

Generalized scalar tensor theory in four and higher dimensional spherically symmetric space-time

SUBENYO CHAKRABORTY and ARABINDA GHOSH*

Department of Mathematics, Jadavpur University, Calcutta 700 032, India

*Danrpur, J K S High School, Vill. and P.O. Danrpur 712 143, Hooghly District, India

Email: subenoy@juphys.ernet.in

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Abstract. In this paper, we have studied generalized scalar tensor theory for spherically symmetric models, both in four and higher dimensions with a bulk viscous fluid. We have considered both exponential and power law solutions with some assumptions among the physical parameters and solutions have been discussed.

Keywords. Spherically symmetric; scalar tensor theory; higher dimension.

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1. Introduction

The Brans–Dicke theory of gravity [1] is the widely studied and best known alternative theory of Einstein gravity. This theory properly accommodate both Mach's principle [2] and Dirac's large number hypothesis [3]. In this theory a scalar field ϕ is coupled to gravity with coupling constant ω .

However, in scalar-tensor theory [4–7] (also known as generalized B-D theory) the coupling parameter ω is not a constant but is a function of the scalar field i.e. $\omega = \omega(\phi)$. So we can speculate that ω is very small during the early stages of the evolution of the Universe and then ω evolves to the present large value ($\omega \geq 500$, from experimental evidence). As a consequence, the predictions of scalar-tensor theory and that of general relativity theory may differ significantly.

Recently, the study of BD theory (scalar tensor theory) has been intensified for the following reasons [8]: (i) the association of scalar fields to the metric seems to be unavoidable in superstring theories [9], (ii) scalar-tensor theories are invariant under a restricted class of conformal transformations [10–14] which is important in string theory, (iii) BD gravity can be derived from a Kaluza–Klein theory [10] in which the scalar field is generated by the presence of compactified extra dimensions, which is an essential feature of all modern unified theories.

There are two ways in which the BD-Lagrangian can be expressed namely: (i) Jordan frame, (ii) Einstein frame. In Jordan frame the BD-Lagrangian is

$$L = \frac{\sqrt{-r}}{16\Pi} \left[\phi \cdot \bar{R} - w \cdot r^{\mu\nu} \cdot \frac{(\partial_\mu \phi)(\partial_\nu \phi)}{\phi} \right] + \frac{16\Pi G}{c^4} L_m. \quad (1a)$$

This Lagrangian is conformal invariant under the conformal transformations [15]

$$g_{\mu\nu} = \Omega^2 r_{\mu\nu}, \quad \Omega = \phi^\lambda, \quad \sigma = \phi^{1-2\lambda}, \quad \bar{w} = \frac{w - 6\lambda(\lambda - 1)}{(2\lambda - 1)^2}, \quad (2a)$$

for $\lambda \neq 1/2$.

For $\lambda = 1/2$, we make the following transformations

$$g_{\mu\nu} = e^{\alpha\sigma} \cdot r_{\mu\nu}, \quad \phi = \frac{8\Pi}{k^2} e^{\alpha\sigma}, \quad (2b)$$

with $k^2 = 8\Pi G$, $\alpha = \beta k$ and $\beta^2 = 2/(2w + 3)$.

Under this conformal transformation (2b) the above Lagrangian (1a) gets transformed to [16]

$$L = \sqrt{-g} \left[\frac{1}{2k^2} R - \frac{(2w + 3)}{2} g^{\mu\nu} \cdot \frac{(\partial_\mu \phi \cdot \partial_\nu \phi)}{\phi^2} \right] + \left(\frac{16\Pi G}{C^4} \right) \cdot L_m, \quad (1b)$$

where R is the Ricci scalar and L_m is the Lagrangian corresponding to the matter distribution. According to Einstein frame, G is a constant but the rest mass of the test particle is a variable. Now by variational principle (see Lagrangian (1b)), the field equations are

$$G_{\mu\nu} = -T_{\mu\nu} - \frac{(2w + 3)}{\phi^2} \left[\phi_{,\mu} \cdot \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \cdot \phi_{,\alpha} \cdot \phi'^\alpha \right], \quad (3a)$$

and the wave equations

$$\square(\ln \phi) = \frac{1}{(2w + 3)} \left[T - \frac{1}{\phi} (\phi_{,\mu} \cdot \phi'^\mu) \frac{dw}{d\phi} \right], \quad (3b)$$

where we have chosen $8\Pi G/C^4 = 1$. Here T is the trace of the energy-momentum tensor corresponding to the matter distribution

$$T_{\mu\nu} = (\mathcal{S} + \bar{p}) u_\mu u_\nu - \bar{p} \cdot g_{\mu\nu}, \quad (4)$$

