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Parametric resonance in neutrino oscillations in matter

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Abstract. Neutrino oscillations in matter can exhibit a specific resonance enhancement — parametric resonance, which is different from the MSW resonance. Oscillations of atmospheric and solar neutrinos inside the earth can undergo parametric enhancement when neutrino trajectories cross the core of the earth. In this paper we review the parametric resonance of neutrino oscillations in matter. In particular, physical interpretation of the effect and the prospects of its experimental observation in oscillations of solar and atmospheric neutrinos in the earth are discussed.

Keywords. Neutrino oscillations; parametric resonance.

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1. Introduction

It is well known that neutrino oscillations in matter can differ significantly from oscillations in vacuum, the best studied example being the Mikheyev–Smirnov–Wolfenstein (MSW) effect [1,2]. It is, however, much less known that the MSW effect is not the sole mechanism by which matter can enhance transitions between neutrinos of different flavor. The MSW effect enhances the probabilities of neutrino flavor transitions by amplifying neutrino *mixing*: the mixing angle in matter θ can become equal to $45°$ even if the vacuum mixing angle θ_0 is very small. It was pointed out about 12 years ago [3,4] that the probabilities of neutrino flavor transitions can also be strongly enhanced if the oscillation *phase* undergoes certain modification in matter. This can happen if the variation of the matter density along the neutrino path is correlated in a certain way with the change of the oscillation phase. This amplification of the neutrino oscillation probability in matter due to specific phase relationships has an interesting property that it can accumulate if the matter density profile along the neutrino path repeats itself, i.e. is periodic. The phenomenon is analogous to the resonance in dynamical systems whose parameters periodically vary with time – parametric resonance. It was therefore dubbed parametric resonance of neutrino oscillations [3,4]. While periodicity of the parameters of the system is useful, it is not really necessary: parametric resonance can occur even in stochastic media (see, e.g., [5]). The stochastic parametric resonance in neutrino oscillations was briefly discussed in [6].

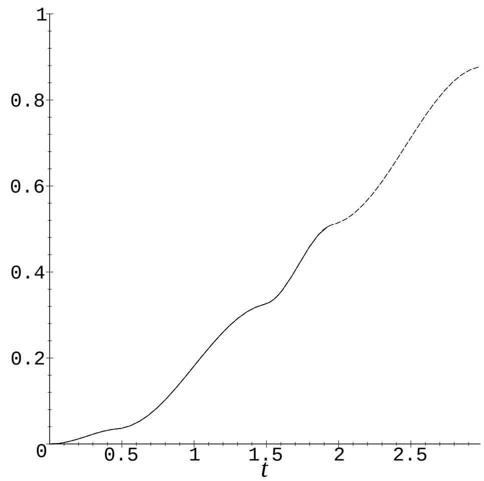

Figure 1. Solid curve: transition probability P for $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations in the earth as a function of the distance t (measured in units of the earth's radius R) along the neutrino trajectory. $\delta \equiv \Delta m^2/4E = 1.8 \times 10^{-13}$ eV, $\sin^2 2\theta_0 = 0.01$, $\Theta_n = 11.5°$. Dashed curve: the same for a hypothetical case of neutrino propagation over full two periods of density modulation ($t_{\max} = 2(R_m + R_c)$).

The parametric resonance can lead to large probabilities of neutrino flavor transition in matter even if the mixing angles *both* in vacuum *and* in matter are small. This happens because each half-wave oscillation of the transition probability is placed on the top of the previous one, i.e. the transition probability builds up (figure 1). If mixing angle in matter is very small (matter density is far from the MSW resonance density), the parametric resonance enhancement of neutrino oscillations can manifest itself only if the neutrinos pass through a large number of periods of density modulation, i.e. travel a sufficiently long distance. However, if matter density is not very far from the MSW resonance one, an interesting interplay between the MSW and parametric effects can occur. In particular, a strong parametric enhancement of neutrino oscillations can take place even if the neutrinos pass only through 1–2 periods of density modulation [6].

For the parametric resonance to occur, the exact shape of the density profile is not very important; what is important is that the change in the density be synchronized with the change of the oscillation phase. In particular, in [3,4] the case of the sinusoidal density profile was considered in which the neutrino evolution equation reduces to the Mathieu equation. In [4] the parametric resonance was also considered for neutrino oscillations in a matter with a periodic step function ('castle wall') density profile, which allows a very simple exact analytic solution. We will discuss this solution in §§2 and 4.

Although the parametric resonance in neutrino oscillations is certainly an interesting physical phenomenon, it requires that very special conditions be satisfied. Unfortunately, these conditions cannot be created in the laboratory because this would require either too long a baseline or neutrino propagation in a matter of too high a density (see §5 below).

Until recently it was also unclear whether a natural object exists where these conditions can be satisfied for any known source of neutrinos. This situation has changed with a very important observation by Liu and Smirnov [7] (see also [8]), who have shown that the parametric resonance conditions can be approximately satisfied for the oscillations of atmospheric ν_μ into sterile neutrinos ν_s inside the earth.

