

Gravitational collapse in higher-dimensional charged-Vaidya space-time

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Abstract. We analyze here the gravitational collapse of higher-dimensional charged-Vaidya space-time. We show that singularities arising in a charged null fluid in higher dimension are always naked violating at least strong cosmic censorship hypothesis (CCH), though not necessarily weak CCH. We show that earlier conclusions on the occurrence of naked singularities in four-dimensional case can be extended essentially in the same manner in 5D case also.

Keywords. Higher dimension; naked singularity; cosmic censorship; gravitational collapse.

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1. Introduction

The investigation on the final fate of gravitational collapse of initially regular distribution of matter is one of the most active field of research in the contemporary general relativity. It is indeed known that under fairly general hypothesis, solutions of the Einstein equation with physically reasonable matter can develop into singularities [1]. The main open issue is whether the singularities, which arise as the end point of collapse, can actually be observed? Various models of spherical collapse have been studied over the last few years on this issue and these show that both black holes and naked singularities arise during gravitational collapse. Genericity and stability of these naked singularities are also discussed in some of the papers [2]. One of the important examples having naked singularities is the Vaidya solution representing an imploding (exploding) null dust fluid with spherical symmetry.

Papapetrou [3] first showed that the solution can give rise to the formation of naked singularities, and thus provide one of the earlier counter examples to the cosmic censorship conjecture [4]. Later, the solution was generalized to the charged case [5]. The charged-Vaidya solution attracted a lot of attention and has been studied in various situations. Lake and Zannias [6] studied the self-similar case and found that, similar to the uncharged case, naked singularities can also be formed from gravitational collapse. Later on Patil *et al* [7] have shown that under certain conditions on mass function, strong curvature naked singularity exists in charged-Vaidya space-time.

Recently, there has been renewed interest in studying higher-dimensional space-times from the point of view of both cosmology [8] and gravitational collapse [9,10]. The results of gravitational collapse in higher-dimensions are of interest in view of the current possibilities being explored for higher-dimensional gravity. An interesting problem that arises is the effect the higher dimension can have on the formation of naked singularity [10–12].

It has been shown in [10] that higher-dimensional inhomogeneous dust (zero pressure) collapse admits naked shell focusing singularities. In the first reference of [10] it has been proved that central singularity of collapse can be a strong curvature or a weak curvature naked singularity depending on the initial density distribution. Second reference of [10] is the generalization of four-dimensional inhomogeneous dust collapse to $(N + 2)$ -dimensional space-time and it is investigated whether the dimensionality of the space-time has any role in the nature of singularities. It has been shown that dimensionality of the space-time does not essentially change the basic nature of the singularity of an inhomogeneous dust collapse.

The present work deals with the spherically symmetric collapse of charged null fluid in higher dimension. We show that just like higher-dimensional inhomogeneous dust collapse, gravitational collapse of spherical charged-Vaidya fluid also admits strong curvature naked singularities, providing yet another counter example to cosmic censorship hypothesis.

Anzhong Wang introduced a more general family of Vaidya space-times which covers monopole solutions, de-Sitter and anti de-Sitter solutions and charged-Vaidya solutions as special cases. In this paper following Wang [13] and Iyer and Vishveshwara [14] we define charged-Vaidya solution in higher dimension and show that singularities arising in this space-time are naked and may satisfy the strong curvature condition under certain restriction on the mass function.

The rest of the paper is organized as follows: In §2 we define charged-Vaidya solution in 5D. In §3 we prove the existence of naked singularity and discuss its strength. The paper ends with the conclusion in §4.

2. Charged-Vaidya solution in five-dimensional space-time

Following [13–15] we write the general spherically symmetric line element in five-dimensional (5D) space-time as

$$ds^2 = - \left(1 - \frac{m(u, r)}{r^2} \right) du^2 + 2du dr + r^2 (d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2), \quad (1)$$

where $m(u, r)$ is usually called the mass function, and related to the gravitational energy within a given radius r [6,16]. Null coordinate u represents the Eddington advanced time, in which r decreases towards the future along a ray $u = \text{constant}$. $d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sin^2 \theta_2 d\theta_3^2$ is a line element on unit 3-sphere.

Non-vanishing components of the Einstein tensor are given by

$$G_0^0 = G_1^1 = -\frac{3m'(u, r)}{2r^3}, \quad G_0^1 = \frac{3\dot{m}(u, r)}{2r^3}, \quad G_2^2 = G_3^3 = G_4^4 = -\frac{m''(u, r)}{2r^2}, \quad (2)$$

where $\{x^\mu\} = \{u, r, \theta_1, \theta_2, \theta_3\}$, $(\mu = 0, 1, 2, 3, 4)$ and $\dot{m}(u, r) = \partial m(u, r)/\partial u$, $m'(u, r) = \partial m(u, r)/\partial r$.

