

Higher dimensional supersymmetric quantum mechanics and Dirac equation

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MS received 9 October 2001

Abstract. We exhibit the supersymmetric quantum mechanical structure of the full 3+1 dimensional Dirac equation considering ‘mass’ as a function of coordinates. Its usefulness in solving potential problems is discussed with specific examples. We also discuss the ‘physical’ significance of the supersymmetric states in this formalism.

Keywords. Supersymmetric quantum mechanics; Dirac equation.

PACS Nos 11.30.Pb; 12.60.Jv; 03.65.Pm

Supersymmetric (SUSY) quantum mechanics, particularly in one-dimension, has been a subject of intensive study in the past [1]. Its generalization to higher space dimensions is an important and interesting problem in its own right. Recently, Das, Okubo and Pernice [2] while discussing generalization of SUSY quantum mechanics to higher dimensions have shown that such generalization forces a spin structure into the theory. Adopting a different approach, Vahle and Ram [3] have established a relationship between the 1+1-dimensional Dirac equation and one-dimensional SUSY quantum mechanics. We, in this paper, exhibit a three-dimensional quantum mechanical structure of the full 3+1 dimensional Dirac equation for both zero and nonzero mass cases. Our result, thus, provides a concrete realization of the conclusions of Das, Okubo and Pernice [2] and generalizes the work of Vahle and Ram [3] to higher dimensions.

It is pertinent to mention here that the connection between the SUSY quantum mechanics and the Dirac equation have been studied earlier in different contexts [4]. Our motivation and approach, however, are different in that we always work with three-dimensional Dirac equation and consider ‘mass’ to be coordinate-dependent. Additionally, we present the physical interpretation of the similar helicity solutions of the Dirac equation as super-partner states which enhances the understanding of SUSY quantum mechanics available through earlier works.

We start with mass $m = 0$ case.

(i) $m = 0$ case

The Dirac equation in this case has the familiar form

$$(\vec{\alpha} \cdot \vec{p})\Psi = i \frac{\partial}{\partial t} \Psi \quad (1)$$

with

$$\alpha = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad (2)$$

where $\vec{\sigma}$ s are the standard Pauli matrices.

$$\Psi = \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix} \quad (3)$$

with Ψ_a and Ψ_b each being two-component spinors. Taking the time dependence of Ψ to be given by e^{-iEt} , eq. (1) is straightforwardly written as a pair of first-order coupled equations

$$\begin{aligned} (\vec{\sigma} \cdot \vec{p})\Psi_b &= E\Psi_a \\ (\vec{\sigma} \cdot \vec{p})\Psi_a &= E\Psi_b. \end{aligned} \quad (4)$$

Uncoupling eq. (4) yields

$$p^2 \Psi_{a,b} = \bar{E} \Psi_{a,b} \quad \text{with} \quad \bar{E} = E^2. \quad (5)$$

These are the Schrödinger equations corresponding to vanishing supersymmetric partner potentials. One can, however, go ahead and define the SUSY Hamiltonian as

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} p^2 \quad (6)$$

with $H_1 = H_2 = p^2$. Defining the SUSY charges as

$$Q = \vec{\sigma} \cdot \vec{p} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad Q^\dagger = \vec{\sigma} \cdot \vec{p} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (7)$$

it follows that

$$H = \begin{pmatrix} p^2 & 0 \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} Q^\dagger Q & 0 \\ 0 & Q Q^\dagger \end{pmatrix}. \quad (8)$$

Further the SUSY algebra involving commutators and anticommutators,

$$\begin{aligned} [Q, H] &= [Q^\dagger, H] = 0 \\ \{Q, Q\} &= \{Q^\dagger, Q^\dagger\} = 0 \\ \{Q, Q^\dagger\} &= H \end{aligned} \quad (9)$$

is satisfied. One further notices that Q converts an upper component spinor $\begin{pmatrix} \Psi_a \\ 0 \end{pmatrix}$ to a lower one and Q^\dagger does the opposite. It is easily verified that if Φ happens to be an eigenstate of $H_1(H_2)$, $Q\Phi(Q^\dagger\Phi)$ is that of $H_2(H_1)$ with equal 'energy'. The difficulty in factorizing the Laplacian in dimensions larger than two as discussed in [2] is circumvented here by the introduction of a matrix valued Q and a two-component wave function which, as is well-known, is reminiscent of the intrinsic spin structure of the theory that is evident from the nonrelativistic reduction of the Dirac equation.

