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Riccion from higher-dimensional space-time with D-dimensional sphere as a compact manifold and one-loop renormalization

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Abstract. Lagrangian density of riccions is obtained with the quartic self-interacting potential using higher-derivative gravitational action in $(4+D)$-dimensional space-time with S^D as a compact manifold. It is found that the resulting four-dimensional theory for riccions is one-loop multiplicatively renormalizable. Renormalization group equations are solved and its solutions yield many interesting results such as (i) dependence of extra dimensions on the enegy mass scale showing that these dimensions increase with the increasing mass scale up to $D=6$, (ii) phase transition at 3.05×10^{16} GeV and (iii) dependence of gravitational and other coupling constants on energy scale. Results also suggest that space-time above 3.05×10^{16} GeV should be fractal. Moreover, dimension of the compact manifold decreases with the decreasing energy mass scale such that $D=1$ at the scale of the phase transition. Results imply invisiblity of S^1 at this scale (which is 3.05×10^{16} GeV).

Keywords. Quantum field theory; higher-dimensional and higher-derivative gravity; one-loop renormalization.

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1. Introduction

Theory of gravity with the action containing higher-derivative terms of curvature tensor is an interesting candidate for the past many years. It obeys basic principles of the general relativity, namely, principle of covariance as well as principle of equivalence. While quantizing gravity (quantizing components of the metric tensor), this theory has problem at the perturbation level, where ghost terms appear in the Feynman propagator of the graviton [1].

Recently a different feature of higher-derivative gravity has been noticed. The present paper deals with the new feature of this theory, where it is obtained that, in the high-energy regime, the Ricci scalar also behaves like a physical field in addition to its usual nature like a geometrical field. Thus, at a high energy level, the Ricci scalar manifests itself in dual manner [2–7].

Here dual roles of the Ricci scalar R (like a matter field as well as a geometrical field) are exploited. The ghost problem does not appear here if coupling constants in the gravitational action is taken properly (the condition to avoid the ghost problem is given in the following section). The matter aspect of R is represented by a scalar field $\tilde{R} = \eta R$ (where η has length dimension in natural units defined below).

In quantum field theory, fields are treated as mathematical concepts describing particles. After the name of the great mathematician Ricci, particle described by $\tilde{R}$ is called as riccion.

In earlier works [2–4,6,7], riccions were obtained from the four-dimensional action for R^2-gravity and, in [5,8], it was obtained from the $(4+D)$-dimensional space-time geometry. In [2–4,6], phase transition for riccions are discussed. In [5], it is discussed that riccions decouple to riccinos and anti-riccinos when parity is voilated. In [7], it is showed that riccions also behave like instantons. The main aim of the present paper is to discuss one-loop renormalization of the theory of riccions.

In what follows, like in ref. [8], riccions are obtained from the higher-dimensional geometry with topology $M^4 \otimes S^D$ (M^4 is the four-dimensional space-time with the signature $(+,-,-,-)$ and S^D is D-dimensional sphere which is an extra-dimensional compact space. The distance function is defined as

$$\mathrm{d}S^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu - \rho^2 \mathrm{d}\Omega^2 \tag{1.1a}$$

with

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\theta_2^2 + \cdots + \sin^2\theta_1 \cdots \sin^2\theta_{(D-1)} \mathrm{d}\theta_D^2. \tag{1.1b}$$

Here $g_{\mu\nu}$ $(\mu,\nu = 0,1,2,3)$ are components of the metric tensor in M^4, ρ is the radius of D-dimensional sphere S^D which is independent of coordinates x^μ and $0 \le \theta_1, \theta_2, \ldots, \theta_{(D-1)} \le \pi$ and $0 \le \theta_D \le 2\pi$. As usual, the space-time manifold is taken to be C^∞-connected, Hausdorff and paracompact without boundary [7,8].

The paper is organized as follows: In §2, taking the action for higher-derivative gravity in $(4+D)$-dimensional space-time, action for the riccion is obtained. Section 3 contains one-loop quantum correction to riccions in the background geometry, calculation of counter-terms and renormalization. Renormalization group equations are obtained and solved in §4. Section 5 is the concluding section where results are discussed.

Natural units are defined as $\kappa_\mathrm{B} = \hbar = c = 1$ (where κ_B is Boltzman's constant, $\hbar$ is Planck's constant divided by 2π and c is the speed of light), which are used throughout the paper.

2. Riccions from (4+D)-dimensional geometry

The action for the higher-derivative gravity is taken as

$$S_g^{(4+D)} = \int \mathrm{d}^4x \mathrm{d}^D y \sqrt{-g_{(4+D)}} \left[\frac{R_{(4+D)}}{16\pi G_{(4+D)}} + \alpha_{(4+D)} R^2_{(4+D)} + \gamma_{(4+D)} R^3_{(4+D)} - 2\eta^2 R_D \Lambda_{(4+D)} \right], \tag{2.1a}$$

where

$$G_{(4+D)} = GV_D, \quad \alpha_{(4+D)} = \alpha V_D^{-1}, \quad \gamma_{(4+D)} = \frac{\eta^2}{3!(D-2)} V_D^{-1},$$

$$R_D = \frac{D(D-1)}{\rho^2} \quad \text{and} \quad \Lambda_{(4+D)} = \frac{\Lambda}{(4+D)V_D}.$$

Here V_D is the volume of S^D, $g_{(4+D)}$ is the determinant of the metric tensor g_{MN} $(M, N = 0, 1, 2, \ldots, (4+D))$ and $R_{(4+D)} = R + R_D$. α is a dimensionless coupling constant, R is the Ricci scalar, $\eta^2 R_D \Lambda/(4+D)$ is the cosmological constant and G is the gravitational constant in a four-dimensional theory.

It is important to mention here that higher-derivative terms in the action given by eq. (2.1a) are significant at the energy mass scale given by

$$M^2 \geq \left[\frac{2\eta^2}{3!(D-2)}\right]^{-1} \left[-\alpha + \sqrt{\alpha^2 + \frac{1}{24\pi G(D-2)}}\right]. \tag{2.1b}$$

M is obtained using the method described in Appendix A. In case $G = G_N$ (Newtonian gravitational constant), $M \geq 2.2 \times 10^9$ GeV. It shows that higher-derivative terms are relevant in the gravitational action at high energy level.