It is known that the earth consists of two main structures – the mantle and the core. Within the mantle and within the core the matter density changes rather slowly (the density variation scale is large compared to the typical oscillation lengths of atmospheric and solar neutrinos), but at their border it jumps sharply by about a factor of two. Therefore to a good approximation one can consider the mantle and the core as structures of constant densities equal to the corresponding average densities (two-layer model). Neutrinos coming to the detector from the lower hemisphere at zenith angles Θ in the range defined by $\cos\Theta = (-1) \div (-0.837)$ traverse the earth's mantle, core and then again mantle. Therefore such neutrinos experience a periodic 'castle wall' potential, and their oscillations can be parametrically enhanced. Even though the neutrinos pass only through '1.5 periods' of density modulations (this would be exactly one period and a half if the distances neutrinos travel in the mantle and in the core were equal), the parametric effects on neutrino oscillations in the earth can be quite strong. Subsequently, it has been pointed out in [9] that the parametric resonance conditions can also be satisfied (and to even a better accuracy) for the $\nu_2 \leftrightarrow \nu_e$ oscillations in the earth in the case of the ν_e–$\nu_{\mu(\tau)}$ mixing [10]. This, in particular, may have important implications for the solar neutrino problem. The parametric resonance in the oscillations of solar and atmospheric neutrinos in the earth was further explored in a number of papers [11–14].

In the present paper we review the parametric resonance in neutrino oscillations and its possible implications for oscillations of solar and atmospheric neutrinos in the earth. In §2 we discuss neutrino oscillations and their parametric enhancement in matter with 'castle wall' density profile. In §3 we discuss the physical interpretation of the parametric resonance in neutrino oscillations. In §4 the parametric resonance in oscillations of solar and atmospheric neutrinos in the earth is discussed. In §5 the parametric resonance conditions for neutrino oscillations in the earth are considered. In the last section the prospects of experimental observation of the parametric resonance in neutrino oscillations are discussed and the conclusions are given.

2. Neutrino oscillations in matter with 'castle wall' density profile

Consider oscillations in a 2-flavor neutrino system in a matter with periodic step function density profile [4,11]. We will be assuming that one period of density modulation consists of two parts of the lengths T_1 and T_2, with the corresponding effective matter densities N_1 and N_2 ('castle wall' density profile, figures 3, 5, 7). For the $\nu_e \leftrightarrow \nu_{\mu(\tau)}$ oscillations the effective matter density coincides with the electron number density, whereas for the $\nu_{e,\mu,\tau} \leftrightarrow \nu_s$ oscillations in an isotopically symmetric matter it is a factor of two smaller. The parametric resonance in such a system occurs when the oscillations phases $2\phi_1$ and $2\phi_2$ acquired over the intervals T_1 and T_2 are odd integer multiples of π [4,7,8]. Let us denote

$$\delta = \frac{\Delta m^2}{4E}\,,$$
$$V_i = \frac{G_{\rm F}}{\sqrt{2}}\,N_i, \quad \omega_i = \sqrt{(\cos 2\theta_0\,\delta - V_i)^2 + (\sin 2\theta_0\,\delta)^2} \quad (i=1,\ 2)\,. \tag{1}$$

Here E, Δm^2 and θ_0 are the neutrino energy, mass squared difference and vacuum mixing angle, respectively. The difference of the neutrino eigenenergies in a matter of density N_i is $2\omega_i$, so that the oscillations phases acquired over the intervals T_1 and T_2 are

$$2\phi_1 = 2\omega_1 T_1\,, \qquad 2\phi_2 = 2\omega_2 T_2\,. \tag{2}$$

The evolution of a system of oscillating neutrinos is conveniently described by the evolution matrix U, which in the case of the 2-flavor system is a 2×2 unitary matrix. For any interval of time over which the matter density is constant the evolution matrix can be trivially found; the evolution matrix in a matter with a step-function density profile is then just the product of the corresponding constant-density evolution matrices. In particular, for one period of density modulation $T = T_1 + T_2$ the evolution matrix is [11]

$$U_T = U_{T_2} U_{T_1} = Y - i\boldsymbol{\sigma}\mathbf{X} = \exp[-i(\boldsymbol{\sigma}\hat{\mathbf{X}})\Phi]\,. \tag{3}$$

Here $\boldsymbol{\sigma}$ are the Pauli matrices in the flavor space,

$$Y = c_1 c_2 - (\mathbf{n_1 n_2}) s_1 s_2\,, \tag{4}$$

$$\mathbf{X} = s_1 c_2\,\mathbf{n}_1 + s_2 c_1\,\mathbf{n}_2 - s_1 s_2\,(\mathbf{n}_1 \times \mathbf{n}_2)\,, \tag{5}$$

$$\Phi = \arccos Y = \arcsin X\,, \qquad \hat{\mathbf{X}} = \frac{\mathbf{X}}{X}\,, \tag{6}$$

and we have used the notation

$$s_i = \sin\phi_i\,, \quad c_i = \cos\phi_i\,, \quad \phi_i = \omega_i T_i\,, \tag{7}$$

$$\mathbf{n}_i = (\sin 2\theta_i,\ 0,\ -\cos 2\theta_i) \qquad (i = 1,\ 2). \tag{8}$$

Here θ_i is the mixing angle in matter at the density N_i. Notice that $Y^2 + \mathbf{X}^2 = 1$ as a consequence of unitarity of U_T.

The evolution matrix for n periods ($n = 1, 2, \ldots$) can be obtained by raising U_T to the nth power:

$$U_{nT} \equiv U(t = nT,\ 0) = \exp[-i(\boldsymbol{\sigma}\hat{\mathbf{X}})n\Phi]\,. \tag{9}$$

Equations (3)–(9) give the exact solution of the evolution equation for any instant of time that is an integer multiple of the period T. In order to obtain the solution for $nT < t < (n+1)T$ one has to evolve the solution at $t = nT$ by applying the evolution matrix

$$U_1(t,\ nT) = \exp[-iH_1\cdot(t - nT)] \tag{10}$$

for $nT < t < nT + T_1$ or

$$U_2(t,\ nT + T_1)U_1 = \exp[-iH_2\cdot(t - nT - T_1)]\exp[-iH_1 T_1] \tag{11}$$

for $nT + T_1 \le t < (n+1)T$, with $H_{1,2}$ being the effective Hamiltonians of neutrino system at the densities N_1 and M_2, respectively.

