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Quantum space-times in the year 2002

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Abstract. We review certain emergent notions on the nature of space-time from noncommutative geometry and their radical implications. These ideas of space-time are suggested from developments in fuzzy physics, string theory, and deformation quantization. The review focuses on the ideas coming from fuzzy physics. We find models of quantum space-time like fuzzy S^4 on which states cannot be localized, but which fluctuate into other manifolds like CP^3. New uncertainty principles concerning such lack of localizability on quantum space-times are formulated. Such investigations show the possibility of formulating and answering questions like the probability of finding a point of a quantum manifold in a state localized on another one. Additional striking possibilities indicated by these developments is the (generic) failure of CPT theorem and the conventional spin-statistics connection. They even suggest that Planck's 'constant' may not be a constant, but an operator which does not commute with all observables. All these novel possibilities arise within the rules of conventional quantum physics, and with no serious input from gravity physics.

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1. Space-time in quantum physics

The point of departure from classical to quantum physics is the algebra $\mathscr{F}(\mathscr{T}^*\mathscr{Q})$ of functions on the classical phase space T^*Q. According to Dirac, quantization can be achieved by replacing a function f in this algebra by an operator $\hat{f}$ and equating $i\hbar$ times the Poisson bracket between functions to the commutator between the corresponding operators.

In classical physics, the functions f commute, so $\mathscr{F}(\mathscr{T}^*\mathscr{Q})$ is a commutative algebra. But the corresponding quantum algebra $\hat{\mathscr{F}}$ is not commutative. Dynamics is on $\hat{\mathscr{F}}$. So quantum physics is a *noncommutative dynamics*.

A particular aspect of this dynamics is *fuzzy phase space* where we cannot localize points, and which has an attendent effective ultraviolet cut-off: The number of states in a phase space volume V is infinite in classical physics and $V/\hbar^{2d}$ in quantum physics when the phase space is of dimension $2d$. The emergence of this cut-off from quantization is of particular importance for the program of fuzzy physics [1].

This brings us to the focus of our talk. In quantum physics, the commutative algebra of functions on phase space is deformed to a noncommutative algebra, leading to a 'noncommutative phase space'. Such deformations, characteristic of quantum theory, are now appearing in different approaches to fundamental physics. The talk will focus on a few such selected approaches and their implications.

Before proceeding further, let us mention a few of these lines of thought leading to noncommutative geometry: (1) Noncommutative geometry has made its appearance as a method for regularizing quantum field theory (qft) and in studies of deformation quantization. This talk will more or less base itself on these aspects. (2) It has turned up in string physics as quantized D-branes. (3) Certain approaches to canonical gravity [2] have also used noncommutative geometry with great effectiveness.

2. Fuzzy physics and quantum field theory

In what follows, we will focus on fuzzy physics both as a means to regularize quantum field theories and in their relation to quantum space-times. But as mentioned already, we will not talk about how they emerge from string physics.

3. Fuzzy space-time as regulator

The original ideas for using fuzzy spaces to regulate qft's are due to Madore [1]. They concern quantizing the underlying space-time itself, making it a *fuzzy space-time*.

We know since Planck and Bose that quantization introduces a short distance cut-off, changing the number of states in a phase space volume V from ∞ to $V/\hbar^{2d}$.

Now qft's on a manifold require regularization. The usual nonperturbative regularization involves lattice qft's. The use of fuzzy space-times can be another approach. The latter has important advantages like maintaining symmetries and avoiding fermion doubling. The particular approach reported here involves many colleagues. Our representative papers are [3,4] and the work of Vaidya, Dolan *et al* and Lopez *et al* in [5]. Related or overlapping work is due to [6] and [7].

We do Euclidean qft's. Quantizing S^4 would be of great physical interest. But for now, we will focus on the simpler case of S^2.

