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Spherically symmetric inhomogeneous dust collapse in higher dimensional space-time and cosmic censorship hypothesis

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Abstract. We consider a collapsing spherically symmetric inhomogeneous dust cloud in higher dimensional space-time. We show that the central singularity of collapse can be a strong curvature or a weak curvature naked singularity depending on the initial density distribution.

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1. Introduction

Cosmic censorship conjecture is still an outstanding open problem and possesses a prime position in the study of general relativity. The issues and intricacies regarding the final fate of gravitational collapse and having a bearing on the cosmic censorship hypothesis have very nicely been brought out in a number of review articles [1]. The existence of strong curvature naked singularities in gravitational collapse of spherically symmetric space-times with matter fields like inhomogeneous dust [2], perfect fluid [3], radiation [4], counter rotating particles with vanishing radial stresses [5] is now a well accepted phenomenon. These are counterexamples to strong cosmic censorship hypothesis in the sense that naked singularities that arise here are at least locally naked. Even then a suitable formulation of cosmic censorship hypothesis and its proof is still far away. Lately there has been growing interest in studying gravitational collapse in higher dimensions [6]. Ilha *et al* [6] have generalized the Oppenhimer–Snyder collapse model to higher dimensions. The idea that space-time should be extended from four to higher dimensions was introduced by Kaluza and Klein [7] to unify gravity and electromagnetism. Five dimensional (5D) space-time is particularly more relevant because both 10 D and 11 D supergravity theories yield solutions where a 5D space-time results after dimensional reduction [8]. Many papers on higher dimensional solutions [9] have appeared lately because of their implications in astrophysics, cosmology, string theory and particle physics.

An interesting problem that arises is the effect that higher dimensions can have on the formation of naked singularities. Sil and Chatterjee [10] studied dust collapse in five dimensional space-time. By considering a self-similar Tolman type model in higher dimensional space-time they showed the occurrence of a naked shell focusing singularity which may develop into a strong curvature singularity. Recently Ghosh and Saraykar [11] showed that strong curvature naked singularities arise as a result of radiation collapse in a five dimensional Vaidya space-time.

In this paper we consider the nature and structure of singularities arising in a non-self-similar dust collapse in 5D. We show that the central singularity of collapse may indeed be a (strong or weak curvature) naked one depending on the conditions of initial density distribution. To this end we follow the method as in [12] where initial data is given by a convergent power series.

Thus in §2, we describe five dimensional Tolman type dust model and show the existence of a naked shell focussing singularity. In §3 we discuss the strength of the naked singularity. We end the paper by giving concluding remarks in §4.

2. Dust collapse in five dimensional Tolman type model

A spherically symmetric inhomogeneous dust cloud in five dimensional space-time [10] is given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{R'^2}{1+f}\mathrm{d}r^2 + R^2(\mathrm{d}\theta_1^2 + \sin^2\theta_1\,\mathrm{d}\theta_2^2 + \sin^2\theta_1\sin^2\theta_2\mathrm{d}\theta_3^2), \quad (1)$$

where $f(r)$ is an arbitrary function of comoving coordinate r, satisfying $f > -1$. $R(t,r)$ is the physical radius at a time t of the shell labeled by r, in the sense that $4\pi R^2(r,t)$ is the proper area of the shell at time t. A prime denotes the partial derivative with respect to r.

The energy momentum tensor is given by

$$T^{ij} = \varepsilon\delta^i_t\delta^j_t, \quad (2)$$

where

$$\varepsilon(t,r) = \frac{3F'}{2R^3R'}. \quad (3)$$

The function $R(r,t)$ is the solution of

$$\dot{R}^2 = \frac{F(r)}{R^2} + f(r), \quad (4)$$

where an over dot denotes partial derivative with respect to t. The functions $F(r)$ and $f(r)$ are arbitrary, and result from the integration of the field equations.

For simplicity we shall confine ourselves to the marginally bound case $f(r) = 0$.