2.1 *Parametric resonance*

Assume that the initial neutrino state at $t = 0$ is a flavor eigenstate ν_a. The probability of finding another flavor eigenstate ν_b at a time $t > 0$ (transition probability) is then $P(\nu_a \to \nu_b,\ t) = |U_{21}(t)|^2$ where $U(t)$ is the evolution matrix. For neutrino oscillations in matter with the 'castle wall' density profile this probability can reach its maximum value when the parametric resonance conditions are satisfied. These conditions can be written as [3,4,6–11,15]

$$\phi_1 = \frac{\pi}{2} + k\pi\,, \qquad \phi_2 = \frac{\pi}{2} + k'\pi\,, \qquad k, k' = 0, 1, 2, ... \tag{12}$$

At the resonance, the transition probability for the evolution over n periods of density modulation takes a simple form

$$P(\nu_a \to \nu_b,\ t = nT) = \sin^2[n(2\theta_2 - 2\theta_1)]\,. \tag{13}$$

Let us first assume that the densities N_1, N_2 are either both below the MSW resonance density $N_{\rm MSW}$ which is determined from $G_{\rm F} N_{\rm MSW}/\sqrt{2} = \cos 2\theta_0\,\delta$ or they are both above it. This means that the mixing angles $\theta_{1,2}$ satisfy $\theta_{1,2} < \pi/4$ or $\theta_{1,2} > \pi/4$, respectively. It is easy to see that in this case the difference $2\theta_2 - 2\theta_1$ is always farther away from $\pi/2$ than either $2\theta_1$ or $2\theta_2$. This means in this case the transition probability for evolution over one period cannot exceed the maximal transition probabilities in matter of constant density equal to either N_1 or N_2, namely, $\sin^2 2\theta_1$ or $\sin^2 2\theta_2$. However, the parametric resonance does lead to an important gain. In a medium of constant density N_i the transition probability can never exceed $\sin^2 2\theta_i$, no matter how long the distance that neutrinos travel. On the contrary, in the matter with 'castle wall' density profile, if the parametric resonance conditions (12) are satisfied, the transition probability can become large provided neutrinos travel large enough distance. It can be seen from (13) that the transition probability can become quite sizeable even for small $\sin^2 2\theta_1$ and $\sin^2 2\theta_2$ provided that neutrinos have traveled sufficiently large distance. This is illustrated in figures 2 and 3 for the case $N_1, N_2 < N_{\rm MSW}$ (the transition probability in the case $N_1, N_2 > N_{\rm MSW}$ has a similar behavior). The number of periods neutrinos have to pass in order to experience a complete (or almost complete) conversion is

$$n \simeq \frac{\pi}{4(\theta_1 - \theta_2)}\,. \tag{14}$$

Consider now the case $N_1 < N_{\rm MSW} < N_2$ ($\theta_1 < \pi/4 < \theta_2$). The transition probability over n periods at the parametric resonance is again given by eq. (13). However in this case, for $\theta_2 > \pi/4 + \theta_1/2$ (which is always satisfied for small mixing in matter), one has $\sin^2(2\theta_2 - 2\theta_1) > \sin^2 2\theta_1,\ \sin^2 2\theta_2$. This means that *even for the time interval equal to one period of matter density modulation the transition probability exceeds the maximal probabilities of oscillations in matter of constant densities N_1 and N_2*. This parametric enhancement is further magnified in the case of neutrinos traveling over 'one and a half' periods of density modulation, which has important implications for neutrinos traversing the earth. The case $N_1 < N_{\rm MSW} < N_2$ is illustrated in figures 1, 4, 5 and 6, 7.

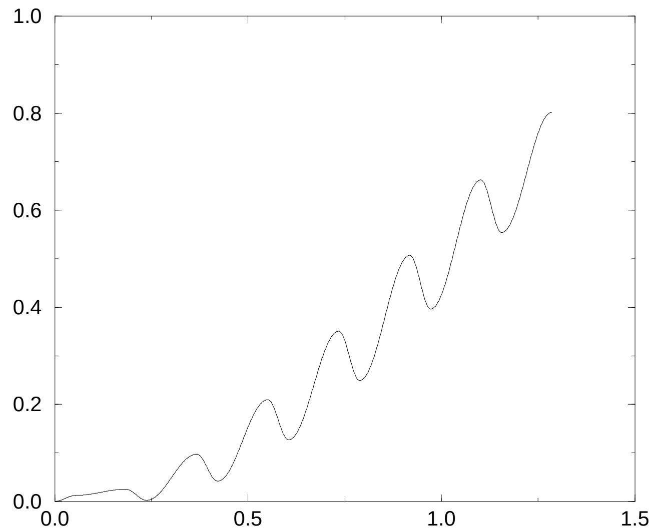

Figure 2. Coordinate dependence of the neutrino flavor transition probability P in a matter with the castle wall density profile. $\sin^2 2\theta_0 = 0.01$, $\delta = 10^{-12}$ eV, $V_1 = 10^{-13}$ eV, $V_2 = 6.33 \times 10^{-13}$ eV, $T_1 = 5.4 \times 10^{-2}$, $T_2 = 0.1296$, all distances are in units of $R = 3.23 \times 10^{13}$ eV^{-1}.