Thus, we have obtained the three-dimensional free particle supersymmetric quantum mechanics starting from zero-mass 3+1 dimensional Dirac equation. Recalling that the solutions of the zero mass Dirac equation are two independent helicity states of the massless particle, our discussion above leads to the interesting conclusion that these two helicity states can be viewed as the superpartners of each other in the SUSY quantum mechanical formalism.

(ii) $m \neq 0$ case

Here the Dirac equation is [5]

$$(\gamma_\mu \partial_\mu + m)\Psi = 0. \quad (10)$$

(a) *Reduction to effective 1-D*: To make the discussion clear we proceed in two steps: we first take the coordinate dependence of the four-component Dirac wave function as

$$\Psi(\vec{X}, t) = \Psi(X_3) e^{i(p_1 X_1 + p_2 X_2 - Et)} \quad (11)$$

and finally consider the three-dimensional generalization of it.

Consideration of eq. (11) reduces the Dirac eq. (10) to the following form:

$$[i(\gamma_1 p_1 + \gamma_2 p_2) + \gamma_3 \partial_3 - \gamma_4 E + m]\Psi = 0. \quad (12)$$

Multiplying γ_3 from left to the above equation gives us

$$[i\gamma_3 \gamma_1 p_1 + i\gamma_3 \gamma_2 p_2 + \partial_3 + \gamma_3(m - \gamma_4 E)]\Psi = 0. \quad (13)$$

We now take $m = m(X_3)$ to be a Lorentz scalar potential [6]. Differentiating eq. (13) with respect to X_3 once more and substituting for ∂_3 from eq. (13), one arrives at the equation

$$[\partial_3^2 + \gamma_3 m' + \tilde{E} - m^2]\Psi = 0 \quad (14)$$

with $m' = \partial_3 m(X_3)$ and $\tilde{E} = E^2 - p_1^2 - p_2^2$.

Using $\gamma_3 = \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix}$ and writing $\Psi = \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix}$ as before, eq. (14) splits into two coupled equations

$$(-\partial_3^2 + m^2)\Psi_a + i\sigma_3 m' \Psi_b = \tilde{E} \Psi_a \quad \text{and} \quad (-\partial_3^2 + m^2)\Psi_b - i\sigma_3 m' \Psi_a = \tilde{E} \Psi_b. \quad (15)$$

One immediately notices that these equations decouple in the space of $\chi_1 = \Psi_a + i\Psi_b$ and $\chi_2 = \Psi_a - i\Psi_b$ to equations

$$(-\partial_3^2 + m^2 + \sigma_3 m')\chi_1 = \tilde{E}\chi_1 \quad \text{and} \quad (-\partial_3^2 + m^2 - \sigma_3 m')\chi_2 = \tilde{E}\chi_2. \quad (16)$$

So χ_1 and χ_2 can be looked upon as the superpartners with superpotentials $\pm\sigma_3 m$. The Hamiltonian now can be written as

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (p_3^2 + m^2) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sigma_3 \partial_3 m = \begin{pmatrix} Q^\dagger Q & 0 \\ 0 & Q Q^\dagger \end{pmatrix} \quad (17)$$

with

$$Q = \begin{pmatrix} 0 & 0 \\ \sigma_3 p_3 + im & 0 \end{pmatrix} \quad \text{and} \quad Q^\dagger = \begin{pmatrix} 0 & \sigma_3 p_3 - im \\ 0 & 0 \end{pmatrix}. \quad (18)$$

The Q and Q^\dagger defined above satisfy the SUSY algebra (eq. (9)).