Since in the present discussion we are concerned with gravitational collapse, we require that

$$\dot{R}(t,r) < 0.$$

Hence we get

$$\dot{R} = \frac{-\sqrt{F}}{R}. \tag{5}$$

After integrating and using scaling freedom $R(r,0) = r$ we get

$$R^2 = r^2 - 2\sqrt{F}t. \tag{6}$$

According to (6) the area radius of the shell r shrinks to zero at the time $t_c(r)$ given by

$$t_c(r) = \frac{r^2}{2\sqrt{F}}. \tag{7}$$

The Kretschmann scalar is given by

$$K = \frac{28F'^2}{R^6R'^2} - \frac{144FF'}{R^7R'} + \frac{288F^2}{R^8}. \tag{8}$$

At $t = t_c(r)$, the Kretschmann scalar and energy density both diverge, indicating the presence of scalar polynomial curvature singularity [13]. The time and radial coordinates are respectively in the ranges $-\infty < t < t_c(r)$ and $0 \leq r < \infty$.

It has been shown that [14] shell crossing singularities (characterized by $R' = 0$ and $R > 0$) are gravitationally weak and hence such singularities need not be considered seriously. We therefore consider only the shell focusing singularity. We thus assume $R' > 0$ in the following discussion. We shall restrict ourselves to the study of future directed radial null geodesics. In order to check whether the singularity is naked, we examine the null geodesic equations for the tangent vectors $K^a = \mathrm{d}x^a/\mathrm{d}k$, where k is an affine parameter along the geodesics. For radial null geodesics, these are

$$K^t = \frac{\mathrm{d}t}{\mathrm{d}k} = \frac{P}{R}. \tag{9}$$

$$K^r = \frac{\mathrm{d}r}{\mathrm{d}k} = \frac{K^t}{R'} = \frac{P}{RR'}, \tag{10}$$

where the function $P(t,r)$ satisfies the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}k} + P^2\left(\frac{\dot{R}'}{R'R} - \frac{\dot{R}}{R^2} - \frac{1}{R^2}\right) = 0. \tag{11}$$

Let $u = r^\alpha (\alpha > 1)$, then

$$\frac{\mathrm{d}R}{\mathrm{d}u} = \frac{1}{\alpha r^{\alpha-1}}\left(\dot{R}\frac{\mathrm{d}t}{\mathrm{d}r} + R'\right).$$

From eq. (1) we see that, for outgoing radial null geodesics, $\mathrm{d}t/\mathrm{d}r = R'$, hence with the help of (5) above equation becomes

$$\frac{\mathrm{d}R}{\mathrm{d}u} = \frac{R'}{\alpha r^{\alpha-1}}\left(1-\frac{\sqrt{F}}{R}\right) = \frac{R'}{\alpha r^{\alpha-1}}\left(1-\frac{\Lambda}{X}\right) = U(X,u), \tag{12}$$

where

$$\Lambda = \frac{\sqrt{F}}{u}, \quad X = \frac{R}{u}. \tag{13}$$

If the null geodesics terminate in the past at the singularity with a definite tangent, then at the singularity the tangent to the geodesic $\mathrm{d}R/\mathrm{d}u$ is positive and must have a finite value. From eq. (12) we note that the $\mathrm{d}R/\mathrm{d}u$ is positive for $X > \Lambda$ i.e. $R > \sqrt{F}$. Thus boundary of the trapped surface i.e. apparent horizon is given by $R = \sqrt{F}$. Using this relation we find from eq. (6) that

$$t_{\mathrm{ah}}(r) = \frac{r^2}{2\sqrt{F}} - \frac{\sqrt{F}}{2} = t_c(r) - \frac{\sqrt{F}}{2},$$

where $t_{\mathrm{ah}}(r)$ denotes the time at which apparent horizon forms.