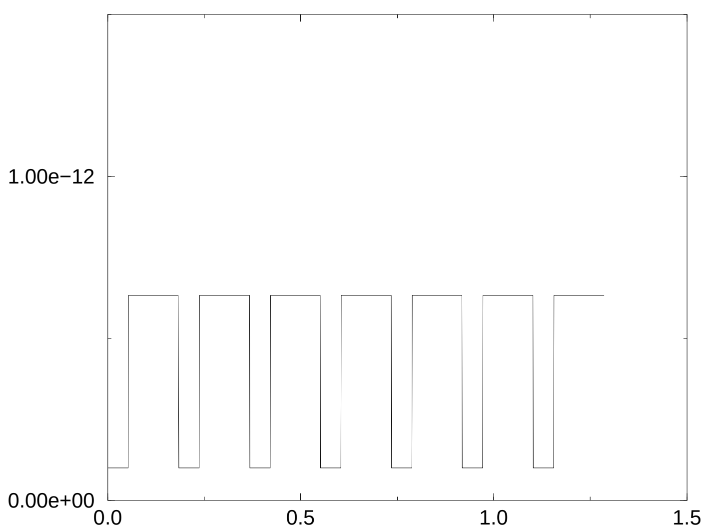

Figure 3. Coordinate dependence of the matter-induced neutrino potential $[(G_{\mathrm{F}}/\sqrt{2}\times$ (density profile)$]$ for the case shown in figure 2.

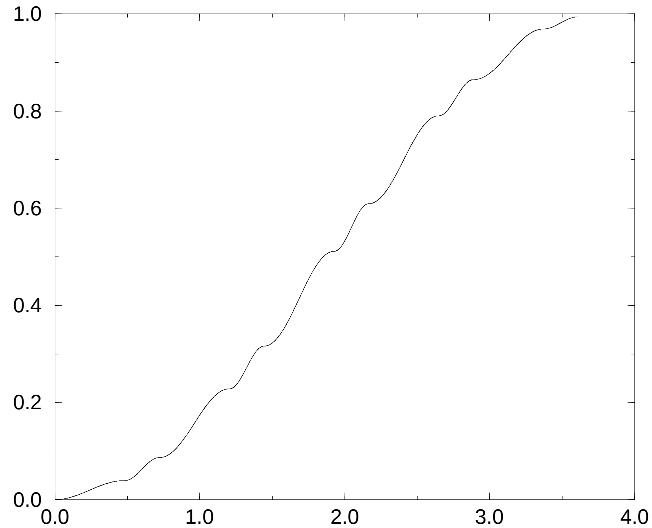

Figure 4. Same as figure 2 but for $\delta = 2\times10^{-13}$ eV, $V_2 = 4\times10^{-13}$ eV, $T_1 = 0.4814$, $T_2 = 0.2408$.

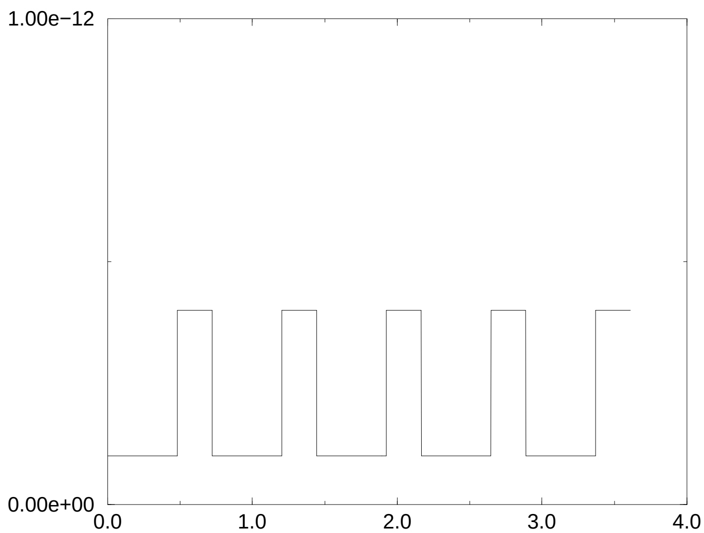

Figure 5. Coordinate dependence of the matter-induced neutrino potential for the case shown in figure 4.

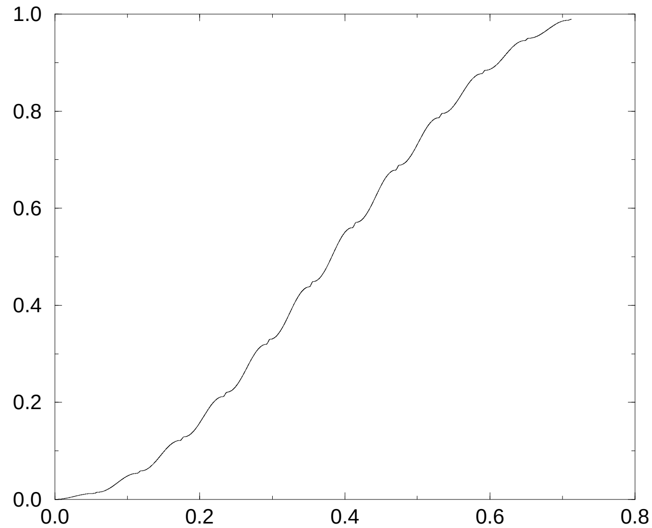

Figure 6. Same as figure 2 but for $\delta = 10^{-12}$ eV, $V_2 = 10^{-11}$ eV, $T_1 = 5.4 \times 10^{-2}$, $T_2 = 5.4 \times 10^{-3}$.