To see the structure at the full four-component level of Ψ , let us take $\Psi_a = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ and $\Psi_b = \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix}$. Equation (16) immediately shows that there are now two sets of superpartners $\Psi_1 + i\Psi_3$ ($\Psi_2 - i\Psi_4$) and $\Psi_2 + i\Psi_4$ ($\Psi_1 - i\Psi_3$) satisfying one-dimensional Schrödinger equation with superpartner potential $m^2 + m'$ and $m^2 - m'$ respectively as obtained by Vahle and Ram [3]. The construction of SUSY Hamiltonian, charges and resulting discussion all follow as in their case. The two-fold degeneracy of superpartner state is a reflection of the degeneracy present at the \tilde{E} level for the solutions involved. It may be recalled that the eigenstates of the helicity operator $\Sigma_3 \equiv \begin{pmatrix} \vec{\sigma} \cdot \hat{p}_3 & 0 \\ 0 & \vec{\sigma} \cdot \hat{p}_3 \end{pmatrix}$ with e.v. = 1 can admit nonzero Ψ_1, Ψ_3 components only and eigenstates with e.v. = -1 only admits nonzero Ψ_2, Ψ_4 . Thus the superpartner states $\Psi_1 \pm i\Psi_3$ and $\Psi_2 \pm i\Psi_4$ involve the components of Ψ associated with positive and negative helicity states respectively.

As an illustration of the physical application of the discussions made above, we consider the example of

$$m(X_3) = \kappa^2 X_3. \quad (19)$$

Substituting eq. (19) in eq. (16) and taking $\chi_1 = \begin{pmatrix} \chi_1^1 \\ \chi_1^2 \end{pmatrix}$, one straightforwardly arrives at

$$\left(\frac{\partial^2}{\partial z^2} + \lambda - z^2 \right) \chi_1^{1,2} = 0 \quad (20)$$

where $z = kX_3$ and $\lambda = (\tilde{E} \mp \kappa^2)/\kappa^2$ with negative sign for χ_1^1 and positive sign for χ_1^2 . Since eq. (20) is of the form of the equation for 1-D oscillator, the eigenvalues (\tilde{E}) of the Hamiltonian $-\partial_3^2 + m^2 \mp m'$ are found to be equal to $2n\omega$ and $2(n+1)\omega$ respectively with $n = 0, 1, 2, \dots$ and $\omega = \kappa^2$. Interestingly this result for \tilde{E} retains the three-dimensional flavor and exactly agrees with the eigenvalues E of a nonrelativistic supersymmetric 3-D oscillator [1]. Similar eigenvalue structure also exists for the two components $\chi_2^1 \equiv (\psi_1 - i\psi_3)$ and $\chi_2^2 \equiv (\psi_2 - i\psi_4)$ of χ_2 confirming the discussion made above about the two sets of degenerate superpartners.

(b) *Generalization to 3-D*: We now go over to the three-dimensional generalization of eq. (11) and take coordinate dependence as

$$\Psi(\vec{X}, t) = \Psi(\vec{X}) e^{-iEt}. \quad (21)$$

The Dirac eq. (10) now reduces to

$$[\gamma_i \partial_i - \gamma_4 E + m(\vec{X})] \Psi = 0. \quad (22)$$

Multiplying this equation by γ_4 from left and operating the resulting equation by $\gamma_j \partial_j$ from left one gets

$$[\gamma_j \gamma_i \partial_j \partial_i + E^2 - m^2 + \gamma_j \partial_j m] \Psi = 0. \quad (23)$$

Interchanging i and j in this equation and adding the resulting equation to it, we obtain

$$[\nabla^2 + E^2 - m^2 + \vec{\gamma} \cdot \vec{m}'] \Psi = 0. \quad (24)$$

In arriving at eq. (24) we have used the Dirac gamma matrix properties $\gamma_i \gamma_j + \gamma_j \gamma_i = 2\delta_{ij}$. Equation (24) is indeed the three-dimensional generalization of eq. (14).