with

$$\bar{p} = p - \eta \cdot \theta.$$

In the above expressions $(\mathcal{S}, p, \eta, \theta)$ stands for matter density, thermodynamic pressure, bulk viscosity coefficients and expansion scalar ($= u^\mu_{,\mu}$) respectively and u_μ is the four velocity vector. In the co-moving co-ordinate system we have

$$u^\mu = \delta_0^\mu$$

with

$$u_\mu \cdot u^\mu = +1 \text{ (in } -2 \text{ signature).}$$

The divergence relation $T^{\mu\nu}_{;\nu} = 0$ yields

$$\dot{\rho} = -(\rho + p)\theta + \eta\theta^2, \quad (4a)$$

where for a physically reasonable fluid $\rho > 0, p > 0$. Further, Hawking's energy condition ($R_{\mu\nu} \cdot u^\mu u^\nu > 0$) is satisfied in this case.

x In the present work, we have considered both power law and inflationary solutions in the generalized scalar tensor theory along with a bulk viscous fluid. Spherically symmetric models in four and five dimensions are considered in §§2 and 3 respectively. The paper ends with a short discussion in §4.

2. Four dimensional spherically symmetric model

The metric ansatz for the model is

$$ds^2 = dt^2 - e^{\lambda(t)}(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2). \quad (5)$$

Considering the energy-momentum tensor for a viscous flow with bulk viscosity the above field equations (see eqs (3a) and (3b)) become

$$\frac{3}{4}\dot{\lambda}^2 = S + ((2\omega + 3)/4)\dot{\psi}^2, \quad (6)$$

$$\ddot{\lambda} + \frac{3}{4}\dot{\lambda}^2 = -p - ((2\omega + 3)/4)\dot{\psi}^2 + \frac{3}{2}\dot{\lambda}\eta, \quad (6a)$$

and

$$\ddot{\psi} + \frac{3}{2}\dot{\lambda}\dot{\psi} = \frac{1}{(2\omega + 3)} \left[(S - 3p) + \frac{9}{2}\dot{\lambda}\eta - \dot{\omega}\dot{\psi} \right]. \quad (7)$$

Here S, p and η are energy density, pressure and coefficient of bulk viscosity and $\psi = \ln \theta$.

As there are altogether three equations containing six unknowns ($\lambda, S, p, \eta, \psi, \omega$), one has to assume three more functional relations among these variables to obtain unique solutions. The assumptions are:

(i) Barotropic equation of state:

$$p = \epsilon S, \quad 0 \leq \epsilon \leq 1. \quad (8)$$

(ii) A particular choice of ω [17,18]

$$2\omega + 3 = \alpha e^{-\psi}. \quad (9)$$

(iii) Scale factor in exponential form:

$$\lambda = bt \quad (10)$$

or scale factor in polynomial form:

$$\lambda = n \ln t + \ln B, \quad (11)$$

where b, B, n, α are constants.

Now eliminating ζ, p and η from eq. (7) using (5) and (6a) we get [19]

$$(2\omega + 3) \left[\ddot{\psi} + \frac{3}{2} \dot{\lambda} \dot{\psi} - \frac{\dot{\psi}^2}{2} + \frac{\dot{\omega}}{(2\omega + 3)} \cdot \dot{\psi} \right] = 3\ddot{\lambda} + 3\dot{\lambda}^2. \quad (12)$$

Case I: Exponential expansion

From (12) eliminating λ and ω using eqs (10) and (9) the differential equation in ψ can be solved at once to give

$$e^\psi = \emptyset = \left[\frac{2b_1 e^{-(3/2)bt}}{3b} - \frac{2bt}{\alpha} - b_2 \right]^{-1}, \quad (13)$$

with b_1, b_2 as integration constants.

The other unknown parameters are then

$$\begin{aligned} S &= \frac{3}{4}b^2 - F(t) \\ p &= \epsilon S \\ \eta &= \frac{b}{2}(1 + \epsilon) + \frac{2}{3} \frac{(1 - \epsilon)}{b} \cdot F(t) \end{aligned} \quad (14)$$

with

$$F(t) = \frac{\alpha}{4} \frac{[2b/\alpha + b_1 e^{-(3/2)bt}]^2}{\left[\frac{2b_1}{3b} e^{-(3/2)bt} - \frac{2bt}{\alpha} - b_2 \right]}.$$

Case II: Power law solution

Here with the choice of λ from (11) and ω from (9), eq. (12) becomes a differential equation in ψ with solution

$$e^\psi = \emptyset = \left[\frac{B_1}{((3n/2) - 1)} t^{(1-(3/2)n)} - \frac{3n(n-1)}{((3n/2) - 1)\alpha} \log t - B_2 \right]^{-1}$$

