

## Hermitian quark matrices

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MS received 25 September 2001; revised 31 January 2002

**Abstract.** Assuming a relation between the quark mass matrices of the two sectors a unique solution can be obtained for the CKM flavor mixing matrix. A numerical example is worked out which is in excellent agreement with experimental data.

**Keywords.** Quark matrices; CKM matrix; unitarity triangle.

**PACS Nos** 12.15.Ff; 11.30.Er

### 1. Introduction

The relation between quark masses and the CKM matrix [1] has been investigated by various authors for the last 30 years. The famous Fritzsch ansatz [2] was proposed in 1978 but with the determination of the top quark mass it had to be abandoned as it cannot accommodate such a high top quark mass. The Fritzsch matrix has been modified in two ways. We may drop the condition of hermiticity to obtain the NNI matrix or we may add a (2,2) element to obtain a Hermitian matrix with four texture zeros. This has been investigated in [3–7]. In this paper given a simple relation between the matrices of the two sectors and a mass spectrum of the two sectors a numerical example is solved giving the flavor mixing CKM matrix [1], the CP violation [8] and the unitarity triangle [4]. The results obtained are in excellent agreement with experimental values.

### 2. The mass matrix

Consider Hermitian mass matrices of the same form for both the  $u$  and  $d$  sectors [6,7]:

$$\begin{pmatrix} 0 & ae^{i\alpha} & 0 \\ ae^{-i\alpha} & b & c \\ 0 & c & d \end{pmatrix} = \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & a & 0 \\ a & b & c \\ 0 & c & d \end{pmatrix} \begin{pmatrix} e^{-i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (1)$$

where  $a, b, c, d$  are real.

Consider the eigenvalue equation of the matrix

$$M = \begin{pmatrix} 0 & a & 0 \\ a & b & c \\ 0 & c & d \end{pmatrix}. \quad (2)$$

If we assume  $a, b, c, d$  to be real and positive, then one of the eigenvalues must be negative. Let the eigenvalues be  $m_1, m_2, m_3$  with  $m_3 \gg m_2 \gg m_1$  since there is a strong hierarchy in both sectors. The eigenvalue equation gives

$$\begin{aligned} b + d &= m_3 + m_2 - m_1; & a^2 d &= m_1 m_2 m_3; \\ bd - a^2 - c^2 &= m_2 m_3 - m_3 m_1 - m_1 m_2. \end{aligned} \quad (3)$$

The elements  $a, b, c$  can be expressed in terms of  $m_1, m_2, m_3$  and  $d$ .

$$\begin{aligned} a^2 &= m_1 m_2 m_3 / d; & b &= m_3 + m_2 - m_1 - d; \\ c^2 &= (d + m_1)(d - m_2)(m_3 - d) / d. \end{aligned} \quad (4)$$

Due to the strong hierarchy of the quark masses in both sectors we have [6]  $d > b, c > a$ . Also from (4) we must have  $m_3 > d > m_2$ .

The orthogonal matrix which diagonalizes the mass matrix (2) has the form [7]  $S = (S_{rs})$  where

$$\begin{aligned} S_{11} &= \sqrt{\frac{m_2 m_3 (d + m_1)}{d(m_2 + m_1)(m_3 + m_1)}} \\ &= \sqrt{\frac{m_2}{m_2 + m_1}} \sqrt{\left(1 + \frac{m_1}{m_3 + m_1} q\right)} \\ S_{12} &= \sqrt{\frac{m_1 m_3 (d - m_2)}{d(m_2 + m_1)(m_3 - m_2)}} \\ &= \sqrt{\frac{m_1}{m_2 + m_1}} \sqrt{\left(1 - \frac{m_2}{m_3 - m_2} q\right)} \\ S_{13} &= \sqrt{\frac{m_1 m_2 (m_3 - d)}{d(m_3 + m_1)(m_3 - m_2)}} \\ &= \sqrt{\frac{m_1 m_2}{(m_3 + m_1)(m_3 - m_2)}} \sqrt{q} \\ S_{21} &= -\sqrt{\frac{m_1 (d + m_1)}{(m_2 + m_1)(m_3 + m_1)}} \\ &= -\sqrt{\frac{m_1}{m_2 + m_1}} \sqrt{\left(1 - \frac{m_3}{m_3 + m_1} p\right)} \\ S_{22} &= \sqrt{\frac{(d - m_2) m_2}{(m_2 + m_1)(m_3 - m_2)}} \end{aligned}$$

