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Korteweg–de Vries hierarchy using the method of base equations

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Abstract. A base-equation method is implemented to realize the hereditary algebra of the Korteweg–de Vries (KdV) hierarchy and the *N*-soliton manifold is reconstructed. The novelty of our approach is that, it can in a rather natural way, predict other nonlinear evolution equations which admit local conservation laws. Significantly enough, base functions themselves are found to provide a basis to regard the KdV-like equations as higher order degenerate bi-Lagrangian systems.

Keywords. KdV hierarchy; base-equations method.

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1. Introduction

More than thirty years ago Lax [1] showed that the Korteweg–de Vries (KdV) initial-value problem for $u = u(x,\ t)$ given by

$$u_t = -u_{xxx} + 6uu_x \tag{1}$$

with

$$u(x,0) = v(x) \tag{2}$$

is but one of the infinite family of equations that leave the eigenvalue of the Schrödinger equation invariant in t. The subscripts of u in eq. (1) denote differentiation with respect to the associated variables. The family of equations discovered by Lax often goes by the name KdV hierarchy and is generated by making use of the recursion operator [2]

$$\Lambda = 1/4\,\partial_x^2 - \partial_x^{-1}u\partial_x - 1/2\,\partial_x^{-1}u_x \tag{3}$$

with $\partial_x = \partial/\partial x$ in the differential relation

$$\frac{\partial u}{\partial t_{2n+1}} = \left(\frac{\partial}{\partial x}\right)\Lambda^n u,\ n = 1,2\dots . \tag{4}$$

Here t_{2n+1}'s refer to group parameters for the hierarchy consisting of equations enumerated by the integers $(2n+1)$. For $n = 1$ we get the KdV equation from eq. (4) and higher order equations are obtained for $n > 1$. The members of the hierarchy, in general, share many common features. In addition to being solvable by the inverse spectral method, all of them have infinitely many conserved quantities and symmetries. They also possess the Painlevé property and can be expressed in bi and/or multi-Hamiltonian form.

In this work we propose to use a method of base equations to construct the KdV hierarchy. Our approach to the problem, on the one hand, serves as a useful supplement to other existing methods [2] and, on the other hand, tends to predict some results which do not follow from earlier treatments. Our treatment may be regarded as a simple variant of the well-known base-function method used for writing the solution of ordinary nonlinear differential equations [3]. The origin of this method, perhaps, dates back to the classic period nineteenth century mathematics [4]. Relatively recently, Reid [5,6] generalized the basic idea to include many types of nonlinearities and even to deal with nonlinear evolution equations. Like most techniques for treating the nonlinear differential equation, the method of base equations, might appear somewhat *ad-hoc*. However, it possesses sufficient generality to qualify as a separate technique. For example, it can be viewed as a nonlinear superposition technique not profoundly different from the Bäcklund transformation [7]. In the following section we introduce the base equations needed for our construction procedure.

2. Base equations

In the inverse spectral method the solution of the KdV equation is given by

$$u(x,t) = -2\partial_x K(x,x,t), \tag{5}$$

where $K(x,x,t)$ is obtained by solving the Gel'fand–Levitan–Marchenko equation [8] with a kernel characterized by only the bound-state spectrum of $v(x)$. The t-evolution of the kernel enters through the normalization constant. For the case of a single-soliton solution at a wave number κ the function $K(x,x,t)$ satisfies the Ricatti equation

$$K_x + K^2 + 2\kappa K = 0. \tag{6}$$

Upon differentiation with respect to x, eq. (6) gives

$$K_{2x} + 2KK_x + 2\kappa K_x = 0. \tag{7}$$

Again differentiating eq. (7) $(n-2)$ times by using the Leibniz theorem we get

$$E_n \equiv \partial_x^{(n)} K + 2\sum_{r=0}^{n-3} \binom{n-2}{r} \partial_x^{(n-r-2)} K \partial_x^{(r+1)} K + (2K + 2\kappa)\partial_x^{(n-1)} K = 0 \tag{8}$$

with $\partial_x^{(n)} = \partial^n/\partial x^n$. Similarly,

$$E_i \equiv \partial_x^i K + 2\sum_{r=0}^{i-3} \binom{i-2}{r} \partial_x^{(i-r-2)} K \partial_x^{(r+1)} K + (2K + 2\kappa)\partial_x^{(i-1)} K = 0\,. \tag{9}$$

We now eliminate $(2K+2\kappa)$ from eq. (8) by using eq. (9) and write

$$E_n^i = \partial_x^n K + 2\sum_{r=0}^{n-3}\binom{n-2}{r}\partial_x^{(n-r-2)}K\partial_x^{(r+1)}K - \left(\partial_x^i K + 2\sum_{r=0}^{i-3}\binom{i-2}{r}\partial_x^{(i-r-2)}K\partial_x^{(r+1)}K\right)\frac{\partial_x^{(n-1)}K}{\partial_x^{(i-1)}K} = 0\,. \tag{10}$$

We refer E_n^i's as base equations in our construction procedure. In particular, we shall see that superposition of E_n^i's generates the equations of the KdV hierarchy. We note that the subscript n in eq. (10) determines the order of the differential expression for E_n^i. Since all equations in the hierarchy are of odd order in u, the first member in the superposed equations should be of even $(2m+2\ ,m=1,2,3,\ldots)$ order in K.

