

## Invariance properties of the Dirac equation with external electro-magnetic field\*

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**Abstract.** In this paper, we attempt to obtain the nature of the external field such that the Dirac equation with external electro-magnetic field is invariant. The Poincaré group, which is the maximal symmetry group for field free case, is constrained by the presence of the external field. Introducing infinitesimal transformation of  $x$  and  $\psi$ , we apply Lie's extended group method to obtain the class of external field which admit of the invariance of the equation. It is important to note that the constraints for the existence of invariance are explicitly on the electric and magnetic field, though only potentials explicitly appears in the equation.

**Keywords.** Dirac equation; electromagnetic field; Lie groups.

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### 1. Introduction

The objective of this short paper is to investigate the invariance properties of the Dirac equation with external electro-magnetic field. There exists a large number of literatures on the problem beginning almost from the formulation of the equation to the present date. But all of these investigations are confined to the case of free particle equation, in the absence of the external electro-magnetic field, perhaps excluding very few special cases.

The Dirac equation is given by

$$\left\{ p_0 + \frac{e}{c}A_0 - \alpha \left( p + \frac{e}{c}A \right) - \beta mc \right\} \psi = 0 \quad (1)$$

$A = (i\phi, \vec{A})$  is the external electro-magnetic field potential [1]. We write the equation in space-time symmetrical form, with the introduction of

$$\begin{aligned} \gamma^0 &= \beta, \quad \gamma^j = i\alpha^j\beta \\ \gamma^k\gamma^l + \gamma^l\gamma^k &= 2\delta^{kl} \end{aligned} \quad (2)$$

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$x^j \equiv (x^0 = ict, x)$ , super and subscripts in latin  $k, j, \dots$ , stand for (0,1,2,3) for space-time indices and those in Greek  $\mu, \nu, \dots$  stand for spinor indices. Summation is implied with repeated indices, except when it is not explicitly stated otherwise. Equation (1) is now given by

$$D \equiv \left\{ \gamma^k \left( \frac{\partial}{\partial x^k} + iA_k \right) + M \right\} \psi = 0 \quad (3)$$

with  $M = mc/\hbar$ . The interaction parameter  $e/\hbar c$  is absorbed as a factor in  $A_k$  to avoid repetition. Without any loss of generality we can work with Hermitian  $\gamma^j$  matrices. Notation

$$(a \cdot b) = a^k b_k, \quad (p \cdot q) = p^\mu q_\mu. \quad (4)$$

Equation (3) involves two kinds of variables: (1) discrete variables  $\beta, \alpha^j(\gamma^j)$ , the Dirac matrices, and (2)  $x^q$ , the continuous space-time variables. The two sets of variables are not explicitly connected (though they are implicitly connected by eq. (3)). Hence, one can investigate their invariant properties independent of one another and attempt to correlate them from the physical stand point. The study of the invariance properties associated with the representation of the Dirac matrices started as early as 1936 by Pauli [2] and subsequently followed by many others. The present author made an almost exhaustive study of this problem earlier [3]. Since, the external el-mag. four potential in eq. (3), annexed to the momentum, is tied up with the matrices  $\gamma^j$ , as scalar product, the external field has a special role to play and obviously the results are not the same as field-free case but are restricted.

In this paper, we try to find the nature of external field for the invariance of the equation in the presence of external field. We introduce infinitesimal transformations of the space time coordinates as well as the spinor wave function and construct the extended group following Lie [4,5,6]. Then apply the operators of the extended group on the equation to obtain the nature of the admissible infinitesimal transformations. Obviously, one can not obtain the character of the admissible infinitesimal transformation without the explicit knowledge of the nature of the external el-mag. field.

In the next section, we develop the Lie's extended group for the problem after obtaining the first prolongation. This leads us to the condition for invariance. The following sections are devoted to obtain the explicit nature of the infinitesimal transformation admissible. Finally, we obtain the general nature of the external electro-magnetic field for the invariance of the equation and the inter-relation between them. Last section is devoted to the discussion on some of the fields which admit of the invariance of the equation.