Since $F(r)$ is strictly positive for $r > 0$, with $F(r) = 0$ at $r = 0$, we have $t_{\mathrm{ah}}(r) < t_c(r)$ for $r > 0$ and $t_{\mathrm{ah}}(0) = t_c(0)$. Thus all other points on the singularity curve, except the point $r = 0$ are covered by the apparent horizon. We therefore consider the singularity of collapse at $r = 0$ i.e. the central singularity. We now find conditions on the initial data so that the central singularity of collapse is naked.

If the outgoing null geodesics are to terminate in the past at the singularity at $r = 0$, which occurs at time $t = t_c$ at which $R(t_c, 0) = 0$, then along these geodesics we have $R \to 0$ as $r \to 0$. After simplifying differential equation (12), we see that the right hand side of this equation is of the form 0/0 in the limit of approach to the singularity $(R = 0, u = 0)$. The point $R = 0, u = 0$ in the (R, u) plane is thus a singularity of the differential equation (12) (cf. [15]).

By using eq. (6) one can write

$$R' = \frac{RF'}{4F} + \left(1-\frac{rF'}{4F}\right)\frac{r}{R}, \tag{14}$$

$$= \frac{\eta X r^{\alpha-1}}{4} + \left(\frac{4-\eta}{4}\right)\frac{1}{r^{\alpha-1}X}, \tag{15}$$

where $\eta = rF'/F$.

The initial state of the spherically symmetric dust cloud is described in terms of the density and velocity profiles specified at an initial epoch of time from which collapse commences. We denote by $\rho(r) = \varepsilon(r, 0)$, the density distribution of the cloud at the starting epoch of collapse.

From eq. (3) we get

$$F(r) = (2/3)\int \rho(r) r^3 \mathrm{d}r. \tag{16}$$

We assume that the density $\rho(r)$ can be expanded [12] in a power series about the central density ρ_0:

$$\rho(r) = \rho_0 + \rho_1 r + \frac{\rho_2 r^2}{2!} + \frac{\rho_3 r^3}{3!} + \cdots \frac{\rho_n r^n}{n!} \cdots, \tag{17}$$

where $\rho_0 > 0$ and ρ_n stands for the nth derivative of ρ at $r = 0$.

Then F becomes

$$F(r) = F_0 r^4 + F_1 r^5 + F_2 r^6 + \cdots, \tag{18}$$

where

$$F_n = \frac{2}{3} \frac{\rho_n}{n!(n+4)} \quad \text{and} \quad n = 0, 1, 2, \ldots. \tag{19}$$

Also η appearing in eq. (15) is given by

$$\eta = \frac{rF'}{F} = \frac{\sum_0^\infty (n+4) F_n r^{n+4}}{\sum_0^\infty F_n r^{n+4}}. \tag{20}$$

Since we are interested in the behavior of η near the center, we can simplify this further to get

$$\eta(r) = 4 + \eta_1 r + \eta_2 r^2 + \eta_3 r^3 + \cdots, \tag{21}$$

where

$$\eta_1 = \frac{F_1}{F_0}, \quad \eta_2 = \frac{2F_2}{F_0} - \frac{F_1^2}{F_0^2}, \quad \eta_3 = \frac{3F_3}{F_0} - \frac{3F_1 F_2}{F_0^2} + \frac{F_1^3}{F_0^3}. \tag{22}$$

If all the derivatives ρ_n of the density vanish for $n \leq (q-1)$, and the qth derivative is the first nonvanishing derivative, then T_η^q, the qth term in the expression for η is

$$T_\eta^q = \boxed{\frac{qF_q}{F_0}} r^q. \tag{23}$$

Here q takes the values 1, 2 etc. and $T_\eta^0 = 4$.