Here B_1, B_2 are the integration constants. So the physical parameters have the expressions

$$\begin{aligned} S &= \frac{3B^2}{4t^2} - G(t), \\ p &= \epsilon S, \\ \eta &= -\frac{2}{3t} + \frac{n}{2t}(1 + \epsilon) + \frac{2t}{3n}(1 - \epsilon)G(t), \end{aligned} \quad (15)$$

with

$$G(t) = (\alpha/4) \cdot \frac{\left[\frac{3n(n-1)}{((3n/2)-1)} \cdot (1/t) + B_1 \cdot t^{-3n/2} \right]^2}{\left[\frac{B_1}{((3n/2)-1)} \cdot t^{(1-3n/2)} - \frac{3n(n-1)}{((3n/2)-1)\alpha} \log t - B_2 \right]}.$$

3. Five dimensional spherically symmetric model

The line element for the five dimensional Kaluza–Klein model is

$$ds^2 = dt^2 - e^{\lambda(t)}(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2) - e^{\mu(t)} dy^2. \quad (16)$$

The field equations (2) and (3) in explicit form for this metric becomes

$$\begin{aligned} \frac{3}{4}(\dot{\lambda}^2 + \dot{\lambda}\dot{\mu}) &= S + \frac{(2\omega + 3)}{4}\dot{\psi}^2, \\ \ddot{\lambda} + \frac{3}{4}\dot{\lambda}^2 + \frac{\ddot{\mu}}{2} + \frac{\dot{\mu}^2}{4} + \frac{\dot{\lambda}\dot{\mu}}{2} &= -p - \frac{(2\omega + 3)}{4}\dot{\psi}^2 + \eta \left(\frac{3\dot{\lambda}}{2} + \frac{\dot{\mu}}{2} \right) \end{aligned} \quad (17)$$

and

$$\begin{aligned} \frac{3}{2}(\ddot{\lambda} + \dot{\lambda}^2) &= -p - \left(\frac{2\omega + 3}{4} \right) \dot{\psi}^2 + \eta \left(\frac{3\dot{\lambda}}{2} + \frac{\dot{\mu}}{2} \right), \\ \ddot{\psi} + (3\dot{\lambda}/2 + \dot{\mu}/2)\dot{\psi} &= (1/(2\omega + 3))[(S - 4p) + 4\eta(3\dot{\lambda}/2 + \dot{\mu}/2) - \dot{\omega}\psi]. \end{aligned}$$

Using the same form of the assumptions as in previous section the explicit solution is as follows.

Case I: Scale factors in exponential form

The choice

$$\lambda = at, \quad \mu = bt \quad (18)$$

gives the restrictions (i) $b = -3a$ or (ii) $b = a$.

For $b = -3a$, the η term has been eliminated from the field equations and the model reduces to a static model. So it is not physically interesting. But with $b = a$, the explicit solution gives

$$\begin{aligned} S &= \frac{3}{2}a^2 - F_1(t), \\ \eta &= \frac{3}{4}a(1 + \epsilon) + \frac{(1 - \epsilon)}{2a}F_1(t), \\ \emptyset &= \left[\frac{3C_1}{4a}e^{-2at} - \frac{45a}{8\alpha}t + (3/2)C_2 \right]^{-2/3}, \\ 2\omega + 3 &= \alpha/\emptyset, \\ F_1(t) &= \left(\frac{\alpha}{4} \right) \cdot \left(\frac{15a}{4\alpha} + C_1 \cdot e^{-2at} \right)^2 / \left(\frac{3C_1}{4}e^{-2at} - \frac{45at}{8\alpha} - \frac{3}{2}C_2 \right), \end{aligned} \quad (19)$$

with C_1, C_2 as integration constants.

Case II: Polynomial expansion of the Universe

The choice for the scale factors is taken to be

$$e^\lambda = \lambda_0 t^a, e^\mu = \mu_0 t^b$$

with (λ_0, μ_0) and (a, b) as constants. To satisfy the field equations a and b are restricted by (i) $a = b$ or (ii) $3a + b = 2$. The explicit solutions are

(A) When $a = b$:

$$\begin{aligned} S &= \frac{3a^2}{2t^2} - \chi(t), \\ \eta &= \frac{3\{a(1+\epsilon) - 1\}}{4t} + \frac{\chi(t)}{2a}(1-\epsilon)t, \\ \emptyset &= \left[\frac{3}{2} \frac{C_1 t^{1-2a}}{(2a-1)} + \frac{9a(5a-4)}{\alpha(1-2a)} \log t - \frac{3}{2} C_2 \right]^{-2/3} \\ \chi(t) &= (\alpha/4), \frac{\left\{ \frac{3a(5a-4)}{2\alpha(2a-1)t} + C_1 t^{-2a} \right\}^2}{\left[\frac{C_1 t^{1-2a}}{2(2a-1)} + \frac{9a(5a-4)}{\alpha(1-2a)} \log t - \frac{3}{2} C_1 \right]}. \end{aligned} \quad (20)$$