## 2. The extended group of Lie

### 2.1 Infinitesimal transformation

Let us consider the infinitesimal transformation of  $x$  and  $\psi$

$$\bar{x}^k = x^k + \varepsilon \xi^k(x, \psi) \quad (5a)$$

$$\bar{\psi}^\mu = \psi^\mu + \varepsilon \Phi^\mu(x, \psi) \quad (5b)$$

$\varepsilon$  is an infinitesimal quantity. The first prolongation induced by the transformation is

$$\delta \left( \frac{\partial \psi^\mu}{\partial x^k} \right) = \overline{\left( \frac{\partial \psi^\mu}{\partial x^k} \right)} - \frac{\partial \psi^\mu}{\partial x^k} \quad (6)$$

$$\delta \left( \frac{\partial \psi^\mu}{\partial x^k} \right) = \varepsilon \left[ \frac{\partial \Phi^\mu}{\partial x^k} + \frac{\partial \Phi^\mu}{\partial \psi^\nu} \frac{\partial \psi^\nu}{\partial x^k} - \frac{\partial \psi^\mu}{\partial x^j} \left( \frac{\partial \xi^j}{\partial x^k} + \frac{\partial \xi^j}{\partial \psi^\nu} \frac{\partial \psi^\nu}{\partial x^k} \right) \right]. \quad (7)$$

The transformations

$$\left( x, \psi, \frac{\partial \psi}{\partial x} \right) \rightarrow \left( \bar{x}, \bar{\psi}, \frac{\partial \bar{\psi}}{\partial \bar{x}} \right)$$

form a group (Lie's extended group). Since the Dirac equation consists of only first-order derivatives, it is sufficient for us to consider the first prolongation.

## 2.2 The condition for the invariance of the equation

The variations of eq. (3) with respect to these transformation are given by

$$\frac{\partial D}{\partial x^k} = i \gamma_{\mu\nu}^j \frac{\partial A_j}{\partial x^k} \psi^\nu \quad (8)$$

$$\frac{\partial D}{\partial \psi^\nu} = i \gamma_{\mu\nu}^j A_j + \delta_{\mu\nu} M \quad (9)$$

$$\frac{\partial D}{\partial (\partial \psi^\nu / \partial x^k)} = \gamma_{\mu\nu}^k. \quad (10)$$

It should be emphasized that in obtaining these relations,  $x^k$ ,  $\psi^\mu$  and  $\partial \psi^\mu / \partial x^k$  are to be considered as independent variables. In order that the Dirac equation is to be invariant with respect to these transformations, we have

$$\xi^k \frac{\partial D}{\partial x^k} + \Phi^\mu \frac{\partial D}{\partial \psi^\mu} + \delta \left( \frac{\partial \psi^\mu}{\partial x^k} \right) \frac{\partial D}{\partial (\partial \psi^\mu / \partial x^k)} = 0. \quad (11)$$

This should be satisfied identically and independently for  $\psi$  and  $\partial \psi / \partial x$ . But they are related by the original eq. (3). To take care of this, the usual way is to supplement eq. (11), by introducing the indeterminate multiplier matrix, depending only on  $x$ ,  $P_\mu^\theta$  and multiplying eq. (3), [4,5].

$$P_\mu^\theta \left\{ \gamma_{\theta\nu}^k \left( \frac{\partial}{\partial x^k} + i A_k \right) + \delta_{\theta\nu} M \right\} \psi^\nu = 0. \quad (12)$$

Equation (11) written in full from eqs (8)–(10),

$$\begin{aligned}
 & i\gamma_{\mu\nu}^k \xi^j \frac{\partial A_k}{\partial x^j} \psi^\nu \\
 & + \Phi^\mu (i\gamma_{\mu\nu}^j A_j + M\delta_{\mu\nu}) \\
 & + \gamma_{\mu\nu}^k \left\{ \frac{\partial \Phi^\nu}{\partial x^k} + \frac{\partial \Phi^\nu}{\partial \psi^\theta} \frac{\partial \psi^\theta}{\partial x^k} - \frac{\partial \psi^\nu}{\partial x^j} \left( \frac{\partial \xi^j}{\partial x^k} + \frac{\partial \xi^j}{\partial \psi^\theta} \frac{\partial \psi^\theta}{\partial x^k} \right) \right\} = 0. \quad (13)
 \end{aligned}$$