Combining eq. (2) with the Einstein field equation

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \quad (3)$$

κ being gravitational constant. We find that the corresponding energy momentum tensor can be written in the form [17]

$$T_{\mu\nu} = T_{\mu\nu}^{(n)} + T_{\mu\nu}^{(m)}, \quad (4)$$

where

$$\begin{aligned} T_{\mu\nu}^{(n)} &= \sigma l_\mu l_\nu, \\ T_{\mu\nu}^{(m)} &= (\rho + P)(l_\mu n_\nu + l_\nu n_\mu) + P g_{\mu\nu} \end{aligned} \quad (5)$$

and

$$\sigma = \frac{3\dot{m}(u, r)}{2\kappa r^3}, \quad \rho = \frac{3m'(u, r)}{2\kappa r^3}, \quad P = \frac{-m''(u, r)}{2\kappa r^2}, \quad (6)$$

with l_μ and n_μ being two null vectors,

$$\begin{aligned} l_\mu &= \delta_\mu^0, \quad n_\mu = \frac{1}{2} \left[1 - \frac{m(u, r)}{r^2} \right] \delta_\mu^0 - \delta_\mu^1, \\ l_\lambda l^\lambda &= n_\lambda n^\lambda = 0, \quad l_\lambda n^\lambda = -1. \end{aligned} \quad (7)$$

The part of EMT, $T_{\mu\nu}^{(n)}$, can be considered as the component of the matter field that moves along the null hyper surface $u = \text{constant}$.

In particular, when $\rho = P = 0$, the solution reduce to the higher-dimensional Vaidya solution with $m = m(u)$ [12, 14]. Therefore, for the general case we consider the EMT of eq. (5) as a generalization of the Vaidya solution, in higher-dimensional space-time.

Following Anzhong Wang [13] we define the most general expression for charged-Vaidya solution in 5D as

$$m(u, r) = f(u) - \frac{q^2(u)}{3r^2}, \quad (8)$$

where the two arbitrary functions $f(u)$ and $q(u)$ represent, respectively the mass function and electric charge at the advanced time u .

Inserting the above expression into eq. (6), we find that

$$\sigma = \frac{3}{2\kappa r^3} \left(\dot{f}(u) - \frac{2q\dot{q}(u)}{3r^2} \right), \quad (9)$$

$$\rho = P = \frac{q^2(u)}{\kappa r^6}. \quad (10)$$

Here $T_{\mu\nu}^{(n)}$ corresponds to the EMT of the Vaidya null fluid and $T_{\mu\nu}^{(m)}$ to the electromagnetic field, $F_{\mu\nu}$, given by

$$F_{\mu\nu} = \frac{q(u)}{r^3} (\delta_\mu^0 \delta_\nu^1 - \delta_\mu^1 \delta_\nu^0).$$

From eq. (9) we can see that $\sigma \geq 0$ gives the main restriction on the choice of the function $f(u)$ and $q(u)$. We note that the stress tensor in general may not obey the weak energy condition. In particular if $df/dq > 0$ then there always exists a critical radius $r_c = [2q(u)\dot{q}(u)/3\dot{f}(u)]^{1/2}$ such that when $r < r_c$ the weak energy condition is always violated. However, in realistic situations, the particle cannot get into the region $r < r_c$ because of the Lorentz force and so the energy conditions are preserved [13].

3. Nature of space-time singularities

The situation being considered here is that of a radially injected radiation flow in an initially empty region of the higher-dimensional Minkowskian space-time. The radiation is focussed into a central singularity at $r = 0$, $u = 0$, of growing mass $f(u)$. Thus $f(u)$ is an arbitrary, non-negative increasing function of u . For $u < 0$, we have $f(u) = q(u) = 0$, i.e., an empty Minkowskian metric, and for $u > T$, $\dot{f}(u) = \dot{q}(u) = 0$, $f(u)$ and $q^2(u)$ are positive definite. The metric for $u = 0$ to $u = T$ is charged-Vaidya, and for $u > T$ it becomes Reissner Nordström solution in 5D.

In order to get an analytical solution for our higher-dimensional case, we choose $f(u) \propto u^2$ and $q^2(u) \propto u^4$.