4. Fuzzy $\mathbb{C}^2$, S^3 and S^2

4.1 *Relation between* $\mathbb{C}^2$, S^3 *and* S^2

Consider $\mathbb{C}^2$ with coordinates $z=(z_1,z_2)$. We have

$$S^3 = \left\langle z : |z|^2 := \sum |z_i|^2 = 1 \right\rangle \subset \mathbb{C}^2 \tag{1}$$

and

$$S^2 = \langle \vec{x} = z^\dagger \vec{\tau} z, z \in S^3 \rangle, \tag{2}$$

$$\vec{\tau} = \text{Pauli matrices}, \tag{3}$$

$$\vec{x}.\vec{x} = 1. \tag{4}$$

S^2 is the quotient of S^3 by the U(1) action $z \to ze^{i\theta}$ (Hopf fibration). The group $\mathrm{SU}(2) = \{g\}$ acts on $\mathbb{C}^2$, S^3 and S^2:

$$z \to gz, x_i \to R_{ij}(g) x_j, R(g) = \text{rotation matrix for } g. \tag{5}$$

4.2 *The fuzzy* $\mathbb{C}^2$

Now we *quantize* $\mathbb{C}^2$ by the replacements

$$z_\alpha \to a_\alpha, \quad \overline{z_\alpha} \to a_\alpha^\dagger, \tag{6}$$

where a_α and $a_\alpha^\dagger$ are harmonic oscillator annihilation and creation operators with the usual commutation relations. SU(2) still acts with generators

$$L_\alpha = a^\dagger \sigma_\alpha a, \tag{7}$$

which commute with the number operator:

$$N = a^\dagger a, \quad [N, L_\alpha] = 0. \tag{8}$$

4.3 *The fuzzy three-sphere*

Consider

$$\begin{aligned} S_\alpha &= a_\alpha \frac{1}{\sqrt{N+1}}, \\ S_\alpha{}^\dagger &= \frac{1}{\sqrt{N+1}} a_\alpha{}^\dagger, \\ S_\alpha{}^\dagger S_\alpha &= \frac{N}{N+1} \to 1 \text{ as } N \to \infty, \\ \left[S_\alpha, S_\beta^\dagger\right] &\to 0 \text{ as } N \to \infty. \end{aligned} \tag{9}$$

So S_α are normalized commuting vectors as $N \to \infty$ and the fuzzy three-sphere S_{F}^3 is the algebra generated by S_α and $S_\beta{}^\dagger$.

It is important to note that $1/(N+1)$ here plays the role of a quantized Planck's constant.

This raises the following important questions: Is it possible that Planck's 'constant' is in reality an operator? How can one experimentally test this possibility?

The representation of the S_{F}^3-algebra on the Fock space is irreducible.

4.4 *The fuzzy two-sphere*

Since $[N, L_\alpha] = 0$, we have that

$$[N, S^\dagger \sigma_\alpha S] = 0. \tag{10}$$

So we can restrict the algebra of the fuzzy sphere S_{F}^2 generated by $S^\dagger\sigma_\alpha S$ to the finite-dimensional vector space spanned by the eigenvectors of N with eigenvalue n. They are spanned by

$$|n_1,n_2\rangle := \frac{1}{\sqrt{n_1!n_2!}}({a_1}^\dagger)^{n_1}({a_2}^\dagger)^{n_2}|0\rangle, \\ n_1+n_2 = n, \tag{11}$$

and carry angular momentum $J_0 = n/2$. As this representation is irreducible, it follows that the fuzzy sphere algebra for angular momentum J_0 is the algebra of $(2J_0+1)\times(2J_0+1)$ matrices $\mathrm{Mat}(2J_0+1)$. We denote its elements by M. It is just the vector space spanned by the tensor operators in the angular momentum $n/2$ representation.

Rotation acts on M by $M\to gMg^{-1}$. So the fuzzy sphere has angular momenta $0,1,\ldots,2J_0$, cut-off at $2J_0$. The algebra of functions on the two-sphere instead has all integral angular momenta up to ∞.

5. Scalar fields on S_{F}^2

Scalar fields are power series in 'coordinates' $S^\dagger\sigma_\alpha S$. So

$$\text{scalar field} = (2J_0+1)\text{-dimensional matrix.} \tag{12}$$

A scalar action, such as

$$\mathscr{A} = \frac{1}{2J_0+1}Tr([L_i,\phi]^\dagger[L_i,\phi]+\phi^k), \tag{13}$$

can be quantized by functional integral methods. Renormalization studies of such actions have been carried out in [5].

