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Stochastic resonance and chaotic resonance in bimodal maps: A case study

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Abstract. We present the results of an extensive numerical study on the phenomenon of stochastic resonance in a bimodal cubic map. Both Gaussian random noise as well as deterministic chaos are used as input to drive the system between the basins. Our main result is that when two identical systems capable of stochastic resonance are coupled, the SNR of either system is enhanced at an optimum coupling strength. Our results may be relevant for the study of stochastic resonance in biological systems.

Keywords. Stochastic resonance; chaotic resonance; bimodal maps; coupled maps.

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Stochastic resonance (SR) has been the focus of intense research over the past decade because of its potential applications in diverse areas [1–7]. It refers to the situation where an increase in input noise improves a system's sensitivity to discriminate weak signals [8–10]. The recent interest in SR is mainly due to the fact that it plays a crucial role in many biological systems in extracting a weak periodic signal embedded in a large amount of background noise [5,7,11]. It is also bound to have a large influence on the future development of nonlinear devices and for use in communications and information transmission processes [8,12].

Recently, the concept of chaotic resonance (CR) has also been introduced where the switching is induced using deterministic chaos rather than Gaussian random noise [13]. This is especially important because chaos is widespread in natural systems and it is often difficult to distinguish between noise and chaos in experimental situations. But most of the studies on SR and CR have been done using the standard bistable potential with a fixed point attractor, even though a few other models have also been used [14–16]. Another interesting problem which remains to be studied in detail is the effect of coupling on SR whether the signal boosting ability can be further improved by a suitable coupling between individual bistable systems.

Here we undertake a detailed numerical study of the phenomenon of SR using random noise as well as deterministic chaos. The two parameter cubic map [17] has been used for this purpose which shows bistability in periodic as well as chaotic attractors. Our results indicate that while SR is present with both types of attractors, the phenomenon is more effective in the environment of random noise compared to that of deterministic chaos. An even more interesting result is that when two identical systems capable of SR individually are coupled, the SNR of both systems increases and reaches a maximum value at a finite coupling strength. This result is in striking similarity with the result obtained by Jung *et al* [18] in the case of globally coupled bistable systems.

We now introduce the two parameter cubic map of the form

$$X_{n+1} = f(X_n) = b + aX_n - (X_n)^3. \tag{1}$$

It has been shown to possess a rich variety of dynamical properties including bistability [17]. In particular, if a_1 is the value of the parameter at which $f'(X_i, a_1, b) = 1$, then for $a > a_1$ there is a window in b, where bistability is observed. The bistable attractors are clearly separated with $X > 0$ being the basin of one and $X < 0$ that of the other. For example, for $a = 1.4$, all initial conditions $X_0 \in [-1.5, 1.5]$ tend to a single asymptotic attractor (of period $1,2,4\ldots$) which remains in $[0, 1.5]$ for $b > 0.1$ and in $[-1.5, 0]$ for $b < -0.1$. But for $b \in [-0.1, 0.1]$, both attractors are stable so that initial values $X_0 \in [0, 1.5]$ tend to X^* in the positive half of X and $X_0 \in [-1.5, 0]$ to $-X^*$ in the negative half of X. In other words, the system shows bistability for $a = 1.4, b = [-0.1, 0.1] \equiv [-b_1, b_1]$ with two attractors of period 1 co-existing.

As we increase the value of a, it can be shown that for $a = 2.1$, two attractors of period 2 are co-existing in a narrow window around $b = 0$, namely, $b = [-b_2, b_2] \subset [-b_1, b_1]$. The basin of the positive attractor is still $X_0 > 0$ and that of the negative attractor $X_0 < 0$. As a is increased further, for $a = 2.25$, two attractors of period 4 co-exist in a further narrow window $[-b_4, b_4] \subset [-b_2, b_2]$ and so on. Finally, for $a = 2.4$, two chaotic attractors co-exist in a very narrow window $b = [-b_\infty, b_\infty]$.

Incidently, it should be mentioned that our system can be considered to be a more general version of the one parameter cubic map

$$X_{n+1} \equiv f(X_n) = (a-1)X_n - aX_n^3 \tag{2}$$

in which SR studies have already been reported [13,19].