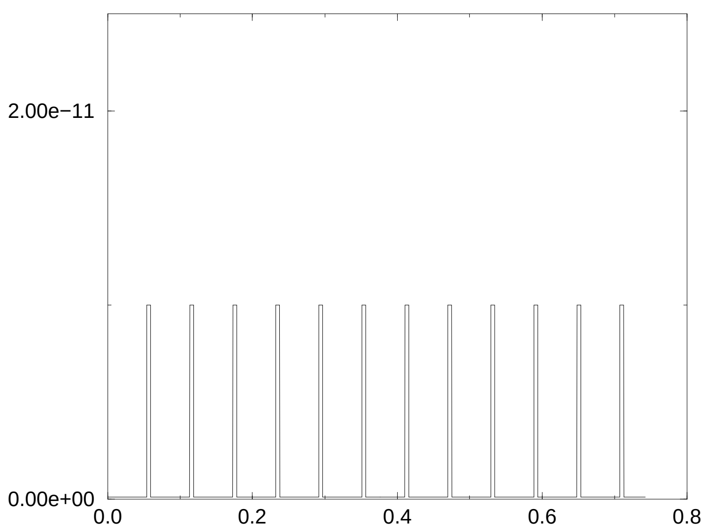

Figure 7. Coordinate dependence of the matter-induced neutrino potential for the case shown in figure 6.

3. Physical interpretation of the parametric resonance in neutrino oscillations

As we have seen, the parametric resonance can strongly enhance the probability of flavor transitions even if the lepton mixing angles both in matter and in vacuum is small. This fact can be given a very simple physical interpretation [16].

Neutrino oscillations in matter of constant density proceed exactly as the oscillations in vacuum, the only difference being that the oscillation amplitude and length are different from those in vacuum. If the vacuum mixing angle θ_0 is small and in addition matter density is not close to the MSW resonance one, the amplitude of neutrino oscillations in matter, $\sin^2 2\theta$, and therefore the transition probability, is small.

The situation can be drastically different in the case of the 'castle wall' density profile. Consider first neutrino evolution during the first part of the period of density modulation (i.e. over the time interval T_1). The matter density during this interval of time is constant: $N(t) = N_1$. If the first of the conditions (12) is satisfied, at the end of this interval the transition probability reaches $\sin^2 2\theta_1$ which is the maximal value possible in a matter of constant density N_1. If the density stayed constant, the transition probability would have started decreasing and would have returned to zero at the time $2T_1$. However, at $t = T_1$ the matter density jumps to a value N_2. If now the second of the conditions in (12) is also satisfied, and if in addition

$$N_1 < N_{\rm MSW} < N_2 \qquad (\theta_1 < \pi/4 < \theta_2)\,, \tag{15}$$

the transition probability will continue increasing instead of decreasing (figures 1, 4, 6). In fact, the second half-wave of neutrino oscillations is similar to the first one. This happens because of the violation of the adiabaticity of neutrino oscillations by a sudden change of the matter density and because this change is correlated with the change of the oscillation phase. The transition probability over one period of density modulation is

$$P(\nu_a \to \nu_b, T) = \sin^2(2\theta_2 - 2\theta_1)\,, \tag{16}$$

which, as was discussed in the previous section, exceeds both $\sin^2 2\theta_1$ and $\sin^2 2\theta_1$ if the condition (15) is satisfied. If the matter density profile is periodic, the increase of the transition probability accumulates: The parametric resonance puts each half-wave increase of the oscillation curve on the top of the previous one, leading to a fast growth of the transition probability (figures 1, 4, 6).

If the parametric resonance conditions (12) are satisfied but the condition (15) is not, the transition probability starts decreasing after the first half-wave increase. However it does not reach zero, and the decrease is followed by another increase. As a result, the transition probability builds up and can reach unity (figure 2). In this case, however, the increase of the transition probability is less fast than when the condition (15) is satisfied. Similar situation takes place if (15) is obeyed, but the parametric resonance conditions (12) are only approximately satisfied, i.e. there is a small detuning.

We shall now illustrate once again the importance of a correlated change of the oscillation phase and matter density profile along the neutrino path. In figures 6 and 7 the coordinate dependence of the transition probability and matter density profile are shown for a specific case in which conditions (12) and (15) are fulfilled. It can be seen from these figures that the probability increase during the time intervals T_2, which correspond to the effective matter density N_2, is very small, and, in addition, in this case $T_2 \ll T_1$. One could therefore conclude that the evolution during these intervals is unimportant. However,

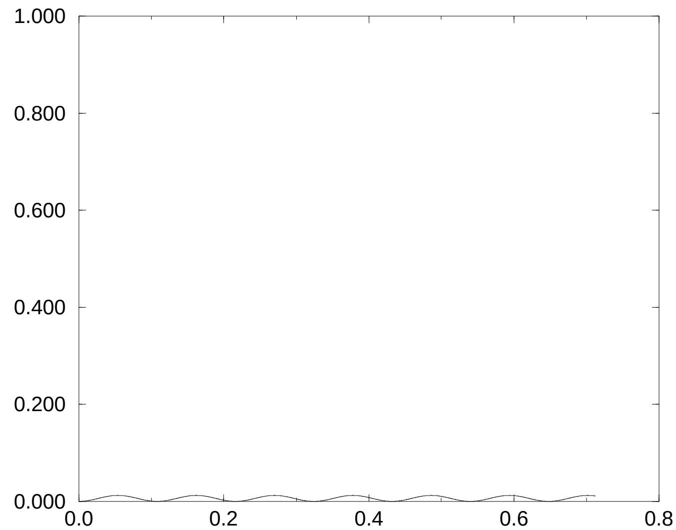

Figure 8. Same as figure 6 but for $V_2 = V_1$ ($V(t) = V_1 = \mathrm{const}$).

this conclusion is wrong: if one removes the ‘spikes’ in the matter density profile of figure 7, i.e. replaces it by the profile $N(t) = N_1 = $ const., the resulting transition probability will be very small at all times (figure 8).

