \alpha^2 a_0^2 + 2\alpha\beta\delta_1 a_0 b_0\right), \tag{31e}$$

$$a_5 = \left(\frac{275d\alpha^2 - 158d^5}{90000}\right) a_0 - \frac{11d\delta_2(\alpha - \beta)^2}{21600} a_0 b_0, \tag{31f}$$

$$b_5 = \left(\frac{275d\beta^2 - 158d^5}{90000}\right) b_0 - \frac{11d\delta_1(\alpha - \beta)^2}{21600} a_0 b_0. \tag{31g}$$

From the coefficients of (τ^2, τ^2) in eq. (2), the condition for both a_6 and b_6 to become arbitrary is obtained as

$$\begin{aligned} &3d(a_5 + b_5) + (\alpha a_4 + \beta b_4) + (\delta_1 + \delta_2)(a_3 + b_3)^2 \\ &\quad + 2(\delta_1 + \delta_2)\left[(a_1 + b_1)(a_5 + b_5) + (a_2 + b_2)(a_4 + b_4)\right] = 0. \end{aligned} \tag{32}$$

Using eqs (31) for a_is and b_is in the above equation we find that both a_6 and b_6 are arbitrary only if $d = 0$, $\alpha = \beta$. This choice is the same as the one given by eq. (27).

4. Analytical solution for the integrable cases

In this section we construct the exact analytical solution for the two integrable choices identified by the Painlevé test. In both the integrable cases we have $\alpha = \beta$. For this choice, under the change of variables $x = u + v$, $y = u - v$ eq. (2) decouples into two single oscillators:

$$\ddot{u} + d\dot{u} + \alpha u + 2\left(\delta_1 + \delta_2\right) u^2 = 0, \tag{33}$$

$$\ddot{v} + d\dot{v} + \alpha v + 2\left(\delta_1 + \delta_2\right) v^2 = 0. \tag{34}$$

Now we consider the case $d = 0$. Multiplying eq. (33) (with $d = 0$) by $\dot{u}$ and then integrating once we obtain

$$\left(\dot{u}\right)^2 + \alpha u^2 + \frac{4}{3}\left(\delta_1 + \delta_2\right) u^3 + u_0 = 0, \tag{35}$$

where u_0 is an integration constant. Defining s_1, s_2 and s_3 as the real roots of the polynomial

$$u^3 + \frac{3\alpha}{4\left(\delta_1 + \delta_2\right)} u^2 + \frac{3u_0}{4\left(\delta_1 + \delta_2\right)} = 0 \tag{36}$$

eq. (35) can be rewritten as

$$\left(\dot{u}\right)^2 = -\frac{4}{3}\left(\delta_1 + \delta_2\right)\left(u - s_1\right)\left(u - s_2\right)\left(u - s_3\right), \tag{37a}$$

where

$$s_1 + s_2 + s_3 = \frac{3\alpha}{4\left(\delta_1 + \delta_2\right)}, \tag{37b}$$

$$s_1 s_2 + s_2 s_3 + s_3 s_1 = 0, \tag{37c}$$

$$s_1 s_2 s_3 = -\frac{3u_0}{4\left(\delta_1 + \delta_2\right)}. \tag{37d}$$

Introducing the change of variable

$$z^2 = \frac{\left(u - s_3\right)}{\left(s_2 - s_3\right)} \tag{38}$$

eq. (37a) becomes

$$\left(\dot{z}\right)^2 = \lambda^2 \left(1 - z^2\right)\left(1 - m^2 z^2\right), \tag{39a}$$

where

$$\lambda^2 = \frac{1}{3}\left(\delta_1 + \delta_2\right)\left(s_1 - s_3\right), \quad m^2 = \frac{\left(s_3 - s_2\right)}{\left(s_3 - s_1\right)}. \tag{39b}$$