Now using $\vec{\gamma} = \begin{pmatrix} 0 & -i\vec{\sigma} \\ i\vec{\sigma} & 0 \end{pmatrix}$ and $\Psi = \begin{pmatrix} \Psi_a \\ \Psi_b \end{pmatrix}$ one immediately obtains the generalized version of eq. (16) as

$$\begin{aligned} (-\nabla^2 + m^2 + \vec{\sigma} \cdot \vec{\nabla} m) \chi_1 &= E^2 \chi_1 \\ (-\nabla^2 + m^2 - \vec{\sigma} \cdot \vec{\nabla} m) \chi_2 &= E^2 \chi_2. \end{aligned} \quad (25)$$

$\chi_1 = (\Psi_a + i\Psi_b)$ and $\chi_2 = (\Psi_a - i\Psi_b)$ are now the superpartners with matrix valued vector superpotentials [2] $W \sim \pm \vec{\sigma} m$. Thus if Schrödinger equations (25) can be solved for a class of supersymmetric partner potentials $V_{\pm}(X) = m^2(X) \pm \vec{\sigma} \cdot \vec{\nabla} m(\vec{X})$ then so can be the Dirac equation for the corresponding potential $m(\vec{X})$. This brings out a greater usefulness of SUSY quantum mechanics.

Defining Q and Q^\dagger as generalization of eq. (18) like

$$Q = \begin{pmatrix} 0 & 0 \\ \vec{\sigma} \cdot \vec{p} + im & 0 \end{pmatrix} \quad \text{and} \quad Q^\dagger = \begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{p} - im \\ 0 & 0 \end{pmatrix} \quad (26)$$

one easily verifies that Hamiltonian can be, once again, written as

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (p^2 + m^2) + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{\sigma} \cdot \vec{\nabla} m = \begin{pmatrix} Q^\dagger Q & 0 \\ 0 & QQ^\dagger \end{pmatrix} \quad (27)$$

and the SUSY algebra (eq. (9)) is satisfied. This establishes the SUSY quantum mechanical structure of the full 3 + 1 dimensional Dirac equation.

At this stage, just writing $\Psi_a = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ and $\Psi_b = \begin{pmatrix} \Psi_3 \\ \Psi_4 \end{pmatrix}$ does not lead to a set of decoupled equations at the four-component level as it happened earlier.

We illustrate the difficulty with the help of an example. We consider the potential with spherical symmetry as has been considered to confine quarks inside the hadronic bag [7]

$$m(\vec{X}) = k^2 r \quad (28)$$

where k is a characteristic wave number and r is the radial coordinate. Equations (25) now become

$$\begin{aligned} (-\nabla^2 + m^2 + k^2 \vec{\sigma} \cdot \hat{r}) \chi_1 &= E^2 \chi_1 \\ (-\nabla^2 + m^2 - k^2 \vec{\sigma} \cdot \hat{r}) \chi_2 &= E^2 \chi_2. \end{aligned} \quad (29)$$

It is the nondiagonality of the operator $\vec{\sigma} \cdot \hat{r}$ that prevents decoupling of these equations at the four-component level in the present example.

We find that $\vec{\sigma} \cdot \hat{r}$ can be diagonalized with the help of a unitary matrix S

$$S = \begin{pmatrix} \cos \frac{\theta}{2} e^{i\phi/2} & \sin \frac{\theta}{2} e^{-i\phi/2} \\ -\sin \frac{\theta}{2} e^{i\phi/2} & \cos \frac{\theta}{2} e^{-i\phi/2} \end{pmatrix} \quad (30)$$

where θ and ϕ refer to polar and azimuthal angles respectively. Evidently, $S\chi_{1,2}$ will have the same eigenvalues as $\chi_{1,2}$ given by eq. (29). However, then the new operator $S(-\nabla^2)S^{-1}$ becomes nondiagonal due to operation of the θ and ϕ dependent part of $-\nabla^2$ ($\equiv -L^2/r^2$, where L is the angular momentum operator) on S^{-1} .