Invariance of $S_g^{(4+D)}$ under transformations $g_{MN} \to g_{MN} + \delta g_{MN}$ yields [9,10]

$$\begin{aligned}(16\pi G_{(4+D)})^{-1} \left(R_{MN} - \tfrac{1}{2} g_{MN} R_{(4+D)}\right) + \alpha_{(4+D)} H_{MN}^{(1)} + \gamma_{(4+D)} H_{MN}^{(2)} \\ + \eta^2 R_D \Lambda_{(4+D)} g_{MN} = 0,\end{aligned} \tag{2.2a}$$

where

$$H_{MN}^{(1)} = 2R_{;MN} - 2g_{MN} \Box_{(4+D)} R_{(4+D)} - \tfrac{1}{2} g_{MN} R_{(4+D)}^2 + 2R_{(4+D)} R_{MN}, \tag{2.2b}$$

and

$$H_{MN}^{(2)} = 3R_{;MN}^2 - 3g_{MN} \Box_{(4+D)} R_{(4+D)}^2 - \tfrac{1}{2} g_{MN} R_{(4+D)}^3 + 3R_{(4+D)}^2 R_{MN} \tag{2.2c}$$

with semi-colon (;) denoting curved space covariant derivative and

$$\Box_{(4+D)} = \frac{1}{\sqrt{-g_{(4+D)}}} \frac{\partial}{\partial x^M} \left(\sqrt{-g_{(4+D)}} \quad g^{MN} \frac{\partial}{\partial x^N}\right).$$

Trace of these field equations is obtained as

$$\begin{aligned}\left[-\frac{D+2}{32\pi G_{(4+D)}}\right] R_{(4+D)} - \alpha_{(4+D)} \left[2(D+3)\Box_{(4+D)} R_{(4+D)} + \tfrac{1}{2} D R_{(4+D)}^2\right] \\ -\gamma_{(4+D)} \left[3(D+3)\Box_{(4+D)} R_{(4+D)}^2 + \tfrac{1}{2}(D-2) R_{(4+D)}^3\right] \\ +(4+D)\eta^2 R_D \Lambda_{(4+D)} = 0\end{aligned} \tag{2.3}$$

In the space-time described by the distance function defined in eq. (1.1),

$$\Box_{(4+D)} R_{(4+D)} = \Box R = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial}{\partial x^\nu} \right) R, \tag{2.4}$$

using the definition of $R_{(4+D)}$ given in eq. (2.1).

As $\Box_{(4+D)} R^2_{(4+D)}$ is a total divergence, using the Gauss's divergence theorem one obtains

$$\int_\Omega \mathrm{d}^{(4+D)} x \sqrt{-g_{(4+D)}} \Box_{(4+D)} R^2_{(4+D)} = \int_{\partial\Omega} \mathrm{d}^{(4+D)} x \sqrt{-g_{(4+D)}} R^2_{(4+D);M} n^M,$$

where Ω is the volume of the space-time manifold, which is taken to be C^∞-connected, Hausdorff and paracompact without boundary as usual [11]. n^M are components of unit vector normal to the $(D+3)$-dimensional hypersurface. So, $\partial\Omega = 0$, being the boundary of the space-time manifold under consideration. As a result

$$\int_{\partial\Omega} \mathrm{d}^{(4+D)} x \sqrt{-g_{(4+D)}} R^2_{(4+D);M} n^M = 0,$$

which implies that

$$\int_\Omega \mathrm{d}^{(4+D)} x \sqrt{-g_{(4+D)}} \Box R^2_{(4+D)} = 0$$

yielding

$$\Box_{(4+D)} R^2_{(4+D)} = 0. \tag{2.5}$$

Connecting eqs (2.3), (2.4) and (2.5) as well as using $R_{(4+D)}, \Lambda_{(4+D)}, G_{(4+D)}, \alpha_{(4+D)}$ and $\gamma_{(4+D)}$ from eq. (2.1), one obtains

$$\left[\frac{D+2}{32\pi G}\right](R+R_D) + \alpha\left[2(D+3)\Box R + \frac{1}{2}D(R+R_D)^2\right] + \frac{\eta^2}{3!(D-2)}\left[3(D+3)\Box R^2 + \frac{1}{2}(D-2)(R+R_D)^3\right] - \eta^2 R_D \Lambda = 0, \tag{2.6}$$

which is re-written as

$$\left[\Box + \frac{1}{2}\xi R + m^2 + \frac{\lambda}{3!}\eta^2 R^2\right] R = \frac{1}{2\alpha(D+3)}\left[\eta^2 R_D \Lambda - \frac{D+2}{32\pi G} R_D - \frac{1}{2}\alpha D R_D^2 - \frac{1}{12}\eta^2 R_D^3\right] \tag{2.7a}$$

with

$$\xi = \frac{D}{2(D+3)} + \eta^2 \lambda R_D \tag{2.7b}$$

$$m^2 = \frac{(D+2)\lambda}{16\pi G} + \frac{D R_D}{2(D+3)} + \frac{1}{2}\eta^2 \lambda R_D^2 \tag{2.7c}$$

$$\lambda = \frac{1}{4(D+3)\alpha}, \tag{2.7d}$$

where $\alpha > 0$ to avoid the ghost problem [12].

Multiplying by η and recognizing ηR as $\tilde{R}$, eq. (2.7a) is re-written as

$$\left[\Box + \frac{1}{2}\xi R + m^2 + \frac{\lambda}{3!}\tilde{R}^2\right]\tilde{R} = \frac{\eta}{2\alpha(D+3)}\left[\eta^2 R_D \Lambda - \frac{D+2}{32\pi G}R_D - \frac{1}{2}\alpha D R_D^2 - \frac{1}{12}\eta^2 R_D^3\right]. \tag{2.7e}$$

The reason for multiplying by η is given below. Now the question arises how to interpret the physical meaning of eq. (2.7e). For this purpose, it is convenient to find an analogy in the existing theories. From field theories, it is known that a scalar field ϕ satisfies an equation

$$\left[\Box\phi + \xi_\phi R + m_\phi^2 + \frac{\lambda_\phi}{(3!)}\phi^2\right]\phi = 0, \tag{2.8}$$

where ξ_ϕ, λ_ϕ are coupling constants and m_ϕ^2 is the (mass)^2 term for ϕ.

Equation (2.7e) can be analogous to eq. (2.8) if

$$\Lambda = \eta^{-2}\left[\frac{D+2}{32\pi G} + \frac{D R_D}{8(D+3)\lambda} + \frac{1}{12}\eta^2 R_D^2\right]. \tag{2.9a}$$

It means that, in a four-dimensional theory, the cosmological constant $\eta^2 R_D(\Lambda/(4+D))$ is caused by the extra-dimensional compact component S^D of the higher-dimensional space-time.

So, eq. (2.7e) looks like

$$\left[\Box + \frac{1}{2}\xi R + m^2 + \frac{\lambda}{3!}\tilde{R}^2\right]\tilde{R} = 0 \tag{2.9b}$$

with ξ, m^2 and λ defined by eqs (2.7b), (2.7c) and (2.7d).

Equation (2.8) is derived from the action

$$S_\phi = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}\{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - (\xi_\phi R + m_\phi^2)\phi^2\} - \frac{\lambda}{4!}\phi^4\right] \tag{2.10}$$

using its invariance under transformation $\phi \to \phi + \delta\phi$.