3. KdV hierarchy

Keeping the above in view we have found that the KdV equation $(m=1)$ and the first member $(m=2)$ of the hierarchy can be obtained from

$$E_{2m+2}^2 + \sum_{i=1}^{m}\lambda_m^i\partial_x^i K E_{2m+1-i}^2 = 0 \tag{11}$$

with

$$\lambda_m^i = (2m+1)\,i\,(i+1)\,. \tag{12}$$

To be more specific let us use $m=1$ in eq. (11) and get

$$E_4^2 + 6\partial_x K E_2^2 = 0\,. \tag{13}$$

From eq. (10) we have

$$E_4^2 = \partial_x^4 K + 6\partial_x^2 K\partial_x K - \frac{\partial_x^2 K\partial_x^3 K}{\partial_x K} = 0 \tag{14}$$

and

$$E_2^2 = \partial_x^2 K - \partial_x^2 K = 0. \tag{15}$$

Using eqs (14) and (15) we reduce eq. (13) in the form

$$\partial_x^4 K + 12\partial_x^2 K\partial_x K = \frac{\partial_x^2 K}{\partial_x K}(\partial_x^3 K + 6(\partial_x K)^2)\,. \tag{16}$$

In view of eq. (5), eq. (16) becomes

$$u_{xxx} - 6uu_x = \frac{u_x}{u}(u_{xx} - 3u^2)\,. \tag{17}$$

The single-soliton solution

$$u(x,t) = -2\kappa^2 \mathrm{sech}^2(\kappa x - 4\kappa^3 t) \tag{18}$$

of the KdV equation can now be used to identify the right side of eq. (17) as $-u_t$ giving the so-called KdV equation. For the first member $(m=2)$ of the hierarchy an equation similar to eq. (17) reads

$$u_{5x} - 10uu_{3x} + 30u^2u_x - 20u_{2x}u_x = \frac{u_x}{u}(u_{4x} - 15u_x^2) \tag{19}$$

with $u_{xx} = u_{2x,}\, u_{xxx} = u_{3x}$ etc. The nth equation of the hierarchy has a soliton solution [9]

$$u_n(x,t) = -2\kappa^2 \mathrm{sech}^2(\kappa x - 4\kappa^{2n+1} t)\,. \tag{20}$$

Specializing eq. (20) for $n = 2$, we can identify right side of eq. (19) with u_t so as to arrive at the first member of the KdV hierarchy.

It is very unfortunate that eq. (11) does not give other higher members of the hierarchy. Even to construct the second member we need to implement a somewhat more complicated superposition of base equations. Clearly, the additive term should be such that they give vanishing contribution for $m = 1$ and 2. The second and third members of the hierarchy can be generated from

$$\begin{aligned} &E^2_{2m+2} + \sum_{i=1}^{m} \lambda^i_m \partial^i_x K E^2_{2m+1-i} + \sum_{j=1}^{m-2} \lambda^j_m \partial^{j+1}_x K E^{j+2}_{2m-j} \\ &\qquad + 2^{m-1} \sum_{k=1}^{m-1} \sum_{l=1}^{m-2} \lambda^{m-k}_m \partial^k_x K \partial^l_x K E^2_{2m-l-k} = 0\,. \end{aligned} \tag{21}$$

Interestingly, eq. (21) is characterized by the set of nonlinear constraints as given by eq. (12). Further, as in the case of KdV equation and the first member of the hierarchy one can verify that the results from eq. (21) are in exact agreement with those obtained from eq. (4) for $n = 3$ and 4 respectively. We have found that all members of the KdV hierarchy can be obtained by making use of a formula

$$\begin{aligned} &E^2_{2m+2} + \sum_{i=1}^{m} \lambda^i_m \partial^i_x K E^2_{2m+1-i} + \sum_{j=1}^{m-2} \lambda^j_m \partial^{j+1}_x E^{j+2}_{2m-j} \\ &\quad + \sum_{j=1}^{m-4} \lambda^j_m \partial^{j+2}_x K E^{j+3}_{2m-j-1} + \cdots + 2^{m-1} \sum_{k=0}^{m-1} \lambda^{m-k}_m \partial^k_x K \\ &\quad \times \left[\sum_{l=1}^{m-2} \partial^l_x K [E^2_{2m-k-l}] + \sum_{n=1}^{m-3} \partial^n_x K [E^2_{2m-k-l-n} + \cdots] \cdots \right] = 0\,. \end{aligned} \tag{22}$$