$$\begin{aligned}
 &= \sqrt{\frac{m_2}{m_2 + m_1}} \sqrt{\left(1 - \frac{m_3}{m_3 - m_2} p\right)} \\
 S_{23} &= \sqrt{\frac{m_3(m_3 - d)}{(m_3 + m_1)(m_3 - m_2)}} \\
 &= \sqrt{\frac{m_3^2}{(m_3 + m_1)(m_3 - m_2)}} \sqrt{p} \\
 S_{31} &= \sqrt{\frac{m_1(d - m_2)(m_3 - d)}{d(m_2 + m_1)(m_3 + m_1)}} \\
 &= \sqrt{\frac{m_1(m_3 - m_2)}{(m_2 + m_1)(m_3 + m_1)}} \sqrt{\left(1 - \frac{m_3}{m_3 - m_2} p\right)} q \\
 S_{32} &= -\sqrt{\frac{m_2(d + m_2)(m_3 - d)}{d(m_2 + m_1)(m_3 - m_2)}} \\
 &= -\sqrt{\frac{m_2(m_3 + m_1)}{(m_2 + m_1)(m_3 - m_2)}} \sqrt{\left(1 - \frac{m_3}{m_3 - m_1} p\right)} q \\
 S_{33} &= \sqrt{\frac{m_3(d + m_1)(d - m_2)}{d(m_3 + m_1)(m_3 - m_2)}} \\
 &= \sqrt{\left(1 + \frac{m_1}{m_3 + m_1} q\right)} \sqrt{\left(1 - \frac{m_3}{m_3 - m_2} p\right)} \quad (5)
 \end{aligned}$$

where  $p$  is given by

$$d = m_3(1 - p) \quad (6)$$

and

$$q = \frac{p}{(1 - p)}. \quad (7)$$

Later when dealing with separate sectors we use the subscripts  $u$  and  $d$ . Also for the  $u$  sector  $m_1 = m_u, m_2 = m_c, m_3 = m_t$  and for the  $d$  sector  $m_1 = m_d, m_2 = m_s, m_3 = m_b$ .

The parameter used in [4] is  $r = c/b$ . In our notation we get approximately

$$p \simeq \frac{r^2 m_2^2}{(m_3 - m_2) m_3}. \quad (8)$$

Since  $r \sim 1$  the parameter  $p$  must be small.

The parameter used in [6] is

$$R = \frac{b}{m_3} = \frac{m_2 - m_1}{m_3} + p. \quad (9)$$

### 3. The CKM matrix

The CKM matrix has the form  $S_u^+ P S_d$  where

$$P = \begin{pmatrix} e^{i\delta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{where } \delta = \alpha_d - \alpha_u \quad (10)$$

$$V_{rs} = S_{u_{1r}} S_{d_{1s}} e^{i\delta} + S_{u_{2r}} S_{d_{2s}} + S_{u_{3r}} S_{d_{3s}} \quad (11)$$

$$= v_{rs} e^{i\delta_{rs}} \quad (12)$$

where  $v_{rs} = |V_{rs}|$  is a positive real number.

### 4. The Euler rotation angles

The orthogonal matrix  $S$  can be written as a product of three rotation matrices [4] (eqs (4)–(15))

$$S = R_{31} R_{23} R_{12} \quad (13)$$

$$= \begin{pmatrix} c_\tau & 0 & s_\tau \\ 0 & 1 & 0 \\ -s_\tau & 0 & c_\tau \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\sigma & s_\sigma \\ 0 & -s_\sigma & c_\sigma \end{pmatrix} \begin{pmatrix} c_\omega & c_\omega & 0 \\ -s_\omega & c_\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (14)$$

$$= \begin{pmatrix} (c_\tau c_\omega + s_\tau s_\omega s_\sigma) & (c_\tau s_\omega - s_\tau s_\sigma c_\omega) & s_\tau c_\sigma \\ -c_\sigma s_\omega & c_\sigma c_\omega & s_\sigma \\ (-s_\tau c_\omega + c_\tau s_\sigma s_\omega) & (-s_\tau s_\omega - c_\tau s_\sigma c_\omega) & c_\tau c_\sigma \end{pmatrix}. \quad (15)$$

Then

$$\tan \tau = \frac{S_{13}}{S_{33}}, \quad \tan \omega = -\frac{S_{21}}{S_{22}}, \quad \sin \sigma = S_{23}. \quad (16)$$