The sum of eqs (12) and (13) should be satisfied independently of  $\psi$  and  $\partial\psi/\partial x$ . But as they stand one cannot do so before stipulating the  $\psi$  dependence of  $\xi$  and  $\Phi$ . On examining the sum one notices that there is only one quadratic term in  $\partial\psi/\partial x$ . Hence, either

$$\frac{\partial \xi^j}{\partial \psi^\theta} \left( \text{coefficient of } \frac{\partial \psi^\nu}{\partial x^j} \frac{\partial \psi^\theta}{\partial x^k} \right) = 0 \quad (14)$$

or introduce higher order indeterminate multiplier. It is shown in Appendix that this does not change the assertion of the above equation. Thus  $\xi^k(x)$  are independent of  $\psi$  and depend only on  $x^j$ .

Again quadratic or higher order terms of  $\psi$  in  $\Phi^\mu$  will give rise to terms like  $(\partial\psi^\lambda/\partial x^j)\psi^\mu$ ,  $(\partial\psi^\lambda/\partial x^j)\psi^\mu\psi^\nu$ , from  $\gamma_{\mu\nu}^k(\partial\Phi^\nu/\partial\psi^\theta)(\partial\psi^\theta/\partial x^k)$  in eq. (13). Hence,  $\Phi^\mu$  may at least be linear in  $\psi^\nu$ . Thus

$$\Phi^\mu(x, \psi) = B_\nu^\mu(x) \psi^\nu + \bar{\Phi}^\mu(x) \quad (15)$$

$\bar{\Phi}^\mu(x)$  is independent of  $\psi^\nu$  and depends only on  $x$ .

Finally, equating the coefficients of (i) terms independent of  $\psi$  and  $\partial\psi/\partial x$  and coefficients of (ii)  $\psi$  and (iii)  $\partial\psi/\partial x$  to zero, one obtains desired equations for determining  $\Phi, B, P$  and  $\xi$ ,

$$\left\{ \gamma_{\mu\theta}^k \left( \frac{\partial}{\partial x^k} + iA_k \right) + M\delta_{\mu\theta} \right\} \bar{\Phi}^\theta = 0 \quad (16)$$

from the above equation, which is the same as eq. (3). It follows that  $\bar{\Phi}^\theta$  should be a linear combination of  $\psi^\theta$  with constant coefficients. But our assertion in eq. (15) is that  $\bar{\Phi}^\theta$  is independent of  $\psi^\mu$ . Hence

$$\bar{\Phi}^\theta(x) = 0 \quad \text{and} \quad \Phi^\mu = B_\nu^\mu(x) \psi^\nu. \quad (17)$$

Next, equating the coefficient of  $\partial\psi^\theta/\partial x^k$ , we get

$$\gamma_{\mu\nu}^k B_\theta^\nu + P_\mu^\nu \gamma_{\nu\theta}^k = \gamma_{\mu\theta}^j \frac{\partial}{\partial x^j} \xi^k. \quad (18)$$

Finally, equating the coefficient of  $\psi^\theta$

$$\begin{aligned}
 & \left\{ \gamma_{\mu\nu}^k \left( \frac{\partial}{\partial x^k} + iA_k \right) + \delta_{\mu\nu} M \right\} B_\theta^\nu + i\gamma_{\mu\theta}^k \xi^j \frac{\partial A_k}{\partial x^j} \\
 & + P_\mu^\nu (i\gamma_{\nu\theta}^k A_k + \delta_{\nu\theta} M) = 0. \quad (19)
 \end{aligned}$$

These two equations can be written concisely in terms of matrices  $B, P$  and vector  $\xi$ .