In particular, we take

$$\begin{aligned} f(u) &= 0, \quad u < 0 \\ &= \lambda u^2, \quad 0 \leq u \leq T \\ &= m_0, \quad u > T, \end{aligned} \tag{11}$$

and

$$\begin{aligned} q^2(u) &= 0, \quad u < 0 \\ &= a^2 u^4, \quad 0 \leq u \leq T \\ &= q_0^2 \quad u > T, \end{aligned} \tag{12}$$

where λ and a are some positive constants. Inserting the expression for $f(u)$ and $q^2(u)$ in eq. (8), we write the mass function for charged-Vaidya region as

$$m(u, r) = \lambda u^2 - \frac{a^2 u^4}{3r^2}. \tag{13}$$

It can be easily seen that with the choice of the above mass function, metric (1) is self-similar [18] admitting a homothetic killing vector ξ^a given by

$$\xi^a = u \frac{\partial}{\partial u} + r \frac{\partial}{\partial r}, \tag{14}$$

which satisfies the condition

$$L_{\xi} g_{ab} = \xi_{a;b} + \xi_{b;a} = 2g_{ab}, \quad (15)$$

where L denotes the Lie derivative. Defining now $K^a = dx^a/dk$ as tangent to null geodesics, where k is an affine parameter, it follows that $\xi^a K_a$ is constant along radial null geodesics and hence a constant of motion:

$$\xi^a K_a = rK_r + uK_u = A. \quad (16)$$

Following the method of Dwivedi and Joshi [19] we now examine under what conditions on $m(u, r)$ such collapse leads to a naked singularity and whether this singularity is a strong curvature one or not. The existence of naked singularity or otherwise can be determined by examining the behavior of radial null geodesic. If they terminate at the singularity in the past with a definite positive tangent vector, then the singularity is naked. If the tangent vector is not positive in the limit as one approaches the singularity, then the singularity is covered.

Geodesics equations of motion for metric (1) on using the null condition $K^a K_a = 0$, takes the simple form

$$\frac{dK^u}{dk} + \left(\frac{m}{r^3} - \frac{m'}{2r^2} \right) (K^u)^2 = 0, \quad (17)$$

$$\frac{dK^r}{dk} + \left(\frac{\dot{m}}{2r^2} + \frac{m}{r^3} - \frac{m'}{2r^2} + \frac{mm'}{2r^4} - \frac{m^2}{r^5} \right) (K^u)^2 + \left(\frac{m'}{r^2} - \frac{2m}{r^3} \right) K^u K^r = 0. \quad (18)$$

Now defining $R(u, r)$ as

$$K^u = \frac{R}{r} \quad (19)$$

and, from the null condition, we obtain

$$K^r = \frac{R}{2r} \left(1 - \frac{m(u, r)}{r^2} \right), \quad (20)$$

where R satisfies the differential equation

$$\frac{dR}{dk} - \frac{R^2}{2r^2} \left(1 - \frac{3m(u, r)}{r^2} + \frac{m'}{r} \right) = 0. \quad (21)$$

Using eqs (13), (19) and (20) in eq. (16) we write the solution of differential equation (21) as

$$R = \frac{6A}{6 - 3X + 3\lambda X^3 - a^2 X^5}, \quad (22)$$

where we have defined the self-similarity variable $X = u/r$.

Further, we note that

$$\frac{dX}{dk} = \frac{1}{r} \frac{du}{dk} - \frac{u}{r^2} \frac{dr}{dk}, \quad (23)$$

which, on inserting the expressions for K^u and K^r becomes

$$\frac{dX}{dk} = \frac{R}{6r^2} (6 - 3X + 3\lambda X^3 - a^2 X^5) = \frac{A}{r^2}. \quad (24)$$

Radial ($K^{\theta_1} = K^{\theta_2} = K^{\theta_3} = 0$) null geodesics of the metric (1) must satisfy the null condition

$$\frac{dr}{du} = \frac{1}{2} \left(1 - \frac{m(u, r)}{r^2} \right), \quad (25)$$

which, upon using eq. (13) turns out to be

$$\frac{dr}{du} = \frac{1}{2} \left(1 - \frac{\lambda u^2}{r^2} + \frac{a^2 u^4}{3r^4} \right). \quad (26)$$

This differential equation (eq. (26)) has singular point at $r = 0, u = 0$. To analyze the nature of this singularity one can try to analyze the outgoing singular geodesics terminating at the singularity in the past.