With the combined effort of the noise and the signal added externally to the system the iterates, originally confined to one basin, shuttles between the two basins in a systematic manner in accordance with the signal, leading to SR. To study the SR with period one attractors of the system, the parameters chosen are $a = 1.4$ and $b = 0.01$. A random Gaussian noise of variance 0.2 and zero mean and a small periodic signal are added to the system so that it becomes

$$f(X) = E\xi(t) + Z\sin(2\pi pt) + b + aX - X^3 \tag{3}$$

where $\xi(t)$ represents the Gaussian noise and $\sin(2\pi pt)$ a small periodic signal with $p = 1/T = 1/8$ as the frequency. E and Z represent the strength of the noise and the signal respectively which can be tuned to give the desired amplitudes for the noise and the signal. It is well known that for SR to occur, the system has to remain in an unstable equilibrium [9]. Physically, it means that the state of the system should be near the basin boundary. If it is deep inside the well, even the combined amplitude of noise and signal will not be able to effect switching, destroying SR in the system. In order to achieve this, we have restricted the X values in the range $[-1.5, 1.5]$, by a proper scaling.

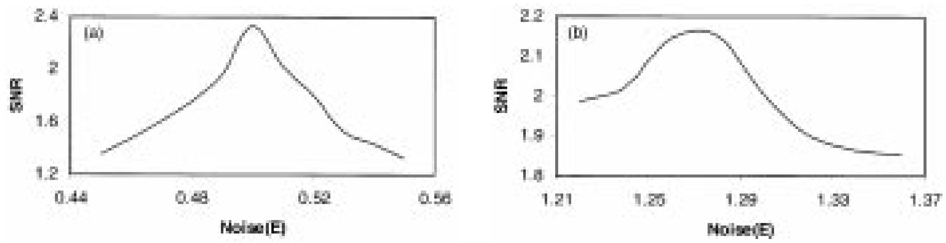

Figure 1. (**a**) Variation of SNR with noise amplitude E for periodic bistable attractors with $a = 1.4$, indicating SR. (**b**) Variation of SNR with noise for chaotic bistable attractors with $a = 2.4$.

We use the most important quantifier to study SR in the system, namely, the signal to noise ratio (SNR) [8,9]. To obtain the SNR, the amplitude of the noise E is varied from 0.45 to 0.55 taking a small signal amplitude, $Z = 0.16$. For these noise inputs the power spectrum of the time series is calculated using the FFT. Then SNR is given by the relation: $\mathrm{SNR} = \log_{10}(S/N)$ where N is the average background noise around the signal S. The noise level N is measured by averaging the values for five bins to the left of the signal and five to the right. The error in the measurement of peak height and noise level is ± 0.0001. Our results are shown in figure 1a. The bell-shaped curve clearly shows SR having a peak value for noise strength $E = 0.5$. If the value of Z is changed from 0.16, there is a corresponding change in the value of E where the peak is obtained.

The above calculations are repeated taking $a = 2.4$ and $b = 0.01$ where the system possess two chaotic bistable attractors. To get the SNR in this case, the signal amplitude is chosen as $Z = 0.8$ and the peak is obtained for $E = 1.28$. This is shown in figure 1b. The SR studies on chaotic attractors are very rare [14]. Here we find that even though the attractor is inherently chaotic in one basin, its inter-well hopping is regular at a finite noise amplitude. If the intrawell chaotic motion is somehow suppressed, and the output is measured only when the system makes an inter-well transition, we get almost periodic signals from strange attractors with the help of finite amount of noise.

Another interesting phenomenon is chaotic resonance, where instead of random noise, a chaotic time series is used along with the periodic signal to drive the system between the wells. For this, we use a chaotic time series generated from the logistic map

$$f(X) = 1 - 2X^2 \tag{4}$$

using an initial random seed and it is used in place of the Gaussian noise to calculate the SNR for the above parameter values a and b. The results, both for periodic as well as chaotic attractors are shown in figure 2. One qualitative difference between the SNR curves for SR and CR is that the SNR profile is in general sharper for SR whereas the data for CR is much more scattered. It probably implies that even though SR occurs both with random noise and deterministic chaos, the phenomenon is more pronounced in a noisy environment, that is, a random noise is more effective in boosting a small periodic signal than deterministic chaos. This result is especially important in the study of biological systems as they are known to extract signals from a natural environment which may be inherently noisy or chaotic. The question whether a neuron can distinguish between noise and chaos has become an important scientific debate of late [20].