### 5. A numerical example

We take the following quark mass values at electroweak scale ( $\mu = M_z$ ) in Mev [7]:

$$\begin{aligned} m_u &= 2.5, & m_c &= 600, & m_t &= 174000; \\ m_d &= 4.0, & m_s &= 80, & m_b &= 3000. \end{aligned} \quad (17)$$

Substituting these in (5) we obtain  $S_u$  and  $S_d$  in terms of  $p_u, q_u$  and  $p_d, q_d$ . We have to determine  $p_u$  and  $p_d$ . For this we assume a simple relation that

$$d = m_3 - \chi m_1 \quad (18)$$

with  $\chi$  having the same value for both sectors, then

$$\frac{p_u}{p_d} = \frac{m_u/m_t}{m_d/m_b} \quad (19)$$

with the values of the quark masses given in (17) we get

$$p_u = 0.01077586 p_d. \quad (20)$$

The coefficient of  $e^{i\delta}$  is smallest in  $V_{33}$ . This is  $S_{u13}S_{d13} \simeq 0.0000\ 0000\ 03$  which may be neglected. The remaining terms  $S_{u22}S_{d23} + S_{u33}S_{d33}$  are real and can be expressed in terms of  $p_d$ . Using the known value  $|V_{33}| = 0.9992$  we obtain

$$p_d = 0.00193537. \quad (21)$$

To obtain  $\alpha$  we use the known value  $|V_{12}| = 0.22$ .

Using (11) and (17) and the expressions for  $S_u$  and  $S_d$

$$\cos \delta = 0.107924, \quad \sin \delta = 0.994519. \quad (22)$$

Then using (5), (11), (12), (17) and (20)–(22) we have

$$(v_{rs}) = \begin{pmatrix} 0.97549496 & 0.22000000 & 0.00255733 \\ 0.21985300 & 0.97471374 & 0.03991681 \\ 0.00845890 & 0.03909204 & 0.99920000 \end{pmatrix} \quad (23)$$

and

$$(\tan \delta_{rs}) = \begin{pmatrix} +8.12273046 & -5.51554088 & -0.10367152 \\ -0.29615569 & +0.01432197 & +0.00044831 \\ -0.00011632 & -0.00000564 & 0.0000000003 \end{pmatrix}. \quad (24)$$

Also we get the Euler angles

$$\begin{aligned} \tan \tau_u &= -0.00000102, \quad \tan \omega_u = +0.06448162, \quad \sin \sigma_u = +0.00457506 \\ \tan \tau_d &= +0.00026626, \quad \tan \omega_d = +0.22359620, \quad \sin \sigma_d = +0.04456976. \end{aligned} \quad (25)$$

The CP-violation [8] is  $\tau = |\text{Im} V_{i\alpha} V_{j\beta} V_{i\beta}^\times V_{j\alpha}^\times|$  with  $i \neq j, \alpha \neq \beta$ .

Let  $i = 2, j = 1, \alpha = 3, \beta = 1$  then we get

$$\tau = |\text{Im} V_{23} V_{11} V_{21}^\times V_{13}^\times| = 2 \times 10^{-5}. \quad (26)$$

The three sides of the unitarity triangles [4] eqs (2)–(8)  $V_{13}^\times V_{11} + V_{23}^\times V_{21} + V_{33}^\times V_{31} = 0$  are

$$|V_{13}^\times V_{11}| = 0.00249466, \quad |V_{23}^\times V_{21}| = 0.00877583, \quad |V_{33}^\times V_{31}| = 0.00845213. \quad (27)$$

The angles of this triangle are

$$89^\circ 02', 74^\circ 21', 16^\circ 37'. \quad (28)$$

This is very close to a right-angled triangle.

We may also note that

$$\frac{v_{13}}{v_{23}} = 0.06406650 \simeq \sqrt{\frac{m_u}{m_c}} = 0.06455 \quad (29)$$

$$\frac{v_{31}}{v_{32}} = 0.21638419 \simeq \sqrt{\frac{m_d}{m_s}} = 0.2236 \quad (30)$$

## 6. Conclusion

In this paper an additional symmetry has been proposed that the  $\chi$  given by  $d = m_3 - \chi m_1$  has the same value for both sectors. This gives a unique solution to the problem. Given a set of quark masses this gives results compatible with experimental values.

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