Gauge theories can be formulated on fuzzy spaces. For brevity, we will not enter into their discussion.

Summarizing, we have the classical descent chain

$$\mathbb{C}\to S^3\to S^2. \tag{14}$$

It becomes after quantization

$$\mathbb{C}_{\mathrm{F}}\to S_{\mathrm{F}}^3\to S_{\mathrm{F}}^2. \tag{15}$$

The algebra dimension is ∞ for all, except S_{F}^2 for which it is $(2J_0+1)^2$.

6. On coherent states and star products

In quantum field theory, we calculate correlation functions like

$$\langle\phi(\vec{n}_1)\phi(\vec{n}_2)\ldots\phi(\vec{n}_j)\rangle, \qquad \vec{n}_j\in S^2. \tag{16}$$

To compare with such expressions, we have to know how to map our operators (finite-dimensional matrices) to functions. This is where coherent states and star products prove important. For us, they will also be particularly important when issues of topology fluctuations are discussed. We now explain how star products can be defined on fuzzy spaces.

We start from the infinte-dimensional Fock space associated with two oscillators, and introduce the standard coherent states

$$|\, z;\infty\rangle = \mathrm{e}^{z_\alpha {a_\alpha}^\dagger - \bar{z}_\alpha a_\alpha}|0\rangle, \tag{17}$$

where ∞ has been inserted in the state vector to indicate that it is associated with the Fock space. (It is omitted from the vacuum state which will be common to both the Fock space and its subspaces which will appear below.)

A theorem asserts that the diagonal matrix elements

$$\langle z;\infty \,|\, \hat{A} \,|\, z;\infty\rangle = A(z) \tag{18}$$

determines the operator $\hat{A}$ (under suitable conditions on $\hat{A}$): the map $\hat{A} \to A$ of operators to functions is one-to-one. Both the Moyal and Perelomov star products follow from this result as we now indicate.

The star product $A * B$ of two functions A and B is defined by

$$A * B(z) := \langle z;\infty \,|\, \hat{A}\hat{B} \,|\, z;\infty\rangle. \tag{19}$$

(It is not the Moyal star product, but equivalent to it.)

If $\hat{A}$ is 0 outside the subspace where $N = n$, then

$$\begin{aligned} &\hat{A} = P\hat{A}P, \\ &P = \text{projector onto this subspace}. \end{aligned} \tag{20}$$

The explicit expression for P is

$$P = \sum_{n_1+n_2=n} |n_1,n_2\rangle\langle n_1,n_2\,|, \tag{21}$$

where we have used the definition (11).

The diagonal coherent state expectation value of $P\hat{A}P$ is (up to a constant in the definition of the state)

$$\begin{aligned} A(z) &= \langle z|\hat{A}|z\rangle, \\ |z\rangle &:= \frac{1}{\sqrt{n!}} \sum_{n_1+n_2=n} (z_\alpha {a_\alpha}^\dagger)^n|0\rangle. \end{aligned} \tag{22}$$

The group SU(2) with generators L_i acts on these states according to

$$\begin{aligned} &|z\rangle \to |gz\rangle, \\ &g \in \mathrm{SU}(2), \end{aligned} \tag{23}$$

preserving $\sum |z_i|^2$, so we can set it equal to 1. This normalizes these states.

The fuzzy sphere operators L_i and $(1/(N+1))L_i$ can be restricted to $N = n$. We can set them equal to 0 on the subspace $N \neq n$, and treat them as the above $\hat{A}$'s.

In this way, we have a star product on S^2_{F}. There are explicit expressions for this star product [4].