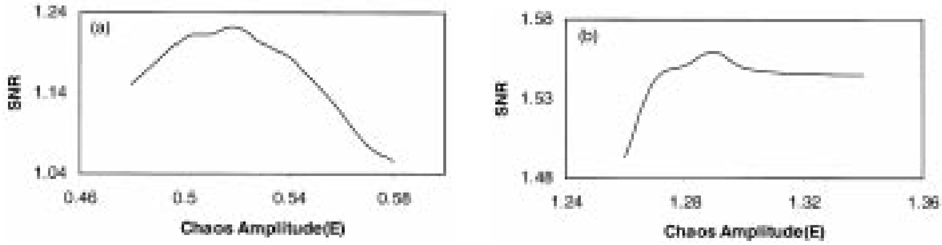

Figure 2. (**a**) Variation of SNR with chaos amplitude E for periodic bistable attractors. (**b**) Variation of SNR with chaos amplitude E for chaotic bistable attractors.

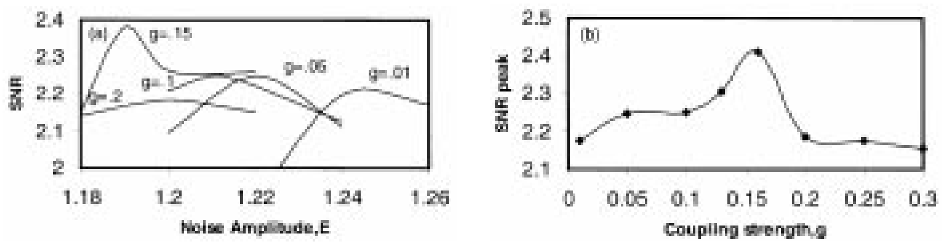

Figure 3. (**a**) SNR variation with amplitude of random noise E for various coupling strengths for the coupled system. Only a few important values are shown for the sake of clarity. (**b**) Variation of SNR peak with coupling strength g as seen in figure 3a. Note the sharp increase in SNR for g around 0.15.

The response of identical systems working in a similar environment can be greatly enhanced if there is an effective coupling between the individual elements so that there is a collective response in place of individual ones [21]. This is particularly important in a phenomenon like SR because it can lead to the amplification of very small signals immersed in noise. Here we consider two identical systems represented by the cubic map and capable of SR separately, but their outputs are coupled:

$$X_{n+1} = b + aX_n - X_n^3 + g(Y_n - X_n) + E\xi(t) + Z\sin(2\pi pt) \tag{5}$$

$$Y_{n+1} = b + aY_n - Y_n^3 + g(X_n - Y_n) + E\xi(t) + Z\sin(2\pi pt). \tag{6}$$

A difference coupling is employed with g representing the coupling strength. Our aim is to study how the presence of an identical system changes the performance or SNR of the original sysem, through coupling. We consider two identical systems evolving independantly starting from different initial conditions, which is the most practical situation to be expected. Note that for the coupling to exist, there should be a finite value for the coupling term. In other words, if X and Y are exactly equal, there is no effect for coupling.

Taking $a = 2.4$ and $b = 0.01$ in the regime of the chaotic bistable attractors the SNR of the system is calculated with $\xi(t)$ as the Gaussian noise for various coupling strengths g. The results are shown in figure 3a. For $g = 0$, they represent two isolated systems without any coupling. As g is increased from zero, the peak value of SNR increases slowly initially, but goes through a sudden maximum around $g = 0.15$, before decreasing once again for large values of g. Figure 3b makes it more clear where SNR peak values are plotted against g. It is evident that at a finite coupling strength, the signal is boosted to its maximum through a collective response by the system. At the same time for too large coupling strength, the SNR peak value decreases below that for $g = 0$ and further increase in g (over coupling) destroys SR in the system.