4. Parametric resonance in neutrino oscillations in the earth

4.1 *Evolution of oscillating neutrinos in the earth*

As was pointed out in the Introduction, the earth consists of two main structures, the mantle and the core, which for the purposes of neutrino oscillations can to a very good approximation be considered as layers of constant density. We shall consider neutrino oscillations in the earth in this two-layer approximation. Neutrinos coming to the detector from the lower hemisphere of the earth at zenith angles Θ in the range $\cos\Theta = (-1) \div (-0.837)$ (nadir angle $\Theta_n \equiv 180^\circ - \Theta \leq 33.17^\circ$) traverse the earth's mantle, core and then again mantle, i.e. three layers of constant density with the third layer being identical to the first one. Therefore such neutrinos experience a periodic ‘castle wall’ potential, and their oscillations can be parametrically enhanced. Although the neutrinos propagate in this case only through three layers (‘1.5 periods’ of density modulation), the parametric enhancement of the transition probability can be very strong.

The evolution matrix in this case is $U = U_{T_1} U_{T_2} U_{T_1}$. It can be parametrized as

$$U = Z - i\boldsymbol{\sigma}\mathbf{W}\,, \qquad Z^2 + \mathbf{W}^2 = 1\,. \tag{17}$$

The matrix U describes the evolution of an arbitrary initial state and therefore contains all the information relevant for neutrino oscillations. In particular, the probabilities of the neutrino flavor oscillations P and of $\nu_2 \leftrightarrow \nu_e$ oscillations P_{2e} are given by [11]

$$P = W_1^2 + W_2^2\,, \qquad P_{2e} = \sin^2\theta_0 + W_1(W_1\cos 2\theta_0 + W_3 \sin 2\theta_0)\,. \quad (18)$$

We have now to identify the effective densities N_1 and N_2 with the average matter densities N_m and N_c in the earth's mantle and core, respectively; similarly, we change the notation $V_{1,2} \to V_{m,c}$, $\phi_{1,2} \to \phi_{m,c}$ and $\theta_{1,2} \to \theta_{m,c}$.

In the two-layer approximation, the parameters Z, $\mathbf{W}$ have a very simple form [11]:

$$Z = 2\cos\phi_m\, Y - \cos\phi_c\,, \quad (19)$$

$$\mathbf{W} = \Big(2\sin\phi_m \sin 2\theta_m\, Y + \sin\phi_c \sin 2\theta_c\,,\ 0\,,$$
$$-\left(2\sin\phi_m \cos 2\theta_m\, Y + \sin\phi_c \cos 2\theta_c\right)\Big)\,. \quad (20)$$

Here the vector $\mathbf{W}$ was written in components, and the parameter Y was defined in (4). At the parametric resonance, i.e. when the conditions (12) are satisfied, the neutrino flavor transition probability takes the value [7,8],

$$P = \sin^2(2\theta_c - 4\theta_m)\,, \quad (21)$$

whereas the probability of the $\nu_2 \leftrightarrow \nu_e$ transitions is [9]

$$P_{2e} = \sin^2(2\theta_c - 4\theta_m + \theta_0)\,. \quad (22)$$

These probabilities can be close to unity (the arguments of the sines close to $\pi/2$) even if the amplitudes of neutrino oscillations in the mantle, $\sin^2 2\theta_m$, and in the core, $\sin^2 2\theta_c$, are rather small. This can happen if the neutrino energy lies in the range $E_c < E < E_m$, where E_m and E_c are the values of the energy that correspond to the MSW resonance in the mantle and in the core of the earth. This condition is equivalent to the one in eq. (15). The probability P_{2e} is relevant for the description of the oscillations of solar neutrinos in the earth [19,20]. In the case of small mixing angle MSW solution of the solar neutrino problem, $\sin^2 2\theta_0 < 10^{-2}$ [21], and P_{2e} practically coincides with P unless both probabilities are very small.

The trajectories of neutrinos traversing the earth are determined by their nadir angle $\Theta_n = 180^\circ - \Theta$. The distances R_m and R_c that neutrinos travel in the mantle (each layer) and in the core are given by

$$R_m = R\left(\cos\Theta_n - \sqrt{r^2/R^2 - \sin^2\Theta_n}\right)\,,$$
$$R_c = 2R\sqrt{r^2/R^2 - \sin^2\Theta_n}\,. \quad (23)$$

Here $R = 6371\,\text{km}$ is the earth's radius and $r = 3486\,\text{km}$ is the radius of the core. The matter density in the mantle of the earth ranges from 2.7 g/cm^3 at the surface to 5.5 g/cm^3 at the bottom, and that in the core ranges from 9.9 to 12.5 g/cm^3 (see, e.g., [22]). The electron number fraction Y_e is close to 1/2 both in the mantle and in the core. Taking the average matter densities in the mantle and core to be 4.5 and 11.5 g/cm^2 respectively, one finds for the $\nu_e \leftrightarrow \nu_{\mu,\tau}$ oscillations involving only active neutrinos the following values of V_m and V_c: $V_m = 8.58 \times 10^{-14}$ eV, $V_c = 2.19 \times 10^{-13}$ eV. For transitions involving sterile neutrinos $\nu_e \leftrightarrow \nu_s$ and $\nu_{\mu,\tau} \leftrightarrow \nu_s$, these parameters are a factor of two smaller.