Mass dimension of ϕ is 1, whereas mass dimension of R is 2 which is a combination of second order derivative as well as squares of first order derivative of metric tensor components with respect to space-time coordinates. So, R is multiplied by η (having length dimension) to get $\tilde{R}$ (as above) with mass dimension 1.

According to discussions given above, eq. (2.9b) is possible only when higher-derivative terms are significantly present in the gravitational action, given by eq. (2.1).

High frequency modes probe the geometry in the small vicinity of a space-time point with coordinates $\{x'^\mu; \mu = 0,1,2,3\}$. Components of the metric tensor $g_{\mu\nu}$ have asymptotic expansion around a point $\{x'\}$ as [9,10]

$$g_{\mu\nu}(x) = g_{\mu\nu}(x') + \tfrac{1}{3}R_{\mu\alpha\nu\beta}(x')y^{\alpha}y^{\beta} - \tfrac{1}{6}\partial_{\gamma}R_{\mu\alpha\nu\beta}(x')y^{\alpha}y^{\beta}y^{\gamma}$$
$$+ \left[\tfrac{1}{20}R_{\mu\alpha\nu\beta;\gamma\delta} + \tfrac{2}{45}R_{\mu\alpha\beta\lambda}R^{\lambda}_{\gamma\nu\delta}\right](x')y^{\alpha}y^{\beta}y^{\gamma}y^{\delta} + \cdots,$$

where $y^{\alpha} = x^{\alpha} - x'^{\alpha}$ $(\alpha = 0,1,2,3)$ and $g_{\mu\nu}(x') = \eta_{\mu\nu}$.

Using these expressions, one obtains the operator

$$\Box = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\right)$$

as

$$\Box = g^{\mu\nu}(x')\frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}} + B^{\nu}(x;x')\frac{\partial}{\partial x^{\nu}}$$

with

$$g^{\mu\nu}(x) = g^{\mu\nu}(x') - \tfrac{1}{3}R^{\mu\nu}_{\alpha\beta}(x')y^{\alpha}y^{\beta} - \tfrac{1}{6}\partial_{\gamma}R^{\mu\nu}_{\alpha\beta}(x')y^{\alpha}y^{\beta}y^{\gamma}$$
$$- \left[\tfrac{1}{20}R^{\mu\nu}_{\alpha\beta;\gamma\delta} + \tfrac{2}{45}R^{\mu}_{\alpha\beta\lambda}R^{\lambda\nu}_{\gamma\delta}\right](x')y^{\alpha}y^{\beta}y^{\gamma}y^{\delta} + \cdots$$

and

$$B^{\nu}(x;x') = \left[\tfrac{1}{6}\partial_{\gamma}R^{\gamma\nu}_{\alpha\beta} - \tfrac{1}{12}\partial^{\nu}R_{\alpha\beta}\right](x')y^{\alpha}y^{\beta} - \left[\tfrac{1}{20}R^{\nu}_{\beta;\gamma\delta}\right.$$
$$\left. + \tfrac{2}{45}R_{\beta\lambda}R^{\lambda\nu}_{\gamma\delta}\right](x')y^{\beta}y^{\gamma}y^{\delta} + \left[\tfrac{1}{20}R^{\nu}_{\alpha;\gamma\delta}\right.$$
$$\left. + \tfrac{2}{45}R_{\beta\lambda}R^{\lambda\nu}_{\gamma\delta}\right](x')y^{\alpha}y^{\gamma}y^{\delta} - \left[\tfrac{1}{20}R^{\mu\nu}_{\alpha;\gamma\delta} + \tfrac{2}{45}R^{\mu}_{\alpha\beta\gamma}R^{\gamma\nu}_{\mu\delta}\right](x')$$
$$\times y^{\alpha}y^{\beta}y^{\delta} - \left[\tfrac{1}{20}R^{\mu\nu}_{\alpha\beta;\gamma\mu} + \tfrac{2}{45}R^{\mu}_{\alpha\beta\lambda}R^{\lambda\nu}_{\gamma\mu}\right](x')y^{\alpha}y^{\beta}y^{\gamma}$$
$$- \tfrac{1}{6}R_{\gamma\delta}(x')\left[\tfrac{1}{6}\partial_{\gamma}R^{\gamma\nu}_{\alpha\beta} - \tfrac{1}{12}\partial^{\nu}R_{\alpha\beta}\right](x')y^{\alpha}y^{\beta}y^{\gamma}y^{\delta} + \cdots.$$

Thus, at high energy level, one can work in the small neighborhood of a point $\{x'\}$, where $\Box$ depends on curvature terms evaluated at this particular point and $\tilde{R}(x)$ is defined at an arbitrary point in its neighborhood. So, at high energy, it is possible to have $\tilde{R}$ independent of $\Box$ and can be treated similar to ϕ.

Moreover, $\Box\tilde{R}$ is a scalar. In a locally inertial coordinate system, where $B^{\mu} = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$

$$\Box\tilde{R} = \eta^{\mu\nu}\frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}\tilde{R}$$

showing that $\Box$ is a similar operator for $\tilde{R}$ as it is for ϕ. According to the principle of equivalence (in the general relativity), this characteristic feature of $\Box$ with $\tilde{R}$ will be maintained at the global scale also [13] as $\Box\tilde{R}$ is a scalar. It means that (i) $\Box\tilde{R}$ is linear in $\tilde{R}$ at local as well as global scales, (ii) the scalar operator $\Box$ is a similar operator for $\tilde{R}$ as it is for ϕ at local as well as global scales.

On the basis of these analyses, it is inferred that, at high energy level, the Ricci scalar not only behaves as a geometrical field, but also as a spinless physical field [2,7]. At the low energy level, where higher-derivative terms are not significant, it behaves like a geometrical field only. Starobinsky has also found this kind of behavior of the Ricci scalar as a particle and termed it as scalaron [12]. The scalaron has mass dimension 2 in the natural units, whereas the riccion has mass dimension 1. It is so as the riccion is represented through the scalar field $\tilde{R} = \eta R$, but the scalaron is represented through the scalar field R.

To exploit the matter aspect of the four-dimensional Ricci scalar R (obtained from the higher-dimensional geometry), $\tilde{R}$ is treated as a basic physical field. Now, one can argue 'If $\tilde{R}$ is a physical field, there should be an action $S_{\tilde{R}}$ yielding eq. (2.9b) when invariance of $S_{\tilde{R}}$, under transformations $\tilde{R} \to \tilde{R} + \delta\tilde{R}$, is used as S_ϕ, given by eq. (2.10), yields eq. (2.8) when it is invariant under $\phi \to \phi + \delta\phi$.' To support this argument, $S_{\tilde{R}}$ is obtained in what follows.