A simple explanation that can be offered for the above observations is that when one system misses the interstate switching, the nearby system may not. The coupling then forces the other system also to switch. This cooperative behavior induces enhanced regularity in the switching and increase the SNR of either systems. In other words, the coupling between two identical systems (having the same Kramer's rate), brings about a collective SR. This is further confirmed by the fact that if the second system has no noise or signal (that is, not capable of SR independantly), the SNR reduces with coupling.

The existence of a critical coupling strength where the SNR is boosted to a maximum can be understood intuitively in the following way. At any stage, if the two systems differ only by a very small amount, either system switches independantly without the help of the other, making the effect of coupling nominal. Coupling becomes most effective, if the states of the two systems differ by an optimum amount. From the view point of SR one can say that the system near the basin boundary can help the other system to switch over, thus enhancing the SNR through coupling. When the difference between X_n and Y_n becomes too large, the coupling once again becomes ineffective. That this is indeed happening as g is increased can be seen from figure 4, where the values of X_n and Y_n are plotted for various values of g. As g is increased, they become more and more scattered, indicating that their difference increases with g. Thus the forced switching is most effective at an optimum value of g.

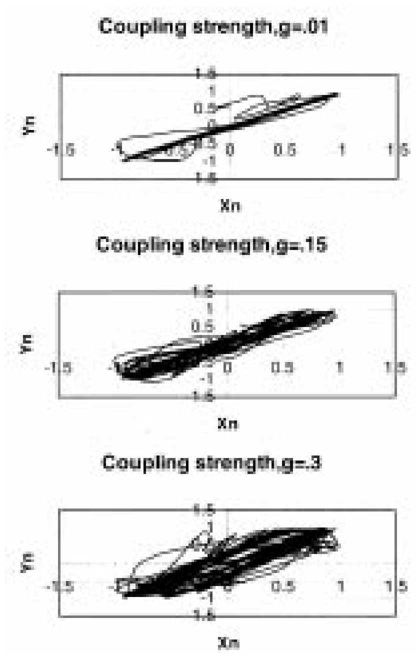

Figure 4. Plot of X_n vs. Y_n for various values of g for the coupled system. Note that $(X_n - Y_n)$ increases on the average as g is increased.

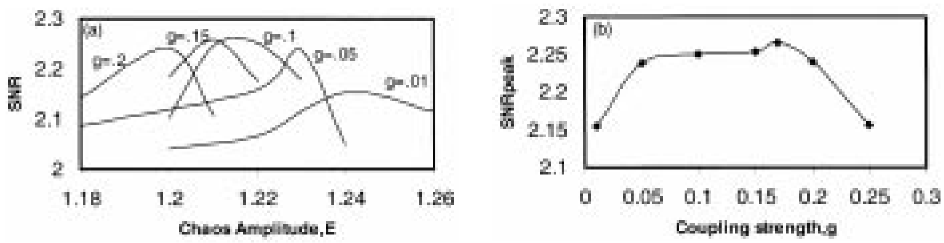

Figure 5. (**a**) SNR variation for the coupled system when it is driven by deterministic chaos for various coupling strengths. (**b**) Variation of SNR peak with g as seen in figure 5a. The peak is less pronounced compared to the case of random noise.

This result is significant in that the amplification of weak signals brought about by SR with the help of noise can be further enhanced by a suitable coupling. A somewhat similar result has been reported in the case of globally coupled bistable systems [18]. Here, there is a collective effect near a spontaneous ordering transition at a critical noise intensity. As a result, there is a collective or coherent SR which can be utilized, for example, for the detection of ultra small signals in biological systems using a network of globally coupled SQUIDS.

The above calculations are repeated with $\xi(t)$ as a chaotic time series from the logistic map instead of the Gaussian noise and the results are presented in figure 5. Even though the SNR peak still goes through a maximum around $g = 0.15$, the increase is smaller compared to noise. It once again confirms that the random noise still has an upper hand in boosting a signal, compared to deterministic chaos.

There are many situations where coupling has been used effectively to enhance the output as in the study of biological systems and in the development of certain nonlinear devices. Recently, SR has been reported in a coupled excitory–inhibitory neural pair [22] for the detection of a signal immersed in noise. We hope our numerical observations can initiate further studies along these directions.

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