$$\left\{ \gamma \cdot \left( \frac{\partial}{\partial \underline{x}} + i\mathbb{A} \right) + M \right\} B + P(i\gamma \cdot \mathbb{A} + M) + i \left( \xi \cdot \frac{\partial}{\partial \underline{x}} \right) (\gamma \cdot \mathbb{A}) = 0 \quad (20)$$

and

$$\gamma^k B + P\gamma^k = \left( \gamma \cdot \frac{\partial}{\partial \underline{x}} \right) \xi^k. \quad (21a)$$

These are the two basic equations for determining  $B, P$  and  $\xi$ . It is important to note that the last equation do not depend of  $\mathbb{A}$ , the potential of the external electro-magnetic field. This pair of eqs (20) and (21) can be further simplified. From eq. (21)

$$(\gamma \cdot \mathbb{A})B + P(\gamma \cdot \mathbb{A}) = A_j \left( \gamma \cdot \frac{\partial}{\partial \underline{x}} \right) \xi^j. \quad (21b)$$

Subtracting this from eq. (20), we get

$$\left( \gamma \cdot \frac{\partial}{\partial \underline{x}} \right) B + M(B + P) + iA_j \left( \gamma \cdot \frac{\partial}{\partial \underline{x}} \right) \xi^j + i \left( \xi \cdot \frac{\partial}{\partial \underline{x}} \right) (\gamma \cdot \mathbb{A}) = 0. \quad (22)$$

### 3. Determination of $\xi, B$ and $P$

Eliminating  $P$ , from eq. (21), one gets

$$\gamma^k B \gamma^k - \gamma^j B \gamma^j = \left( \gamma \cdot \frac{\partial}{\partial \underline{x}} \right) (\xi^k \gamma^k - \xi^j \gamma^j). \quad (23)$$

(No sum and  $k \neq j$ ). Since, on both sides of eq. (23) there are even factors of  $\gamma$  matrices, the matrix  $B$  is necessarily of the form

$$B = B_0 + B_{pq} \gamma^p \gamma^q, \quad (B_{pq} + B_{qp} = 0). \quad (24)$$

$B_0$  is a scalar. Since there are no terms on the left-hand side of eq. (23) with scalar ( $B_0$ ) and  $\gamma^k \gamma^j$ , one obtains

$$\frac{\partial \xi^k}{\partial x^k} = \frac{\partial \xi^j}{\partial x^j} \equiv \Lambda \text{ (say)} \quad (25)$$

(no sum) and

$$\frac{\partial \xi^k}{\partial x^j} + \frac{\partial \xi^j}{\partial x^k} = 0 (k \neq j) \quad (26)$$

$k, j$  are running indices and finally

$$B_{kq} = + \frac{1}{2} \frac{\partial \xi^k}{\partial x^q} (q \neq k). \quad (27)$$

Proceeding similarly after eliminating  $B$  from eq. (21),

$$P_{kq} = -\frac{1}{2} \frac{\partial \xi^k}{\partial x^q}. \quad (28)$$

Further from eqs (21), (27) and (28)

$$B_0 + P_0 = B + P = \frac{\partial \xi^k}{\partial x^k} = \Lambda. \quad (29)$$

One may point out that eq. (21) is independent of the external field and so eqs (23)–(29) are the same as those for free particle.

Operating  $(\partial/\partial x^i)(\partial/\partial x^k)$  on eq. (26), one gets

$$\frac{\partial^2 \Lambda}{\partial x^{i^2}} + \frac{\partial^2 \Lambda}{\partial x^{k^2}} = 0 \text{ (no sum)}. \quad (30)$$

Writing equations with  $k, q$  and  $j, q$  (all distinct) one obtains

$$\frac{\partial^2 \Lambda}{\partial x^{k^2}} = 0 = \frac{\partial^2 \Lambda}{\partial x^{j^2}}. \quad (31)$$

Thus

$$\Lambda = a_p x^p + b \quad (32)$$

$a_p, b$  are constant. Again operating on eq. (26),  $(\partial/\partial x^q)$  ( $q \neq k, j$ ), one obtains, ( $k, j, q$  distinct)

$$\frac{\partial^2 \xi^k}{\partial x^j \partial x^q} = 0 \text{ } (k \neq j, q, j \neq q). \quad (33)$$

From eqs (30)–(33)

$$\xi^k = (a_j x^j) x^k - \frac{a_k R^2}{2} + \Gamma_j^k x^j + b x^k + \tau^k \quad (34a)$$

$$R^2 = x^q \cdot x^q \quad \text{and} \quad \Gamma_j^k + \Gamma_k^j = 0. \quad (34b)$$

Since this is a well-known result, we avoid detail working on this.