If such an action exists, one can write

$$\delta S_{\tilde{R}} = -\int \mathrm{d}^4x\sqrt{-g}\,\delta\tilde{R}\left[\Box + \frac{1}{2}\xi R + m^2 + \frac{\lambda}{3!}\tilde{R}^2\right]\tilde{R} \tag{2.11a}$$

which yields eq. (2.9b) if $\delta S_{\tilde{R}} = 0$ under transformations $\tilde{R} \to \tilde{R} + \delta\tilde{R}$. From eq. (2.11a)

$$\begin{aligned}
\delta S_{\tilde{R}} &= -\int \mathrm{d}^4x\sqrt{-g}\,\delta\tilde{R}\left[\Box + \frac{1}{2\eta}\xi \tilde{R} + m^2 + \frac{\lambda}{3!}\tilde{R}^2\right]\tilde{R} \\
&= \int \mathrm{d}^4x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\tilde{R}\partial_\nu(\delta\tilde{R}) - \left(\frac{1}{2\eta}\xi\tilde{R}^2 + m^2\tilde{R} + \frac{\lambda}{3!}\tilde{R}^3\right)\delta\tilde{R}\right] \\
&= \int \mathrm{d}^4x\,\delta\left\{\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\tilde{R}\partial_\nu\tilde{R} - \left(\frac{1}{3!\eta}\xi\tilde{R}^3 + \frac{1}{2}m^2\tilde{R}^2 + \frac{\lambda}{4!}\tilde{R}^4\right)\right]\right\} \\
&\quad - \int \mathrm{d}^4x\sqrt{-g}\frac{\delta(\sqrt{-g})}{\sqrt{-g}}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\tilde{R}\partial_\nu\tilde{R} - \left(\frac{1}{3!\eta}\xi\tilde{R}^3 + \frac{1}{2}m^2\tilde{R}^2 + \frac{\lambda}{4!}\tilde{R}^4\right)\right] \\
&\quad - \int \mathrm{d}^4x\sqrt{-g}\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\tilde{R}\partial_\nu\tilde{R}.
\end{aligned} \tag{2.11b}$$

As in the integral

$$\int \mathrm{d}^4x\sqrt{-g}\frac{\delta(\sqrt{-g})}{\sqrt{-g}}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\tilde{R}\partial_\nu\tilde{R} - \left(\frac{1}{3!\eta}\xi\tilde{R}^3 + \frac{1}{2}m^2\tilde{R}^2 + \frac{\lambda}{4!}\tilde{R}^4\right)\right],$$

$\delta(\sqrt{-g})/\sqrt{-g}$ is invariant under cooordinate transformations being a scalar. So, there is no harm, if it is evaluated in a locally inertial coordinate system (because a scalar is not different at local as well as global scales), where

$$\frac{\delta(\sqrt{-g})}{\sqrt{-g}} = \frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu} = \frac{1}{2}\eta_{\mu\nu}\delta\eta^{\mu\nu} = 0$$

with $\eta_{\mu\nu}$ being components of the Minkowskian metric (which are components of the metric tensor in a locally inertial coordinate system). Thus, one obtains

$$\int \mathrm{d}^4x\sqrt{-g}\frac{\delta(\sqrt{-g})}{\sqrt{-g}}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\tilde{R}\partial_\nu\tilde{R} - \left(\frac{1}{3!\eta}\xi\tilde{R}^3 + \frac{1}{2}m^2\tilde{R}^2 + \frac{\lambda}{4!}\tilde{R}^4\right)\right] = 0. \tag{2.11c}$$

Similarly

$$\int \mathrm{d}^4x\sqrt{-g}\frac{1}{2}\delta g^{\mu\nu}\partial_\mu\tilde{R}\partial_\nu\tilde{R} = 0 \tag{2.11d}$$

as $\delta g^{\mu\nu}\partial_\mu\tilde{R}\partial_\nu\tilde{R} = 0$ in a locally inertial coordinate system which is true at global scales also according to the principle of equivalence.

Now, using eqs (2.11c) and (2.11d), eq. (2.11b) reduces to

$$\delta S_{\tilde{R}} = \int \mathrm{d}^4x\delta\left\{\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\tilde{R}\partial_\nu\tilde{R} - \left(\frac{1}{3!\eta}\xi\tilde{R}^3 + \frac{1}{2}m^2\tilde{R}^2 + \frac{\lambda}{4!}\tilde{R}^4\right)\right]\right\}$$

yielding

$$S_{\tilde{R}} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\tilde{R}\partial_\nu\tilde{R} - \left(\frac{1}{3!\eta}\xi\tilde{R}^3 + \frac{1}{2}m^2\tilde{R}^2 + \frac{\lambda}{4!}\tilde{R}^4\right)\right]. \tag{2.12}$$

It is important to mention here that although $\tilde{R}$ behaves like other scalar fields ϕ, results obtained below for $\tilde{R}$ are novel. Such results are not possible for ϕ. The main reason for this difference to happen is the dependence of (mass)^2 for $\tilde{R}$ on the gravitational constant, dimensionality of the space-time and the coupling constant α which is given by eq. (2.7c), whereas mass of ϕ does not depend on these constants. Moreover, $\tilde{R} = \eta R$, whereas there exists no such relation between R and ϕ.

3. One-loop quantum correction and renormalization

The $S_{\tilde{R}}$ with the Lagrangian density, given by eq. (2.9), can be expanded around the classical minimum $\tilde{R}_0$ in powers of quantum fluctuation $\tilde{R}_q = \tilde{R} - \tilde{R}_0$ as

$$S_{\tilde{R}} = S^{(0)}_{\tilde{R}} + S^{(1)}_{\tilde{R}} + S^{(2)}_{\tilde{R}} + \cdots,$$

where

$$S^{(0)}_{\tilde{R}} = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\tilde{R}_0\partial_\nu\tilde{R}_0 - \left(\frac{1}{3!\eta}\xi\tilde{R}_0^3 + \frac{1}{2}m^2\tilde{R}_0^2 + \frac{\lambda}{4!}\tilde{R}_0^4\right)\right]$$

$$S^{(2)}_{\tilde{R}} = \int \mathrm{d}^4x\sqrt{-g}\tilde{R}_q\left[\Box + \frac{1}{2}\xi R + m^2 + \frac{\lambda}{2!}\tilde{R}_0^2\right]\tilde{R}_q$$

and

$$S^{(1)}_{\tilde{R}} = 0$$

as usual, because this term contains the classical equation.