7. Fluctuating topologies: A novel space-time uncertainty principle

We now pass to the 4-sphere S^4 and consider the fuzzy S^4, which is denoted by ${S_F}^4$. (See in this connection [7].) We realize S^4 as a sphere in $\mathbb{R}^5$:

$$S^4 = \left\langle x := (x_1, x_2, .., x_5) : \sum |x_i|^2 = 1 \right\rangle. \tag{24}$$

For quantization, it is necessary to introduce $\mathbb{C}^4$ with coordinates $z = (z_1, z_2, ..., z_4)$. The unit sphere in $\mathbb{C}^4$ is S^7:

$$S^7 = \left\langle z : \sum |z_i|^2 = 1 \right\rangle. \tag{25}$$

We also introduce the five gamma matrices γ_λ:

$$\gamma_\lambda = \text{ standard } \gamma \text{ matrices.} \tag{26}$$

Then we can set

$$x_\lambda = z^\dagger \gamma_\lambda z, \quad z \in S^7. \tag{27}$$

Now instead of 2, we have 4 sets of creation-annihilation operators $a_\alpha, {a_\alpha}^\dagger$ $(\alpha = 1,2,3,4)$. S^4 will be thought of as a sphere in $\mathbb{R}^5$ and its quantized fuzzy version will hence be formulated.

Coherent states appropriate for S^4 are generalizations of those for S^2:

$$|z\rangle := \frac{1}{\sqrt{n!}} \sum_{n_1+n_2+n_3+n_4=n} (z_\alpha {a_\alpha}^\dagger)^n |0\rangle, \tag{28}$$

where the sum on α is over 4 values.

The quantized version of x_λ is $S^\dagger \sigma_\alpha S$ with S's defined as before. A simple calculation shows that

$$\langle z|S^\dagger \sigma_\alpha S|\rangle = \left(\frac{n}{n+1}\right) z^\dagger \gamma_\lambda z. \tag{29}$$

Or it gives S^4 as imbedded in $\mathbb{R}^5$.

But we now show that the fuzzy S^4_F emerges only approximately, having fluctuations of the order of $1/n$ into the six-dimensional fuzzy manifold $\mathbb{C}P^3_F$, the fuzzy version of $\mathbb{C}P^3$.

To demonstrate this, consider the correlation function $\langle z|S^\dagger \sigma_\alpha S S^\dagger \sigma_\beta |z\rangle$. A short calculation shows that it is

$$z^\dagger \gamma_\alpha z z^\dagger \gamma_\beta z + O\left(\frac{1}{n}\right) z^\dagger \gamma_{\alpha\beta} z, \tag{30}$$

where $\gamma_{\alpha\beta}$ is $[\Gamma_\alpha, \Gamma_\beta]$. Now $z^\dagger \gamma_{\alpha\beta} z$ is *not* a function on S^4. It is a function only on $\mathbb{C}P^3$. (It is invariant under the phase change of $z : z \to \mathrm{e}^{i\theta} z$.) By definition,

$$\langle z|S^\dagger \sigma_\alpha S S^\dagger \sigma_\beta |z\rangle = \left(\frac{n}{n+1}\right) z^\dagger \gamma_\alpha z * \left(\frac{n}{n+1}\right) z^\dagger \gamma_\beta z. \tag{31}$$

A measure of the lack of localization of the star product on S^4 is possible to construct, but we will not discuss that here.

The behaviour of $z^\dagger \gamma_\alpha z$ under star product is generic for most manifolds, notable exceptions being $\mathbb{C}P^N$.

Thus we have an algebra of observables, such as that generated by $z^\dagger \gamma_\alpha z$ under $*$, which is only *approximately* localized on a manifold M. It has fluctuations $O(n/(n+1))$ into another manifold $M' \subset M$ which disappear in the classical limit $1/n \to 0$.

8. More on quantum topologies

The fuzzy space S_{F}^2 has representations in all dimensions.

In contrast, the fuzzy space S_{F}^4, or rather $\mathbb{C}P_{\mathrm{F}}^3$, has representations in symmetric products of its four-dimensional representation, namely, in dimensions $4, 10, 20, \ldots$.

So 20×20 matrices can approximate *either S^2 or $\mathbb{C}P^3$*.

So what we see in these matrices depends on the operators we examine [8]. If we work with S^2-coherent states and operators appropriate for them, then they will approximate S^2. But if instead we work with states and operators appropriate for $\mathbb{C}P_{\mathrm{F}}^3$, then these matrices will approximate that manifold.