The parameters  $a_j$  generate conformal transformations,  $\Gamma_j^k$  generates space rotation and Lorentz transformation,  $b$  dilation and  $\tau^k$  translation.

#### 4. The constraints due to external field

The second basic equation (eqs (22) and (29))

$$\left( \gamma \cdot \frac{\partial}{\partial x} \right) B(x) + M(B + P) + i A_j \left( \gamma \cdot \frac{\partial}{\partial x} \right) \xi^j + i \left( \xi \cdot \frac{\partial}{\partial x} \right) (\gamma \cdot A) = 0. \quad (35)$$

All the terms, excluding that with  $M$ , are linear in  $\gamma$  matrices. Hence

$$B + P = B_0 + P_0 = 0. \quad (36)$$

This implies all the parameters  $a_j$ 's and  $b$  are zero, so that matrix part of  $B$  is constant. Hence the only remaining terms are  $\xi^k$  and  $\Gamma_j^k$  (space rotation and Lorentz transformation) and  $\tau^k$  (space time translation). It needs to be emphasized that if  $m = 0$  and  $A_k = 0$  these constraints are no longer there. So massless Dirac equation may still admit conformal transformation and dilation.

Let  $\Gamma_j^k = Q \neq 0$  and all the other parameters are zero. Equation (35) is then

$$\left( \gamma^p \frac{\partial}{\partial x^p} \right) B_0 + iQ \left\{ (\gamma^j A_k - \gamma^k A_j) + \left( x^j \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^j} \right) (\gamma \cdot A) \right\} = 0. \quad (37)$$

Equating coefficients of  $\gamma^p$ 's

$$\frac{\partial B_0}{\partial x^k} + iQ \left\{ \left( x^j \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^j} \right) A_k - A_j \right\} = 0 \quad (38a)$$

$$\frac{\partial B_0}{\partial x^j} + iQ \left\{ \left( x^j \frac{\partial}{\partial x^k} - x^k \frac{\partial}{\partial x^j} \right) A_j + A_k \right\} = 0 \quad (38b)$$

$$\frac{\partial B_0}{\partial x^p} + iQ \left( x^j \frac{\partial}{\partial x^p} - x^k \frac{\partial}{\partial x^i} \right) A_p (p \neq k, j) = 0. \quad (38c)$$

(i)  $x^k, x^j$  are both space coordinates, say,  $(x, y)$ . The compatibility condition for the existence of  $B_0$  leads to (with  $\tan \phi = y/x$ )

$$\begin{aligned} \frac{\partial H_z}{\partial \phi} &= 0, & \frac{\partial E_z}{\partial \phi} &= 0 \\ \frac{\partial H_x}{\partial \phi} + H_y &= 0, & \frac{\partial H_y}{\partial \phi} - H_x &= 0 \\ \frac{\partial E_x}{\partial \phi} + E_y &= 0, & -\frac{\partial E_y}{\partial \phi} + E_x &= 0. \end{aligned} \quad (39)$$

Similarly for the other space coordinates.

(ii) One of the  $x^k, x^j$  is time coordinate say,  $(x, x^0)$ . The compatibility conditions now take the form (with  $\tan h\theta = x/ct$ )

$$\begin{aligned} \frac{\partial E_x}{\partial \theta} &= 0, & \frac{\partial H_x}{\partial \theta} &= 0 \\ \frac{\partial H_z}{\partial \theta} &= E_y, & \frac{\partial H_y}{\partial \theta} &= -E_z \\ -\frac{\partial E_z}{\partial \theta} &= H_y, & \frac{\partial E_y}{\partial \theta} &= H_x. \end{aligned} \quad (40)$$

Similar relation may be obtained with  $(y, x^0)$  and  $(z, x^0)$ .