In this case we can write $\eta(r)$ as

$$\eta(r) = 4 + \frac{qF_q}{F_0} r^q + O(r^{q+1}). \tag{24}$$

We use expression for $4 - \eta$ from eq. (24) keeping only the terms up to the order r^q and substituting in eq. (15) to get

$$R' = \frac{\eta X}{4} r^{\alpha - 1} - \frac{qF_q r^q}{4F_0 r^{\alpha-1} X}. \tag{25}$$

With the help of (25), eq. (12) becomes

$$\frac{\mathrm{d}R}{\mathrm{d}u} = \frac{1}{\alpha}\left(1 - \frac{\Lambda}{X}\right)\left(\frac{\eta X}{4} - \frac{\Theta}{X}\right) = U(X, u), \tag{26}$$

where

$$\Theta = \frac{qF_q r^q}{4F_0 r^{2(\alpha-1)}}. \tag{27}$$

Let us consider the limit X_0 of the tangent X along the null geodesic terminating at the singularity at $R = 0$, $u = 0$. Using L'Hospital's rule we get

$$X_0 = \lim_{\substack{R\to 0\\ u\to 0}} \frac{R}{u} = \lim_{\substack{R\to 0\\ u\to 0}} \frac{\mathrm{d}R}{\mathrm{d}u} = \lim_{\substack{R\to 0\\ u\to 0}} U(X,u) = U(X_0,0). \tag{28}$$

The necessary condition that the null geodesic emanates from the central singularity is the existence of the positive real root X_0 of the equation,

$$V(X_0) = 0, \tag{29}$$

where

$$V(X) = U(X,0) - X \tag{30}$$

$$= \frac{1}{\alpha}\left(1 - \frac{\Lambda_0}{X}\right)\left(\frac{\eta_0 X}{4} - \frac{\Theta_0}{X}\right) - X, \tag{31}$$

where $\Lambda_0 = \lim_{r\to 0}\Lambda, \eta_0 = \lim_{r\to 0}\eta$.

The constant α is to be determined by the requirement that Θ_0, the limiting value of Θ as $r \to 0$, should not be equal to zero or infinite, which gives

$$q = 2(\alpha - 1) \Rightarrow \alpha = 1 + q/2, \quad \Theta_0 = \frac{qF_q}{4F_0}. \tag{32}$$

Limiting value of the function $\Lambda = \sqrt{F}/r^\alpha$ is found by using (18) and (32) to be

$$\begin{aligned} \Lambda_0 &= 0, && q < 2 \\ &= \sqrt{F_0}, && q = 2 \\ &= \infty, && q > 2. \end{aligned} \tag{33}$$

Since Λ_0 takes different values for different choices of q, the nature of roots depends on the first nonvanishing derivative of density at the center. So we analyse the various cases one by one.

Case 1: $\rho_1 \neq 0$. In this case $q = 1, \alpha = 3/2$, and eq. (31) gives

$$X_0^2 = \frac{-F_1}{2F_0} = \frac{-2\rho_1}{5\rho_0}. \tag{34}$$

We assume the density to be decreasing outward, i.e $\rho_1 < 0$ and hence X_0 will be positive and thus singularity is naked.

Case 2: $\rho_1 = 0, \rho_2 \neq 0$. In this case, $q = 2, \alpha = 2$ and $\Lambda_0 = \sqrt{F_0}$. Since density is decreasing outward we take $\rho_2 < 0$.

Then equation $V(X_0) = 0$, gives

$$\frac{1}{2}\left(1-\frac{\sqrt{F_0}}{X_0}\right)\left(X_0-\frac{F_2}{2F_0X_0}\right)=X_0$$

i.e.

$$\frac{2X_0^3}{F_0^{3/2}}+\frac{2X_0^2}{F_0}+\frac{F_2X_0}{F_0^{5/2}}-\frac{F_2}{F_0^2}=0.$$

Define $y=X_0/\sqrt{F_0}, \xi=F_2/F_0^2$. The last equation then becomes

$$2y^3+2y^2+y\xi-\xi=0. \tag{35}$$

If this equation admits real positive roots then the singularity is naked. Numerical calculations show that the above equation has positive real roots if

$$\xi\leq(1-\sqrt{5})/(9-4\sqrt{5})\quad \text{i.e.}\quad \xi\leq -22.18033. \tag{36}$$

Thus whenever $\xi\leq -22.18033$, the central singularity is naked and it is covered if ξ is greater than this number. In the analogous four dimensional case, one gets a quartic equation and the shell focusing singularity is naked iff $\xi\leq -25.9904$ [12].