The effective action of the theory is expanded in powers of $\hbar$ (with $\hbar = 1$) as

$$\Gamma(\tilde{R}) = S_{\tilde{R}} + \Gamma^{(1)} + \Gamma'$$

with one-loop correction given as [9]

$$\Gamma^{(1)} = \frac{i}{2} \ln \mathrm{Det}(D/\mu^2), \tag{3.1a}$$

where

$$D \equiv \left. \frac{\delta^2 S_{\tilde{R}}}{\delta \tilde{R}^2} \right|_{\tilde{R}=\tilde{R}_0} = \Box + \frac{1}{2}\xi R + m^2 + \frac{\lambda}{2!}\tilde{R}_0^2 \tag{3.1b}$$

and Γ' is a term for higher-loop quantum corrections. In eq. (3.1), μ is a mass parameter to keep $\Gamma^{(1)}$ dimensionless.

To evaluate $\Gamma^{(1)}$, the operator regularization method [14] is used. Up to adiabatic order 4 (potentially divergent terms are expected up to this order only in a four-dimensional theory), one-loop correction is obtained as

$$\begin{aligned}\Gamma^{(1)} = {} & (16\pi^2)^{-1} \frac{\mathrm{d}}{\mathrm{d}s} \left[\int \mathrm{d}^4x \sqrt{-g(x)} \left(\frac{\tilde{M}^2}{\mu^2} \right)^{-s} \left\{ \frac{\tilde{M}^4}{(s-2)(s-1)} \right. \right. \\ & + \frac{\tilde{M}^2}{(s-1)} \left(\frac{1}{6} - \frac{1}{2}\xi \right) R + \left[\frac{1}{6} \left(\frac{1}{5} - \frac{1}{2}\xi \right) \Box R + \frac{1}{180} R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \right. \\ & \left. \left. \left. - \frac{1}{180} R^{\mu\nu} R_{\mu\nu} + \frac{1}{2} \left(\frac{1}{6} - \frac{1}{2}\xi \right)^2 R^2 \right] \right\} \right] \Bigg|_{s=0},\end{aligned} \tag{3.2a}$$

where

$$\tilde{M}^2 = m^2 + (\lambda/2)\tilde{R}_0^2. \tag{3.2b}$$

Here it is important to note that both matter as well as geometrical aspects of the Ricci scalar are used in eq. (3.2). The matter aspect is manifested by $\tilde{R}$ and the geometrical aspect by R. Ricci tensor components $R_{\mu\nu}$ and curvature tensor components $R_{\mu\nu\alpha\beta}$ are the same as mentioned above.

After some manipulations, the Lagrangian density in $\Gamma^{(1)}$ is obtained as

$$\begin{aligned}\Gamma^{(1)} = {} & (16\pi^2)^{-1} \left[(m^2 + (\lambda/2)\tilde{R}_0^2)^2 \left\{ \frac{3}{4} - \frac{1}{2} \ln \left(\frac{m^2 + (\lambda/2)\tilde{R}_0^2}{\mu^2} \right) \right\} \right. \\ & - \left(\frac{1}{6} - \frac{1}{2}\xi \right) R (m^2 + (\lambda/2)\tilde{R}_0^2) \left\{ 1 - \ln \left(\frac{m^2 + (\lambda/2)\tilde{R}_0^2}{\mu^2} \right) \right\} \\ & - \ln \left(\frac{m^2 + (\lambda/2)\tilde{R}_0^2}{\mu^2} \right) \left\{ \frac{1}{6} \left(\frac{1}{5} - \frac{1}{2}\xi \right) \Box R + \frac{1}{180} R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \right. \\ & \left. \left. - \frac{1}{180} R^{\mu\nu} R_{\mu\nu} + \frac{1}{2} \left(\frac{1}{6} - \frac{1}{2}\xi \right)^2 R^2 \right\} \right].\end{aligned} \tag{3.3}$$

Now the renormalized form of Lagrangian density can be written as

$$\begin{aligned}L_{\mathrm{ren}} = {} & \frac{1}{2} g^{\mu\nu} \partial_\mu \tilde{R}_0 \partial_\nu \tilde{R}_0 - \frac{\xi}{3!\eta} \tilde{R}_0^3 - \frac{1}{2} m^2 \tilde{R}_0^2 - \frac{\lambda}{4!} \tilde{R}_0^4 + \Lambda \\ & + \varepsilon_0 R + \frac{1}{2} \varepsilon_1 R^2 + \varepsilon_2 R^{\mu\nu} R_{\mu\nu} + \varepsilon_3 R^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \\ & + \varepsilon_4 \Box R + \Gamma^{(1)} + L_{\mathrm{ct}}\end{aligned} \tag{3.4a}$$

with bare coupling constants $\lambda_i \equiv (m^2, \lambda,\ \Lambda, \xi, \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4), \Gamma^{(1)}$ given by eq. (3.3) and $L_{\rm ct}$ given as

$$L_{\rm ct} = -\frac{1}{2}\delta\xi R\tilde{R}_0^2 - \frac{1}{2}\delta m^2 \tilde{R}_0^2 - \frac{\delta\lambda}{4!}\tilde{R}_0^4 + \delta\Lambda + \delta\varepsilon_0 R + \frac{1}{2}\delta\varepsilon_1 R^2 + \delta\varepsilon_2 R^{\mu\nu}R_{\mu\nu} + \delta\varepsilon_3 R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta} + \delta\varepsilon_4 \Box R. \tag{3.4b}$$

In eq. (3.4b), $\delta\lambda_i \equiv (\delta m^2, \delta\lambda, \delta\Lambda, \delta\xi, \delta\varepsilon_0, \delta\varepsilon_1, \delta\varepsilon_2, \delta\varepsilon_3, \delta\varepsilon_4)$ are counter-terms, which are calculated using the following renormalization conditions [15,16]

$$\Lambda = L_{\rm ren}\big|_{\tilde{R}_0 = \tilde{R}_{(0)0}, R=0} \tag{3.5a}$$

$$\lambda = -\frac{\partial^4}{\partial\tilde{R}_0^4} L_{\rm ren}\Big|_{\tilde{R}_0 = \tilde{R}_{(0)1}, R=0} \tag{3.5b}$$

$$m^2 = -\frac{\partial^2}{\partial\tilde{R}_0^2} L_{\rm ren}\Big|_{\tilde{R}_0 = 0, R=0} \tag{3.5c}$$

$$\frac{1}{2}\xi = -\eta\frac{\partial^3}{\partial R\partial\tilde{R}_0^2} L_{\rm ren}\Big|_{\tilde{R}_0 = \tilde{R}_{(0)2}, R=0} \tag{3.5d}$$

$$\varepsilon_0 = \frac{\partial}{\partial R} L_{\rm ren}\Big|_{\tilde{R}_0 = 0, R=0} \tag{3.5e}$$

$$\varepsilon_1 = \frac{\partial^2}{\partial R^2} L_{\rm ren}\Big|_{\tilde{R}_0 = 0, R=R_5} \tag{3.5f}$$

$$\varepsilon_2 = \frac{\partial}{\partial(R^{\mu\nu}R_{\mu\nu})} L_{\rm ren}\Big|_{\tilde{R}_0 = 0, R=R_6} \tag{3.5g}$$

$$\varepsilon_3 = \frac{\partial}{\partial(R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta})} L_{\rm ren}\Big|_{\tilde{R}_0 = 0, R=R_7} \tag{3.5h}$$

$$\varepsilon_4 = \frac{\partial}{\partial(\Box R)} L_{\rm ren}\Big|_{\tilde{R}_0 = 0, R=R_8}. \tag{3.5i}$$

As $\tilde{R} = \eta R$, when $R = 0$, $\tilde{R}_{(0)0} = \tilde{R}_{(0)1} = \tilde{R}_{(0)2} = 0$ and $R_5 = R_6 = R_7 = R_8 = 0$ when $\tilde{R}_0 = 0$.