Let

$$X_0 = \lim_{\substack{r \rightarrow 0 \\ u \rightarrow 0}} X = \lim_{\substack{r \rightarrow 0 \\ u \rightarrow 0}} \frac{u}{r}. \quad (27)$$

Using (26) and L'Hospital's rule we get

$$X_0 = \lim_{\substack{r \rightarrow 0 \\ u \rightarrow 0}} X = \lim_{\substack{r \rightarrow 0 \\ u \rightarrow 0}} \frac{u}{r} = \lim_{\substack{r \rightarrow 0 \\ u \rightarrow 0}} \frac{du}{dr} = \frac{6}{3 - 3\lambda X_0^2 + a^2 X_0^4}, \quad (28)$$

which implies

$$a^2 X_0^5 - 3\lambda X_0^3 + 3X_0 - 6 = 0. \quad (29)$$

The variable X can be interpreted as the tangent to the outgoing geodesic. Hence eq. (29) governs the behavior of the tangent vector near the singular point. Singularity will be naked if eq. (29) admits one or more positive real roots. When there are no positive real roots to eq. (29), the singularity is not naked because in that case there are no outgoing future directed null geodesics from the singularity. Hence in the absence of positive real roots, the collapse ends into a black hole.

To analyze the nature of the roots of eq. (29), the following rule in the theory of equation may be useful. 'Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term, the coefficient of the first term being positive.' As in eq. (29), coefficient of the first term (i.e. a^2) is positive and the last term is negative, the equation must have at least one positive real root for all values of λ and a^2 . Hence the singularity is always naked.

In particular if we take $a^2 = 0.0001$ and $\lambda = 0.01$, then one of the roots of eq. (29) is $X_0 = 2.0899588$.

Note that if the electric charge $q(u)$, in the mass function (13) is zero, then solution reduces to higher-dimensional Vaidya solution and in that case singularity is naked if $0 < \lambda < 1/27$ which is in agreement with the result in [12].

On inserting the expression for $m(u, r)$ from eq. (13) in eq. (1), we find that the Kretschmann scalar for the metric (1) takes the form

$$K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{A\lambda^2X^4}{r^4} - \frac{B\lambda a^2X^6}{r^4} + \frac{Ca^4X^8}{r^4}, \quad (30)$$

where A, B, C are some constants.

It is clear from eq. (30) that Kretschmann scalar diverges at the naked singularity and hence singularity is a scalar polynomial singularity.

3.1 Strength of the singularity

I now discuss the strength of this singularity by considering the curvature growth near it. Following Clarke and Krolak [20], a sufficient condition for a singularity to be strong in the sense of Tipler [21], is that, at least along one null geodesic (with affine parameter k) we should have in the limit of approach to the singularity

$$\lim_{k \rightarrow 0} k^2 \psi = \lim_{k \rightarrow 0} k^2 R_{ab} K^a K^b > 0, \quad (31)$$

where K^a is the tangent to the null geodesics and R_{ab} is the Ricci tensor.

Using eqs (19) and (20) we write

$$k^2 R_{ab} K^a K^b = k^2 \left[\frac{3\dot{m}}{2r^3} (K^u)^2 \right] \quad (32)$$

$$k^2 R_{ab} K^a K^b = X(3\lambda - 2a^2X^2) \left(\frac{kR}{r^2} \right)^2. \quad (33)$$

Using the fact that as singularity is approached, $k \rightarrow 0$, $r \rightarrow 0$ and $X \rightarrow X_0$ and using L'Hospital's rule, we find that

$$\begin{aligned} \lim_{k \rightarrow 0} \frac{kR}{r^2} &= \frac{3}{3 - 3\lambda X_0^2 + a^2 X_0^4}, \quad \text{if } \lim_{k \rightarrow 0} R = R_0 \neq 0, \infty \\ \lim_{k \rightarrow 0} \frac{kR}{r^2} &= \frac{2}{1 + \lambda X_0^2 - a^2 X_0^4}, \quad \text{if } R_0 = 0, \infty. \end{aligned} \quad (34)$$

Hence eq. (33) gives

$$\begin{aligned} \lim_{k \rightarrow 0} k^2 \psi &= \frac{9X_0(3\lambda - 2a^2X_0^2)}{(3 - 3\lambda X_0^2 + a^2 X_0^4)^2}, \quad \text{if } \lim_{k \rightarrow 0} R = R_0 \neq 0, \infty \\ \lim_{k \rightarrow 0} k^2 \psi &= \frac{4X_0(3\lambda - 2a^2X_0^2)}{(1 + \lambda X_0^2 - a^2 X_0^4)^2}, \quad \text{if } R_0 = 0, \infty. \end{aligned} \quad (35)$$

Thus along radial null geodesics strong curvature condition is satisfied if $3\lambda - 2a^2X_0^2 > 0$.