The solution of eq. (39a) is [29]

$$z = \mathrm{sn}(\lambda t, m), \tag{40}$$

where sn is the Jacobian elliptic function. Then the solution of eq. (33) is obtained as

$$u = s_3 + (s_2 - s_3)\,\mathrm{sn}^2(\lambda t, m). \tag{41}$$

Next, we construct the solution of eq. (33) for the choice given by eq. (28). General methods are not existing to find analytic solution of damped non-linear differential equations of arbitrary order. However, for certain second-order non-linear differential equations, methods are available in the literature. In ref. [29] general solution of the equation

$$\ddot{W} + 6W^2 = 0 \tag{42}$$

is given as

$$W(z) = c^2 \left[\frac{k^2}{1+k^2} - \frac{1}{\mathrm{sn}^2\left\{c(z-z_0), k\right\}} \right], \tag{43}$$

where c and z_0 are arbitrary constants and k^2 is a root of the equation $1-k^2+k^4 = 0$. The change of variables

$$u = \frac{6d^2}{25\,(\delta_1+\delta_2)} W(z)\,\mathrm{e}^{-2dt/5}, \quad z = \mathrm{e}^{-dt/5} \tag{44}$$

with $d^2 = 25\alpha/6$ transform eq. (33) into eq. (42). Hence the solution of eq. (33) is given by eq. (44) with $W(z)$ given by eq. (43).

Applying dynamic Lie symmetries approach [2,3,30] we obtained two linearly independent integrals of motion for each of the two integrable cases. For the integrable condition given by eq. (27) the integrals of motion are

$$I_1 = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\dot{y}^2 + \frac{\alpha}{2}\left(x^2+y^2\right) + \frac{1}{3}\left(\delta_1 x^3 + \delta_2 y^3\right) + \delta_2 x^2 y + \delta_1 x y^2, \tag{45a}$$

$$I_2 = \dot{x}\dot{y} + \frac{\delta_2}{3}x^3 + \frac{\delta_1}{3}y^3 + \alpha xy + \delta_1 x^2 y + \delta_2 x y^2. \tag{45b}$$

I_1 is simply the Hamiltonian of the system. For the integrable choice given by eq. (28) we obtained

$$\begin{aligned} I_1 = \mathrm{e}^{6dt/5} \bigg[& \dot{x}^2 + \dot{y}^2 + \frac{4}{5}d\left(x\dot{x} + y\dot{y}\right) + \frac{2}{3}\alpha\left(x^2+y^2\right) + \frac{2}{3}\delta_2 y^3 \\ & + \frac{2}{3}\delta_1 x^3 + 2\delta_2 x^2 y + 2\delta_1 x y^2 \bigg], \end{aligned} \tag{46a}$$

$$\begin{aligned} I_2 = \mathrm{e}^{6dt/5} \bigg[& \dot{x}\dot{y} + \frac{2}{5}d\left(\dot{x}y + x\dot{y}\right) + \frac{1}{3}\delta_2 x^3 + \frac{1}{3}\delta_1 y^3 + \frac{2}{3}\alpha xy \\ & + \delta_2 x y^2 + \delta_1 x^2 y \bigg]. \end{aligned} \tag{46b}$$

One can easily verify that $\mathrm{d}I_1/\mathrm{d}t = \mathrm{d}I_2/\mathrm{d}t = 0$.

5. Conclusions

In this paper we considered a damped two-coupled non-linear oscillators eq. (2) which is found to exhibit hard in amplitude transition to chaos. Painlevé analysis is applied to explore the integrable nature of the system. We identified three distinct sets of Laurent series solutions with leading orders given by (12). For Cases 1 and 2 we verified the existence of sufficient number of arbitrary constants in the series solution. The parametric restrictions obtained in these two cases are given by eqs (27) and (28). For Case 3 the series has only three arbitrary constants and parametric restrictions are $d = 0$ and $\alpha = \beta$. Thus, we identified two integrable choices given by eqs (27) and (28). For these two integrable cases we obtained exact analytical solution. Further, for the integrable choices eq. (2) admits two linearly independent integrals of motion. The explicit forms of these integrals of motion obtained by dynamic Lie symmetries method are given by eqs (45) and (46). It is of interest to investigate the effect of non-linear damping and addition of periodic force and various non-linear phenomena in eq. (2).

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