Case 3: $\rho_1=0, \rho_2=0$. This happens when the first two derivatives of the density are zero at the center, then $q\geq 3, \alpha\geq 5/2$. In this case $\Lambda_0=\infty$ and positive value of X_0 cannot satisfy eq. (31) and the collapse ends into a black hole.

Stability of occurrence of a naked singularity under small perturbations of initial density distributions (in an appropriate metric space) can be discussed along the lines of [16].

3. Strength of the singularity

We now examine the strength of this naked singularity. The strength of the singularity is an important issue because there have been attempts to relate it to stability [17]. The naked singularity is said to be strong if at least along one null geodesic with affine parameter k, with $k=0$ at the singularity, the following condition is satisfied in the limit of approach to the singularity:

$$S=\lim_{k\to 0}k^2R_{ab}K^aK^b>0, \tag{37}$$

where K^a is the tangent to the null geodesic and R_{ab} is the Ricci tensor.

In the dust case we find that

$$\begin{aligned} S=\lim_{k\to 0}k^2R_{ab}K^aK^b &= \lim_{k\to 0}k^2\frac{3F'(k^t)^2}{2R^3R'} \\ &= \frac{3\eta_0\Lambda_0^2}{2\alpha X_0^6}\lim_{k\to 0}\left(\frac{kP}{r^{2\alpha}}\right)^2, \end{aligned} \tag{38}$$

where $K^t=P/R$ and P satisfies eq. (11).

Using L'Hospital's rule and eqs (9)–(13) and the fact that at the singularity $r\to 0$, $X\to X_0$, we get

$$\lim_{k\to 0}\frac{kP}{r^{2\alpha}} = \frac{\alpha X_0^3}{\alpha X_0 + 2\Lambda_0(\alpha-1)}, \quad \text{if} \quad P_0 = \lim_{k\to 0} P = 0, \infty$$
$$= \frac{X_0^2}{2}, \qquad \text{elsewhere.} \tag{39}$$

Hence, we get

$$S = \lim_{k\to 0} k^2 R_{ab} K^a K^b = \frac{6\alpha\Lambda_0^2}{(\alpha X_0 + 2\Lambda_0(\alpha-1))^2} \quad \text{for} \quad P_0 = 0, \infty$$
$$= \frac{3\Lambda_0^2}{2\alpha X_0^2}, \quad \text{elsewhere.} \tag{40}$$

Hence by using eqs (32) and (33) we can write

$$S = \lim_{k\to 0} k^2 R_{ab} K^a K^b = 0, \quad \text{for} \quad q = 1$$
$$> 0, \quad \text{for} \quad q \geq 2.$$

However, from our earlier conclusions (from §2) naked singularity occurs only when $q \leq 2$, therefore the strong curvation condition is satisfied along singular geodesics only for the classes where $q = 2$.

Thus combining this result with the results in §2 one may write

i) If $\rho_1 < 0$, the singularity is naked and weak.
ii) If $\rho_1 = 0$ and $\rho_2 < 0$, the singularity is naked if $\xi = F_2/F_0^2 = 2\rho_2/\rho_0^2$ is less than the critical value $\xi_c = -22.18033$, and it is covered if $\xi > \xi_c$. Further, naked singularity is a strong curvature singularity.
iii) If $\rho_1 = 0, \rho_2 = 0$, the singularity is covered.

4. Conclusion

The Tolman-Bondi metric in the 4D case has been extensively used to study the formation of naked singularities in spherical gravitational collapse [2]. We have extended this study to a higher dimensional Tolman-Bondi metric and found that strong curvature naked singularities do arise for a different critical value. Also we have examined the strength of the naked singularity and found it gravitationally strong under certain conditions.

In conclusion, this offers a counter example to the cosmic censorship conjecture.

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