For our particular case (i.e. $a^2 = 0.0001$, $\lambda = 0.01$ and $X_0 = 2.0899588$) we find that $3\lambda - 2a^2X_0^2 > 0$, hence naked singularity arising in this case is a strong curvature one.

4. Conclusion

Present work shows that naked singularities do occur as the end stage of gravitational collapse in higher-dimensional charged-Vaidya space-time.

An important thing we observe is that in uncharged case [12,19,22] naked singularity occurs only for specific value of the parameter. But here in charged case we have shown that naked singularities always occur irrespective of the values of the parameters λ and a . In other words one may conclude that in the spherically symmetric collapse of a charged null fluid strong CCH always violates though not necessarily weak CCH.

We have measured the curvature growth along radial null geodesics and found that naked singularities arising in this case are strong curvature singularities under certain restriction on the mass function.

It is observed that five-dimensional charged-Vaidya space-time admits a strong curvature naked singularity, which seems to suggest that the dimension of space-time does not play any fundamental role in the formation of singularity.

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References

- [1] S W Hawking and G F R Ellis, *The large scale structure of spacetime* (Cambridge University Press, 1973)
- [2] R V Saraykar and S H Ghate, *Class Quantum Gravit.* **16**, 281 (1999)
P S Joshi, *Pramana – J. Phys.* **56**, 529 (2000)
H Iguchi, K Nakao and T Harada, *Phys. Rev.* **D57**, 7262 (1998)
F Mena, R K Tavakol and P S Joshi, gr-qc /0002062
S H Ghate, R V Saraykar and K D Patil, *Pramana – J. Phys.* **53**, 253 (1999)
- [3] A Papapetrou, in *A random walk in relativity and cosmology* edited by N Dadhich, J K Rao, J V Narlikar and C V Vishveshwara (Wiley, New York, 1985) pp. 184–191
- [4] R Penrose, *Riv. Nuovo Cimento* **1**, 252 (1969)
- [5] R W Lindquist, R A Schwarz and C W Minsner, *Phys. Rev.* **B137**, 1364 (1965)
C W Minster, *Phys. Rev.* **B137**, 1364 (1965)
W Israel, *Phys. Lett.* **A24**, 184 (1967)
W B Bonnor and P C Vaidya, *Gen. Relativ. Gravit.* **2**, 127 (1970)
- [6] K Lake and T Zannias, *Phys. Rev.* **D43**, 1798 (1991)
- [7] K D Patil, R V Saraykar and S H Ghate, *Pramana – J. Phys.* **52**, 553 (1999)
- [8] S Ferrara, M Porrati and A Zaffaroni, *Lett. Math. Phys.* **47**, 255 (1999)
S E P Bergliaffa, *Mod. Phys. Lett.* **A15**, 531 (2000)

- [9] J Soda and K Hirata, *Phys. Lett.* **B387**, 271 (1996)
V Frolov, *Class Quantum Gravit.* **16**, 407 (1999)
- [10] K D Patil, S H Ghate and R V Saraykar, *Pramana – J. Phys.* **56**, 503 (2001)
K D Patil, S H Ghate and R V Saraykar, *Indian J. Pure Appl. Math.* **33**, 379 (2002)
- [11] A Sil and S Chatterjee, *Gen. Relativ. Gravit.* **26**, 999 (1994)
- [12] S G Ghosh and R V Saraykar, *Phys. Rev.* **D62**, 107502 (2000)
- [13] Anzhong Wang, *Gen. Relativ. Gravit.* **31**, 1 (1999)
- [14] B R Iyer and C V Vishveshwara, *Pramana – J. Phys.* **32**, 745 (1989)
- [15] C Barrabes and W Israel, *Phys. Rev.* **D43**, 1129 (1991)
- [16] E Poisson and W Israel, *Phys. Rev.* **D41**, 1796 (1990)
- [17] V Husian, *Phys. Rev.* **D53**, R1759 (1996)
- [18] A spherical symmetric spacetime is self-similar if $g_{tt}(ct, cr) = g_{tt}(t, r)$ and $g_{rr}(ct, cr) = g_{rr}(t, r)$,
for every $c > 0$
- [19] I H Dwivedi and P S Joshi, *Class Quantum Gravit.* **6**, 1599 (1989)
- [20] C J S Clarke and A Krolak, *J. Geom. Phys.* **2**, 127 (1986)
- [21] F J Tipler, *Phys. Lett.* **A64**, 8 (1977)
- [22] I H Dwivedi and P S Joshi, *Class Quantum Gravit.* **8**, 1339 (1991)
J P S Lemos, *Phys. Rev.* **D59**, 044020 (1999)