Let  $\tau^k = \tau \neq 0$  and all other parameters are zero. Equation (35) is now

$$\left( \gamma \cdot \frac{\partial}{\partial x} \right) B_0 + i \left( \tau \cdot \frac{\partial}{\partial x^k} \right) (\gamma \cdot A) = 0. \quad (41)$$

Thus

$$\frac{\partial B_0}{\partial x^j} + i \tau \frac{\partial}{\partial x^k} A_j = 0. \quad (42)$$

These equations lead to

$$\frac{\partial}{\partial x^k} E = 0, \quad \frac{\partial}{\partial x^k} H = 0 \quad (43)$$

(irrespective of whether  $B_0$  is a constant or not), i.e., the field (note  $B_{pq} = 0$ ) is invariant with respect to the translation of space and time.

Among the usually encountered field in physical problems, the constant  $E$  and  $H$ , axial symmetric  $E$ ,  $H$ , radial electric (not magnetic) field, unidirectional radiation field etc. all satisfy one or the other constraints.

## 5. Discussion

The maximal symmetry group, namely the Poincaré group of the Dirac equation in the absence of the external field is constrained by the presence of the external field. Thus, for the variance, ten parameters (six  $\Gamma^k$  and four  $\tau^k$ ) impose respective restriction on the field.

It is instructive to note that though only the four-potential appears explicitly in the equation, but for the existence of the invariance the constraints are directly on the electric and magnetic field intensities, which are of physical significance.

Lastly, the massless field free Dirac equation may admit of transformations similar to the conformal transformation with  $R^2 = c^2 t^2 - r^2$ .

## Appendix

In order to balance the terms  $(\partial \psi^\nu / \partial x^k)(\partial \psi^\mu / \partial x^j)$ , let us try to introduce higher order indeterminate multiplier with  $Q_\mu^{\alpha\beta}(x)$  as multiplier,

$$Q_\mu^{\alpha\beta} \left\{ \gamma_{\alpha\theta}^k \left( \frac{\partial}{\partial x^k} + i A_k \right) + \delta_{\alpha\theta} M \right\} \psi^\theta \left\{ \gamma_{\beta\lambda}^j \left( \frac{\partial}{\partial x^j} + i A_j \right) + \delta_{\beta\lambda} M \right\} \psi^\lambda = 0. \quad (44)$$

Since there is no term like  $(\partial \psi^\theta / \partial x^k)(\partial \psi^\lambda / \partial x^j) \psi^\nu$  in the above equation, from eq. (13), it follows that

$$\frac{\partial \xi^j}{\partial \psi^\theta} = K_\theta^j(x). \quad (45)$$



But from eq. (44)

$$Q_{\mu}^{\alpha\beta} \gamma_{\alpha\lambda}^k \gamma_{\beta\nu}^j = \gamma_{\mu\nu}^k K_{\lambda}^j(x). \quad (46)$$

Again for fixed  $k, j, \lambda, \nu$ , one can write

$$S_{\mu}(k, j, \lambda, \nu) = Q_{\mu}^{\alpha\beta} \gamma_{\alpha\lambda}^k \gamma_{\beta\nu}^j. \quad (47)$$

$S_{\mu}$  being a four-vector, it is always possible to find  $e^{\mu}$ , such that,  $e^{\mu} S_{\mu} = 0$  leading to

$$e^{\mu} \gamma_{\mu\nu}^k K_{\lambda}^j = 0. \quad (48)$$

Since  $\gamma$ 's do not have any null vector

$$e^{\mu} \gamma_{\mu\nu}^k \neq 0 \quad \text{implying} \quad K_{\lambda}^j = 0. \quad (49)$$

Hence,  $\xi^k(x)$  is a function  $x$ 's only.

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