(B) When $3a + b = 2$:

$$\begin{aligned} S &= \frac{3a(1-a)}{2t^2} - H(t), \\ p &= \epsilon S, \\ \eta &= (1-\epsilon) \left[\frac{3a(a-1)}{2t} + t \cdot H(t) \right], \\ \emptyset &= [(3/4)\{C_2 + C_1 \cdot \ln t - (15/2\alpha)(\ln t)^2\}]^{-2/3} \\ H(t) &= (\alpha/6t^2) \frac{(C_1 + (15/\alpha) \ln t)^2}{[C_2 + C_1 \ln t - (15/2\alpha)(\ln t)^2]}, \end{aligned} \quad (21)$$

where as before C_1, C_2 are integration constants.

4. Discussion

In this paper we have obtained several sets of explicit solutions for general scalar tensor theory both in four dimensional model and in higher dimension, considering a viscous flow with bulk viscosity. The solutions can be classified into two groups namely: (a) exponential expansion, (b) power-law form of expansion of these models.

In four dimensional spherically symmetric model, both exponential and power-law form of expansion is possible. For five dimensional model, though exponential expansion is possible but anisotropic nature has been lost with no way of contracting the extra dimension. However, in power law form of expansion when $3a + b = 2$, it is possible to have contraction of higher dimension.

The solutions (14) and (19) for exponential expansion in four and five dimensional space-time model behave in a similar way asymptotically. The scalar field ϕ starts from zero value at $t = -\infty$ and ends at zero value at $t = +\infty$, with finite maximum at a finite t . The functions F and F_1 in eqs (14) and (19) respectively, approach to infinity as $t \rightarrow \pm\infty$ and has a minima at finite time. Thus S, p, η and w blow up asymptotically. So the Universe starts from an initial singularity where all the physical parameters have infinite value, and then expands infinitely.

For power-law expansion both in four and higher dimension, ϕ behaves in the same way as above while the functions G and χ (see eqs (15) and (20)) approach to zero as $t \rightarrow +\infty$ and have infinite large value at $t = 0$. Thus the Universe starts from an initial singularity at $t = 0$ and then expands infinitely, while the physical parameters start from infinite values and then gradually approaches to finite constant value. The solution (21) for power law form of expansion in five dimension indicates contraction of extra dimension but the behaviour of the physical parameters are just the opposite i.e. they start from finite value and then blow up asymptotically.

Further, we have used here Dicke's revised units i.e. $g_{\mu\gamma} = \emptyset \cdot \gamma_{\mu\gamma}$ (see eq. (2b)) so the actual form of the metric coefficient (in Jordan frame) in atomic units are as follows:

Solutions in §2:

$$e^\lambda = B_0 \cdot t^n [B_1 / ((3^n/2) - 1) t^{(1-3n/2)} - (3n(n-1)/(3n/2-1)\alpha) \log t - B_2],$$

or

$$e^\lambda = e^{bt} [(2b_1/3b)e^{-(3/2)bt} - 2bt/\alpha - b_2]^{-1}.$$

Solutions in §3:

Case I:

$$e^\lambda = e^\mu = e^{at} \left[\frac{3C_1}{4a} e^{-2at} - \frac{45a}{8\alpha} t - \frac{3C_2}{2} \right]^{2/3}$$

Case II:

$$e^\lambda = e^\mu = A_0 \cdot t^a \left[\frac{3C_1 \cdot t^{1-2a}}{2(2a-1)} + \frac{9a(5a-4)}{\alpha(1-2a)} \ln t - \frac{3}{2} C_2 \right]^{2/3}$$

or

$$e^\lambda = \lambda_0 t^a [(3/4)\{C_2 + C_1 \cdot \ln t - (15/2\alpha)(\ln t)^2\}]^{2/3}$$

$$e^\mu = \mu_0 t^{2-3a} [(3/4)\{C_2 + C_1 \cdot \ln t - (15/2\alpha)(\ln t)^2\}]^{2/3}.$$

Thus the solutions are not simple exponential functions or power functions of time but rather a combination of them (for inflationary model) or a combination of polynomial functions and logarithmic function (for power form). Therefore, for generalized scalar tensor theory with bulk viscosity it is possible to have some solution for the 'graceful exist' problem by a proper choice of $\omega(\emptyset)$.

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