Equations (3.4) and (3.5) yield counter-terms as

$$16\pi^2\delta\Lambda = \frac{m^4}{2}\ln(m^2/\mu^2) \tag{3.6a}$$

$$16\pi^2\delta\lambda = -3\lambda^2\ln(m^2/\mu^2) \tag{3.6b}$$

$$16\pi^2\delta m^2 = -\lambda m^2\ln(m^2/\mu^2) \tag{3.6c}$$

$$16\pi^2\delta\xi = -3\lambda\left(\xi - \frac{1}{3}\right)\ln(m^2/\mu^2) \tag{3.6d}$$

$$16\pi^2\delta\varepsilon_0 = \frac{m^2}{2}\left(\xi - \frac{1}{3}\right)\ln(m^2/\mu^2) \tag{3.6e}$$

$$16\pi^2\delta\varepsilon_1 = \frac{1}{4}\left(\xi - \frac{1}{3}\right)^2 \ln(m^2/\mu^2) \tag{3.6f}$$

$$16\pi^2\delta\varepsilon_2 = -\frac{1}{180}\ln(m^2/\mu^2) \tag{3.6g}$$

$$16\pi^2\delta\varepsilon_3 = \frac{1}{180}\ln(m^2/\mu^2) \tag{3.6h}$$

$$16\pi^2\delta\varepsilon_4 = \frac{1}{6}\left(\frac{1}{5} - \frac{1}{2}\xi\right)\ln(m^2/\mu^2). \tag{3.6i}$$

4. Renormalization group equations and their solutions

The effective renormalized Lagrangian can be improved further by solving renormalization group equations for coupling constants $\lambda_{i(R)}$ (suffix R stands for renormalization, which is dropped onwards). For this purpose one-loop β-functions, defined by the equation [1,8,9]

$$\beta_{\lambda_i} = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}(\lambda_i + \delta\lambda_i)\Big|_{\lambda_i} \tag{4.1}$$

with counter-terms $\delta\lambda_i$ from eqs (3.6a)–(3.6i), are obtained as

$$\beta_\Lambda = -\frac{m^4}{16\pi^2} \tag{4.2a}$$

$$\beta_\lambda = \frac{3\lambda^2}{8\pi^2} \tag{4.2b}$$

$$\beta_{m^2} = \frac{\lambda m^2}{8\pi^2} \tag{4.2c}$$

$$\beta_\xi = \frac{3\lambda\left(\xi - \frac{1}{3}\right)}{16\pi^2} \tag{4.2d}$$

$$\beta_{\varepsilon_0} = -\frac{m^2\left(\xi - \frac{1}{3}\right)}{16\pi^2} \tag{4.2e}$$

$$\beta_{\varepsilon_1} = -\frac{\left(\xi - \frac{1}{3}\right)^2}{32\pi^2} \tag{4.2f}$$

$$\beta_{\varepsilon_2} = \frac{1}{1440\pi^2} \tag{4.2g}$$

$$\beta_{\varepsilon_3} = -\frac{1}{1440\pi^2} \tag{4.2h}$$

$$\beta_{\varepsilon_4} = -\frac{1}{48\pi^2}\left(\frac{1}{5} - \frac{1}{2}\xi\right) \tag{4.2i}$$

using the fact that $\mu(\mathrm{d}/\mathrm{d}\mu)\lambda_i = 0$ for bare coupling constants λ_i.

The renormalization group equations are given as

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}t} = \beta_{\lambda_i}, \tag{4.3}$$

where $t = \frac{1}{2}\ln(m_c^2/\mu^2)$ with μ being a mass parameter and m_c being a reference mass scale such that $\mu \geq m_c$. Using β functions for different coupling constants given by eqs (4.2a)–(4.2i), solutions of differential equation (4.3) are derived as

$$\Lambda = \Lambda_0 + \frac{m_0^4}{2\lambda_0}\left[\left(1 - \frac{3\lambda_0 t}{8\pi^2}\right)^{1/3} - 1\right] \tag{4.4a}$$

$$\lambda = \lambda_0\left[1 - \frac{3\lambda_0 t}{8\pi^2}\right]^{-1} \tag{4.4b}$$

$$m^2 = m_0^2\left[1 - \frac{3\lambda_0 t}{8\pi^2}\right]^{-1/3} \tag{4.4c}$$

$$\xi = \frac{1}{3} + \left(\xi_0 - \frac{1}{3}\right)\left[1 - \frac{3\lambda_0 t}{8\pi^2}\right]^{-1} \tag{4.4d}$$

$$\varepsilon_0 = \varepsilon_{00} + \frac{m_0^2\left(\xi_0 - \frac{1}{3}\right)}{2\lambda_0}\left[1 - \left(1 - \frac{3\lambda_0 t}{8\pi^2}\right)^{-1/3}\right] \tag{4.4e}$$

$$\varepsilon_1 = \varepsilon_{10} + \frac{\left(\xi_0 - \frac{1}{3}\right)^2}{4\lambda_0}\left[1 - \left(1 - \frac{3\lambda_0 t}{8\pi^2}\right)^{-1}\right] \tag{4.4f}$$

$$\varepsilon_2 = \varepsilon_{20} + \frac{t}{1440\pi^2} \tag{4.4g}$$

$$\varepsilon_3 = \varepsilon_{30} - \frac{t}{1440\pi^2} \tag{4.4h}$$

$$\varepsilon_4 = \varepsilon_{40} - \frac{t}{1440\pi^2} - \frac{\left(\xi_0 - \frac{1}{3}\right)}{36\lambda_0}\ln\left(1 - \frac{3\lambda_0 t}{8\pi^2}\right), \tag{4.4i}$$

where $\lambda_{i0} = \lambda_i(t=0)$ and $t = 0$ at $\mu = m_c$ according to definition of t given above.

These results show that as $\mu \to \infty$ ($t \to -\infty$), $\lambda \to 0$, $m^2 \to 0$ and $\frac{1}{2}\xi \to \frac{1}{6}$. Thus it follows from these expressions that in the limit $\mu \to \infty$, the theory is asymptotically free.