4.2 *Parametric resonance conditions for neutrino oscillations in the earth*

If the parametric resonance conditions (12) are satisfied, strong parametric enhancement of the oscillations of core crossing neutrinos in the earth can occur [7–13], see figure 1 [23]. We shall now discuss these conditions. The phases ϕ_m and ϕ_c depend on the neutrino parameters Δm^2, θ_0 and E and also on the distances R_m and R_c that the neutrinos travel in the mantle and in the core. The path lengths R_m and R_c vary with the nadir angle; however, as can be seen from (23), their changes are correlated and they cannot take arbitrary values. Therefore if for some values of the neutrino parameters a value of the nadir angle Θ_n exists for which, for example, the first condition in eq. (12) is satisfied, it is not obvious if at the same value of Θ_n the second condition will be satisfied as well. In other words, it is not clear if the parametric resonance conditions can be fulfilled for neutrino oscillations in the earth for at least one set of the neutrino parameters Δm^2, θ_0 and E. However, as was shown in [9,11], not only the parametric resonance conditions are satisfied (or approximately satisfied) for a rather wide range of the nadir angles covering the earth's core, they are fulfilled for the ranges of neutrino parameters which are of interest for the neutrino oscillations solutions of the solar and atmospheric neutrino problems. In particular, the conditions for the principal resonance ($k = k' = 0$) are satisfied to a good accuracy for $\sin^2 2\theta_0 \lesssim 0.1$, $\delta \simeq (1.1 \div 1.9) \times 10^{-13}$ eV^2, which includes the ranges relevant for the small mixing angle MSW solution of the solar neutrino problem and for the subdominant $\nu_\mu \leftrightarrow \nu_e$ and $\nu_e \leftrightarrow \nu_\tau$ oscillations of atmospheric neutrinos [24].

The fact that the parametric resonance conditions can be satisfied so well for neutrino oscillations in the earth is rather surprising. It is a consequence of a number of remarkable numerical coincidences. It has been known for some time [7,26,27] that the potentials V_m and V_c corresponding to the matter densities in the mantle and core, the inverse radius of the earth R^{-1}, and typical values of $\delta \equiv \Delta m^2/4E$ of interest for solar and atmospheric neutrinos, are all of the same order of magnitude – (3×10^{-14}–3×10^{-13}) eV. It is this surprising coincidence that makes appreciable earth effects on the oscillations of solar and atmospheric neutrinos possible. However, for the parametric resonance to take place, a coincidence by an order of magnitude is not sufficient: the conditions (12) have to be satisfied at least within a 50% accuracy [11]. This is exactly what takes place. In addition, in a wide range of the nadir angles Θ_n, with changing Θ_n the value of the parameter δ at which the resonance conditions (12) are satisfied slightly changes, but the fulfillment of these conditions is not destroyed.

In this row of mysterious coincidences, at least the last one – the stability of the parametric resonance conditions with respect to variations of the nadir angle – has a simple explanation. It is related to the fact that, due to the spherical geometry of the earth, with increasing nadir angle R_m increases and R_c decreases so that in a large interval of the nadir angles covering the earth's core that the sum $1/R_c + 1/R_m$ is almost constant. For more details, see ref. [14].

The parametric enhancement of neutrino oscillations in the earth can also occur when either k or k' in eq. (12) or both are different from zero (higher-order parametric resonances). However, the corresponding resonance conditions can only be satisfied for the values of neutrino mass squared differences and mixing angles which are of no practical interest for any known source of neutrinos, possible exception being the hep component of the solar neutrino flux [14].

5. Can the parametric resonance in neutrino oscillations be observed?

Besides being an interesting physical phenomenon, the parametric resonance in neutrino oscillations can provide us with an important additional information about neutrino properties. Therefore experimental observation of this effect would be of considerable interest. We shall now discuss the prospects for experimental observation of the parametric resonance in neutrino oscillations in the earth, having in mind mainly the principal resonance. There are two main sources of neutrinos for which the parametric resonance can be important – atmospheric neutrinos and solar neutrinos. Both sources have their advantages and disadvantages from the point of view of the possibility of observation of the parametric resonance. We shall now briefly discuss them.

We start with atmospheric neutrinos. The parametric resonance can occur in the $\nu_\mu \leftrightarrow \nu_s$ [7,8] and also in the subdominant $\nu_e \leftrightarrow \nu_{\mu(\tau)}$ channels of oscillations [12]. It can affect the distributions of μ-like events and also (in the case of the $\nu_e \leftrightarrow \nu_{\mu(\tau)}$ oscillations) lead to interesting peculiarities in the zenith angle distributions of the multi-GeV e-like events.