Looking at eq. (22) it appears that we could not derive any calculational simplicity for the problem under study. In fact, our objective in this work was somewhat different. We tried to demonstrate that higher order derivatives of the simplest nonlinear differential equation for $K(x,x,t)$, via a postulated superposition principle with nonlinear constraints, form the building block for the entire KdV hierarchy. In the following we note two important aspects of the present construction procedure.

(i) Since all E_n^i's are zero, we can delete terms from the summations in eqs (11), (21) or (22) to write new integrable evolution equations. For example, if the term multiplying λ_2^1 is dropped from eq. (11) we will arrive at

$$u_t = u_{5x} - 5uu_{3x} - 25u_xu_{2x} + 15u^2u_x. \tag{23}$$

Understandably, in writing eq. (23) we have made use of eq. (5). A conservation law associated with eq. (23) can be written as

$$u_t = \partial_x X \tag{24}$$

with flux of $u(x,t)$ given by

$$X = u_{4x} - 5uu_{2x} - 10u_x^2 + 5u^3. \tag{25}$$

Similar equations could also be constructed from eqs (21) or (22). Calogero and Nucci [10] pointed out that existence of a local conservation law is an important element to have Lax pair or equivalently a bi-Hamiltonian structure. Recently a strong relationship has been observed [11] between Ricatti equations and Bäcklund transformations for integrable nonlinear partial differential equations. Here we suggest a method to construct the N-soliton KdV manifold using derivatives of Ricatti equations as base equations.

(ii) In a broad sense one knows that equations of the KdV hierarchy are higher order degenerate Lagrangian systems [12]. The canonical formulation of these equations requires the use of Dirac's theory of constraints [13]. In this context we demonstrate that the base equation can be used to introduce a new Lagrangian density for the KdV equation in addition to what exists in literature. Similar construction might also be possible for other members of the hierarchy.

The Lagrangian density for the KdV equation is given by [14]

$$L^{(2)}(K_t, K_x, K_{xx}) = 4[K_tK_x - K_{xx}^2 + 4K_x^3]. \tag{26}$$

Superscript 2 was used on L to indicate that the Lagrangian density is of second order. It is degenerate because the momenta cannot be inverted for velocities [13]. Starting from $E_3^2 = 0$, it is easy to show that $L^{(2)}(K_t, K_x, K_{xx})$ is equal to a third order degenerate Lagrangian given by

$$L^{(3)}\left(K_t, K_x, K_{3x}\right) = 4\left[K_tK_x + 2K_x^3 - K_xK_{3x}\right]. \tag{27}$$

It is interesting to discover that eq. (27) when substituted in the Euler–Lagrange equation

$$\sum_{i=1}^{m}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^i\left(\frac{\partial L}{\partial\left(\mathrm{d}^iK/\mathrm{d}t^i\right)}\right) + \left(\frac{\delta L}{\delta K}\right) = 0 \tag{28}$$

for the higher order field reproduces the KdV equation. Here $\delta/\delta K$ stands for the functional derivative

$$\frac{\delta}{\delta K} = \sum_{i=0}^{n}(-D)^i\frac{\partial}{\partial K^{(i)}} \tag{29}$$

with $D = \mathrm{d}/\mathrm{d}x$ and $K^{(i)} = \partial^iK/\partial x^i$. We have verified that both $L^{(2)}$ and $L^{(3)}$ in conjunction with the Ostrogradski formalism [15] lead to zero order Hamiltonian densities that satisfy Zakharov–Faddeev–Gardner Hamiltonian form of the KdV equation [16].

4. Summary

Pioneered by Gardner, Green, Kruskal and Miura [17] and by Lax [1] the construction procedure for the equations of the KdV hierarchy has been extensively discussed [2]. In this work we have used the method of base equations for a similar study. We have already pointed out in the text that our approach can be viewed as a nonlinear superposition technique bearing some similarity with the so-called Bäcklund transformation. Any member of the KdV hierarchy can be obtained from our general equation (22) for a particular value of m. Since all E_n^i's are zero, one can judiciously delete terms from the summation of these equations to write the new equations, some of which might turn out to be integrable. This appears to be the merit of the present approach. In addition to this, base equations themselves are seen to be useful to study the Lagrangian structure which via Ostrogradski formalism [15] helps one to write Hamiltonian densities that characterize the Zakharov–Faddeev–Gardner equation [16].

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