Using these limits in eq. (2.7b), it is obtained that $D \to 6$ as $\mu \to \infty$. Also eqs (2.1a) and (2.7b) imply that

$$R_D = \frac{D(D-1)}{\rho^2} = (1/\eta^2\lambda)\left[\xi - \frac{D}{2(D+3)}\right]. \tag{4.5}$$

Moreover, eq. (2.7c) yields

$$\frac{m^2}{2\lambda} = \frac{(D+2)}{32\pi G} + \frac{DR_D}{4\lambda(D+3)} + \frac{1}{4}\eta^2 R_D^2 \tag{4.6a}$$

and

$$\Lambda\eta^2 = \frac{(D+2)}{32\pi G} + \frac{DR_D}{8\lambda(D+3)} + \frac{1}{12}\eta^2 R_D^2 \tag{4.6b}$$

respectively. Connecting eqs (4.5) and (4.6), one gets a quadratic equation for D as

$$(1+A_1+A_2)D^2+3(2A_1+A_2)D+9A_1=0, \tag{4.7a}$$

where

$$A_1=6\left[8\eta^2\lambda^2\left(\frac{m^2}{2\lambda}-\Lambda\eta^2\right)-\frac{4}{3}\xi^2\right] \tag{4.7b}$$

$$A_2=2\xi. \tag{4.7c}$$

As D is the dimension of the extra-dimensional space, the physically relevant root of eq. (4.7) is

$$D=1.5\left|\frac{-(2A_1+A_2)+\sqrt{A_2^2-4A_1}}{(1+A_1+A_2)}\right| \tag{4.8}$$

with $D_0=1$ at $\mu=m_c$. Here A_1 and A_2 are defined in eqs (4.7b) and (4.7c). Dependence of m^2,λ and ξ on the energy mass scale μ is given by eqs (4.4a)–(4.4i). It is given below how η^2 depends on the energy mass scale μ.

Using eq. (4.8) in eqs (2.7b)–(2.7e) at $\mu=m_c$

$$\xi_0=\frac{1}{8} \tag{4.9a}$$

$$\lambda_0=\frac{1}{16\alpha_0}=\frac{1}{16} \tag{4.9b}$$

$$m_0^2=\frac{3\lambda_0}{16\pi G_0}=\frac{3M_{\mathrm{P}}^2}{256\pi} \tag{4.9c}$$

$$\Lambda_0=\frac{3}{32\pi G_0\eta^2}=\frac{3M_{\mathrm{P}}^2}{32\pi\eta^2} \tag{4.9d}$$

taking $\alpha_0=1$ and $G_0=G_N=M_P^{-2}$ where G_0 is the Newtonian gravitational constant.

From eq. (4.5), it is obtained that

$$\rho^2=\frac{D(D-1)}{16\lambda(D+3)}\left(\frac{m^2}{2\lambda}-\Lambda\eta^2\right)^{-1}\times\left[1+\sqrt{1+\frac{128\lambda^2\eta^2(D+3)^2}{3D^2}\left(\frac{m^2}{2\lambda}-\Lambda\eta^2\right)}\right] \tag{4.10a}$$

$\rho=\rho_0$ at $\mu=m_c$ can be evaluated from eq. (4.10a) re-writing the same as

$$\rho^2=\frac{D(D-1)}{8(D+3)m^2}\left(1+\frac{2\lambda\Lambda\eta^2}{m^2}+\cdots\right)\times\left[1+\sqrt{1+\frac{128\lambda^2\eta^2(D+3)^2}{3D^2}\left(\frac{m^2}{2\lambda}-\Lambda\eta^2\right)}\right].$$

As given above at $\mu = m_c, D = 1$ and $(m^2/2\lambda) = \Lambda\eta^2$. So, eq. (4.10a) implies that

$$\rho_0^2 = 0. \tag{4.10b}$$

From table 1, it is found that as the energy mass scale μ comes down $(\Lambda - \Lambda_0)$ decreases. When μ comes down from $1.000001 m_c$ to m_c, a huge amount of energy of the order of 3.05×10^{16} GeV (with density 8.7×10^{66} GeV^4) is released. This abrupt change shows a phase transition at

$$m_c = 3.05 \times 10^{16} \text{ GeV}. \tag{4.11}$$

Equations (4.10) and (4.11) imply that at $\mu = m_c = 3.05 \times 10^{16}$ GeV, $D = 1$ and the radius of extra-dimensional compact space S^D is vanishing. It is difficult to think of a S^1 (circle) with vanishing radius. However, on physical grounds, the smallest length that can be thought of is the Planck length L_{P}. So, it is reasonable to take

$$\rho_0^2 = L_{\mathrm{P}}^2 = 10^{-38} \text{ GeV}^{-2} \approx 0 \tag{4.12}$$

at the energy mass scale m_c given by eq. (4.11), as the length of the order of Planck length is invisible at 3.05×10^{16} GeV.

It means that when phase transition takes place, S^D shrinks to an unobservable tiny circle leading to compactification of the higher-dimensional space-time to the observable four-dimensional space-time.

But at $\mu > m_c, \rho$ should be observed, otherwise higher-dimensional space-time will be redundant. It is possible only when

$$\rho^2 \geq m_c^{-2}. \tag{4.13}$$

Table 1. $(\Lambda - \Lambda_0), G/G_N$ and dimension of the space-time $(4+D)$ are tabulated below against μ/m_c with $m_c = 1.76 \times 10^{16}$ GeV taking $\lambda_0 = 1$.

μ/m_c	$(\Lambda - \Lambda_0)$ in GeV^4	G/G_N	$(4+D)$	ρ/L_{P}
1	0×1	4	1	
1.000001	6.74×10^{65}	1.000000007	5.000000251	1.156×10^3
1.00001	6.59×10^{65}	1.000000013	5.000000448	1.544×10^3
1.0001	6.59×10^{66}	1.000000149	5.000004556	4.924×10^3
1.001	6.00×10^{67}	1.000002188	5.000047621	1.592×10^4
1.01	6.5×10^{68}	1.000015127	5.000454132	4.918×10^4
1.1	6.28×10^{69}	1.000143985	5.004346587	1.29×10^5
2	4.55×10^{70}	1.001001195	5.03143937	4.168×10^5
10	1.494×10^{71}	1.002933661	5.102936483	7.855×10^5
100	3.45×10^{71}	1.008601892	5.213725181	1.15×10^6
348.48	3.713×10^{71}	1.0055123958	5.25371926	1.25×10^6
10^5	7.06×10^{71}	1.005638664	5.475523461	1.98×10^6
10^{10}	1.33×10^{72}	1.001327407	5.868452973	3.23×10^6
10^{20}	2.392×10^{72}	0.958094676	7.479989382	5.83×10^6
10^{30}	3.3×10^{72}	0.91729725	7.928639028	7.68×10^6
10^{50}	4.7×10^{72}	0.824345225	8.553063893	1.195×10^7
∞	∞	0	10	∞