In high dimensions, such approximate manifolds proliferate.

We can answer questions like: the probability of finding a $\mathbb{C}P_{\mathrm{F}}^3$-localized state such as

$$\begin{aligned} &\rho = |z\rangle\langle z|, \\ &z \in \mathbb{C}P^3 \end{aligned} \tag{32}$$

($|z\rangle$ being the coherent state for $\mathbb{C}P_{\mathrm{F}}^3$) in a $\mathbb{C}P_{\mathrm{F}}^1$-localized state

$$\begin{aligned} &\tilde{\rho} = |\tilde{z}\rangle\langle \tilde{z}|, \\ &\tilde{z} \in \mathbb{C}P^1, \text{ that is } S^2, \end{aligned} \tag{33}$$

($|\tilde{z}\rangle$ being the coherent state for $\mathbb{C}P_{\mathrm{F}}^1$). The answer is

$$|\mathrm{Tr}\rho^\dagger \tilde{\rho}|^2. \tag{34}$$

We can see from this discussion that questions with fantasy about quantum space-times become accessible in fuzzy physics.

9. Causality and CPT violation

Fuzzy models can be formulated for space-times with Minkowsky metric, not just for Euclidean space-times like S^2. They are natural models to quantize time, preserving many symmetries.

A popular model for quantizing Minkowsky space-time which has been carefully studied by Doplicher, Fredenhagen and Roberts [9] and others [10,11] is governed by the commutation relation

$$[x^{\mu}, x^{\nu}] = i\theta^{\mu,\nu},$$
$$\theta^{\mu,\nu} = -\theta^{\nu,\mu} = \text{ a real constant.} \tag{35}$$

With space-time fuzzy, the meaning of 'space-like separation' loses exact meaning and leads to causality and hence generically to CPT violation [11].

A way to understand this is as follows: In $1+1$-dimensions, we can set

$$\theta^{\mu,\nu} = \theta\varepsilon^{\mu,\nu}, \tag{36}$$

where $\varepsilon^{\mu,\nu}$ is the Levi–Civita symbol with $\varepsilon^{0,1} = 1$. Then the fuzzy version of a free field of mass M has the representation [10]

$$\Phi = \int \psi(k) \mathrm{e}^{ik_0 x_0} \mathrm{e}^{ik_1 x_1}, \tag{37}$$

where

$$\begin{aligned} k_0^2 - k_1^2 &= M^2, \\ \psi(k) &= \delta(k^2 - M^2) a(k), k_0 > 0, \\ &= \delta(k^2 - M^2) a(-k_0, k_1)^{\dagger}, k_0 < 0, \end{aligned} \tag{38}$$

$a(k)$ and $a(k)^{\dagger}$ being the usual annihilation and creation operators.

We can regard $(x_0 + ix_1)/\sqrt{2\theta}$ and its adjoint as annihilation and creation operators. If $|z\rangle$ denotes the corresponding coherent state, then the square of the fuzzy field localized approximately at $(\mathrm{Re}\, z, \mathrm{Im}\, z)$ is

$$\langle z|(\Phi)^2|z\rangle := (\bar{\Phi})^2(z). \tag{39}$$

One checks that when z and z' correspond to space-like points and α and β are generic states,

$$\langle \alpha|[(\bar{\Phi})^2(z), (\bar{\Phi})^2(z')]|\beta\rangle \approx \mathrm{e}^{-|z-z'|^2}. \tag{40}$$

Such causality violations are *generic* in such models. A fundamental question, inspired by these models then is: *How can we test them?* [12]. We can in principle do so using forward dispersion relations, but they are expected to be small and it is not clear if these dispersion relations can be evaluated with sufficient accuracy to establish significant limitations on possible causality violation.

10. Winding up

The following broad observations are suggested by the preceding discussions:

- Models of space-time based on noncommutative geometry are suggested by quantum physics itself, string physics and attempts to regularize quantum field theories.
- They lead to strikingly novel space-time models.
- But lacking contact with experiments, these models for now remain metaphysical models, just as quantum gravity and string models.

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