Caught between these two impasse if one, for brevity, puts $L_x = L_y = 0$, standard manipulation with eq. (29) leads to equations

$$\begin{aligned} \left(\frac{d^2}{dr^2} + E^2 - k^2 - k^4 r^2 - \frac{1}{4r^2} \right) \Psi'_{1+i3} &= 0 \\ \left(\frac{d^2}{dr^2} + E^2 + k^2 - k^4 r^2 - \frac{1}{4r^2} \right) \Psi'_{2+i4} &= 0 \end{aligned} \quad (31)$$

and two similar equations for Ψ'_{2-i4} and Ψ'_{1-i3} respectively where the prime refers to transformed functions after application of matrix S on $\chi_{1,2}$. The eigenvalues of such equations cast in the form

$$\left[\frac{d^2}{dr^2} + (a - br^2 - c/r^2) \right] f = 0 \quad (32)$$

with $a = E^2 \mp k^2$, $b = k^4$, $c = 1/4$ are obtained as

$$E^2/k^2 = \pm 1 + 2(2n + l - 1/2) \quad (33)$$

with $l(l+1) = c = 1/4$ (i.e., $l = 0.207$) and $n = 1, 2, 3, \dots$. The eigenvalues for allowed n values thus are obtained as $E^2/k^2 = 2.414, 6.414, 10.414, \dots$ and $4.414, 8.414, 12.414, \dots$ for $n = 1, 2, 3$, respectively. These values are in the vicinity of values obtained in [7]. However the usual signature of unbroken supersymmetry of pairing of energy eigenvalues is missing in the above values. This discrepancy could very well have been caused by the approximations made in neglecting the L_x and L_y to obtain eigenvalues. Thus we have not been able to exhibit the supersymmetric partnership of χ_1 and χ_2 as exhibited by eqs (29)

in terms of their eigenvalues for a spherically symmetric potential like $k^2 r$ used for quark confinement.

In conclusion we wish to mention that we have formally deciphered a three-dimensional SUSY quantum mechanical structure starting from the full 3+1 dimensional Dirac equation involving a potential ($m(\vec{X})$). It is to be emphasized that our work involving a scalar potential as opposed to a vector potential used by Das, Okubo and Pernice [2] can be viewed as complementary to their work. Further, the exhibition of an underlying SUSY quantum mechanical structure of the Dirac equation which has an intrinsic spin structure associated with it, provides a concrete realization of the observation of Das, Okubo and Pernice [2] that higher dimensional SUSY quantum mechanics introduces a spin structure into the theory. We also observe that the linear superposition of the components of Dirac solution Ψ associated with same helicity states basically form the superpartners of each other in the C-number formalism of SUSY quantum mechanics.

Further, we have studied the examples of $m(X) = k^2 X_3$ and $m(X) = k^2 r$. While in the case of the former, we were able to obtain the 3-D oscillator energy values nicely paired starting with the minimum value of zero signifying unbroken SUSY, we did not succeed in obtaining similar result in the latter case primarily due to nondiagonality of the operators operating on the superpartner states. This, we suspect, is connected with the plane-wave solutions of Dirac equation being not the eigenstates of the operator $\vec{\Sigma} \cdot \hat{r} = \begin{pmatrix} \vec{\sigma} \cdot \hat{r} & 0 \\ 0 & \vec{\sigma} \cdot \hat{r} \end{pmatrix}$ unless $\hat{r} = \hat{p}$ [8]. A satisfactory resolution of this problem will complete the understanding of physical implication of SUSY structure of the Dirac equation and could shed light on higher dimensional SUSYQM independent of it being anchored to Dirac equation.

Acknowledgements

The authors gratefully acknowledge the warm hospitality of IUCAA, Pune, India where this work was done. The authors also thank Institute of Physics, Bhubaneswar, India, for providing facility of the computer centre.

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