This condition can be easily satisfied by eq. (4.10a) if

$$\left(\frac{m^2}{2\lambda} - \Lambda\eta^2\right) = 1.32 \times 10^{-9} m_c^2. \tag{4.14a}$$

Connecting eqs. (4.4b), (4.4c) and (4.9c) with eq. (4.14a), it is obtained that

$$\begin{aligned}\eta^2 &= \frac{8m_0^2 X^{2/3} - 1.32 \times 10^{-9} m_c^2}{8m_0^2 X^{1/3}} \\ &= \frac{8m_0^2 X^{2/3} - 3.29 \times 10^{-12} m_0^2}{8m_0^2 X^{1/3}} \\ &\simeq \frac{X^{1/3}}{m_0^2}\end{aligned} \tag{4.14b}$$

with

$$\eta_0^2 = m_0^{-2} \tag{4.14c}$$

and

$$X = 1 + \frac{3\lambda_0}{8\pi^2} \ln(\mu/m_c). \tag{4.14d}$$

From eq. (4.8), it is obtained that $D \to 6$, when $\mu \to \infty$ as $\lambda^2\eta^2 \to 0$ and $\xi \to 1/6$. Also from eqs (4.4), (4.5), (4.6a), (4.9) and (4.14), one obtains

$$\begin{aligned}G = \frac{(D+2)G_0}{3X^{2/3}} &\left[1 - \frac{128\pi D^2(D-1)X^{1/3}L_\mathrm{P}^2}{3(D+3)\rho^2}\right. \\ &\left. - \frac{2048\pi^2 D^2(D-1)^2 X^{-1/3} L_\mathrm{P}^4}{9\rho^4}\right]^{-1}\end{aligned} \tag{4.15}$$

where D and ρ are given by eqs (4.8), (4.10a) and (4.14d) respectively. Here G_0 is the Newtonian gravitational constant which is $\simeq M_\mathrm{P}^{-2}$.

5. Concluding remarks

The higher-dimensional space-time, considered here, has the topology $M^4 \otimes S^D$ with M^4 as the four-dimensional space-time and S^D as the D-dimensional sphere (a compact manifold). In §2, it is shown that Λ is caused by the geometry of the compact manifold which behaves as vacuum energy density. It is evident from table 1 that as the energy mass scale comes down, $(\Lambda - \Lambda_0)$ decreases (here $\Lambda_0 = \Lambda(\mu = m_c)$). Moreover, dimension and radius of S^D also decrease with the decreasing mass scale. When the phase transition takes place at $\mu = m_c$, extra-dimensional space becomes too small to be observed. As a result, only four-dimensional space-time is observable at this energy mass scale. It means that with the fall in the mass scale, energy is transferred from S^D to M^4 which is the presently observed component of the $(4+D)$-dimensional space-time.

In the Kaluza–Klein type theories, extra-dimensional space in a $(4+D)$-dimensional theory is supposed to be caused when the other three fundamental forces (other than gravity) after unification also behave like gravity described by the geometry of a compact manifold S^D. Here extra-dimensional space is observable at $\mu > m_c = 3.05\times 10^{16}$ GeV. It means that the behavior of other forces like gravity described by the geometry of a compact manifold is possible at $\mu > m_c$. Shrinking of the compact manifold to an observable tiny circle at $\mu \leq m_c$ shows that the other forces become distinct from the four-dimensional gravity at this scale. Thus with the phase transition at 3.05×10^{16} GeV, spontaneous symmetry breaking takes place, and other forces dissociate from gravity. As a result, higher-dimensional gravity reduces to the four-dimensional gravity. Moreover, energy of the order of 3.05×10^{16} GeV is transferred from the compact manifold to the observable four-dimensional space-time. Also, at this stage, Λ_0 is obtained as

$$\Lambda_0 \simeq 1.1\times 10^{72}\ \text{GeV},$$

but the effective four-dimensional cosmological constant

$$\frac{\eta_0^2 R_D \Lambda_0}{D+4} = \frac{(D-1)}{(D+4)m_0^2 L_{\rm P}^2}\Lambda_0 = 0$$

as $D = 1$ at $\mu = m_c$.

It is interesting to see that as the energy mass scale goes above 3.05×10^{16} GeV, dimension of the compact manifold becomes non-integral showing its fractal nature [17,18]. It means that above 3.05×10^{16} GeV, S^D is not a smooth surface whereas M^4 is smooth. As discussed above, S^D is caused by forces other than gravity (according to Einstein's theory) which are effective forces among elementary particles. So the reason for non-smoothness of S^D may be the zig-zag paths followed by these particles. As the energy scale increases, the nature of particles to move in zig-zag manner is aggravated leading to more fractality due to more and more closeness between particles as well as high momentum.

When the energy mass scale comes down to m_c, fluctuations in M^4 will be large with S^D hidden in it and the energy of S^D will be transferred to M^4, which is our observed universe.

Appendix A

The four-dimensional higher-derivative gravitational action is taken as [2–4]

$$S_g = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{16\pi G} + \tilde{\alpha}R^{\mu\nu}R_{\mu\nu} + \tilde{\beta}R^2 - (1/3!)\lambda\eta^2R^3\right], \tag{A1}$$

where $\tilde{\alpha}$, $\tilde{\beta}$ and λ are dimensionless coupling constants.

To decide the energy mass scale M where higher-derivative terms will dominate over $R/16\pi G$, mass scale representation of these terms can be useful. In natural units, $R/16\pi G$ corresponds to $M^2M_{\rm P}^2/16\pi$. ($M_{\rm P}$ is the Planck mass and $G = M_{\rm P}^{-2}$, $[\tilde{\alpha}R^{\mu\nu}R_{\mu\nu} + \tilde{\beta}R^2]$ corresponds to $[\tilde{\alpha} + \tilde{\beta}g]M^4$ as well as $(1/3!)\lambda\eta^2R^3$ corresponds to $(1/3!)\lambda\eta^2M^6$ because R and $R_{\mu\nu}$ are linear combinations of second derivatives and squares of first derivatives of components of the metric tensor $g_{\mu\nu}$ (being defined through $\mathrm{d}S^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$) w.r.t.

space-time coordinates. $g_{\mu\nu}$ are dimensionless. Thus, it is found that higher-derivative terms are significant only when

$$M^2 \geq \frac{3[\tilde{\alpha}+\tilde{\beta}] + \sqrt{9(\tilde{\alpha}+\tilde{\beta})^2 + (3/8\pi)\lambda\eta^2 M_{\mathrm{P}}^2}}{\lambda\eta^2}. \tag{A2}$$

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