The observation of the parametric effects is hampered by the loose correlation between the directions of the momenta of atmospheric neutrinos and of the charged leptons which they produce and which are actually detected. Because of this the trajectories of neutrinos coming to the detector are not known very precisely. In addition, the data are presented for certain samples of events (sub-GeV, multi-GeV, upward through-going, upward stopping) which includes collecting data over rather wide energy intervals. The contributions of the parametric peaks may therefore be integrated over together with other possible enhancement peaks – due to the MSW resonances in the mantle and in the core, making the distinction between these effects difficult. Also, strong resonance enhancement effects (both parametric and MSW) can only occur either for neutrinos or for antineutrinos, depending on the sign of Δm^2 [28]. The present atmospheric neutrino experiments do not distinguish between neutrinos and antineutrinos, therefore possible matter effects are 'diluted' in the sum of the ν- and $\bar{\nu}$-induced events. At certain values of the ratio of the muon and electron neutrino fluxes $r(E, \Theta_n)$ depending on the value of mixing angle θ_{23} the parametric effects on e-like events are suppressed [12].

Atmospheric neutrinos have some advantages for observation of the parametric resonance in neutrino oscillations in the earth. Neutrinos come to the detectors from all directions, which means that practically the whole solid angle covering the earth's core will contribute to the effect. There are no additional suppression factors due to a specific composition of the incoming neutrino flux which may quench the earth's matter effect on the oscillations of solar neutrinos (see below). Parametric effects may provide a sensitive probe of the neutrino mixing angle θ_{13} with sensitivity possibly going beyond that of the long-baseline accelerator and reactor experiments [12,29]. Possible ways of improving the prospects of the experimental observation of the parametric effects in the atmospheric neutrino oscillations include using various energy cuts, finer zenith angle binning and detectors capable of detecting the recoil nucleon, which would enable one to reconstruct the direction of an incoming neutrino [12,29]. It would also be highly desirable to have detectors that can determine the charge of the observed electrons and muons, i.e. discriminate between neutrinos and antineutrinos.

Solar neutrinos can experience a strong parametric enhancement of their oscillations in the earth if the small mixing angle MSW effect is the correct explanation of the the solar neutrino deficit [9,11]. The parametric enhancement can occur in a wide range of values of $\sin^2 2\theta_0$ and for the nadir angles Θ_n almost completely covering the core of the earth. The trajectory of each detected neutrino is exactly known. For boron neutrinos the resonance occurs at the values of Δm^2 which correspond to the central part of the allowed interval for the small mixing angle MSW solution of the solar neutrino problem.

However, there are some disadvantages, too. Unfortunately, due to their geographical location, the existing solar neutrino detectors have a relatively low time during which solar neutrinos pass through the core of the earth to reach the detector every calendar year. The super-Kamiokande detector has a largest fractional core coverage time equal to 7%. In [26] it was suggested to build a new detector close to the equator in order to increase the sensitivity to the earth regeneration effect; this would also maximize the parametric resonance effects in oscillations of solar neutrinos in the earth. In the case of the MSW solutions of the solar neutrino problem the probability $P_{\rm SE}$ of finding a solar ν_e after it traverses the earth depends sensitively on the average ν_e survival probability in the sun $\bar{P}_{\rm S}$ [19,20]:

$$P_{\rm SE} = \bar{P}_{\rm S} + \frac{1-2\bar{P}_S}{\cos 2\theta_0}\left(P_{2e} - \sin^2\theta_0\right). \tag{24}$$

The probability P_{2e} can experience a strong parametric enhancement, but in the case of small mixing angle MSW solution of the solar neutrino problem the probability $\bar{P}_{\rm S}$ for the super-Kamiokande and SNO experiments turns out to be rather close to 1/2. This means that the effects of passage through the earth on solar neutrinos should be strongly suppressed. The current best fit of the solar neutrino data is not far from the line in the parameter space where $\bar{P}_{\rm S}$ is exactly equal to 1/2 and $P_{\rm SE} = \bar{P}_{\rm S}$ (i.e. the earth matter effects are absent). Whether or not it will be possible to observe the parametric resonance in the oscillations of solar neutrinos in the earth depends on how close to this line the true values of $\sin^2 2\theta_0$ and Δm^2 are. By now the super-Kamiokande experiment has not observed, within its experimental accuracy, any enhancement of neutrino signal for earth core crossing neutrinos [30]. This can be because the parametric enhancement of the neutrino oscillations in the earth does not occur (e.g. if the true solution of the solar neutrino problem is vacuum oscillations or large mixing angle MSW effect), or because the values of Δm^2 and $\sin^2 2\theta_0$ are too close to those at which $\bar{P}_{\rm S} = 1/2$. Hopefully, with accumulated statistics of the super-Kamiokande and forthcoming data from the SNO experiment the situation will soon be clarified.

It is interesting to note that the super-Kamiokande data on the zenith angle dependence of the solar neutrino events seems to indicate some deficiency of the events due to the core-crossing neutrinos rather than an excess [30], although it is not statistically significant. Should this deficiency be confirmed by future data with better statistics, it could have a natural explanation in terms of the parametric resonance of neutrino oscillations. As follows from (24), the parametric enhancement of P_{2e} for core crossing neutrinos can lead to a deficiency of the events if the neutrino parameters are in the small-$\sin^2 2\theta_0$ part of the allowed region which corresponds to $P_{\rm S} > 1/2$ (see, e.g., figure 10 in ref. [31]). In this case one should also have an 'opposite sign' overall day–night effect (fewer events during the night than during the day). In any case, given the current experimental constraints on the neutrino parameters, if the small mixing angle MSW effect is the true solution of the solar neutrino problem, the only hope to observe earth matter (day–night) effects on solar neutrinos seems to be through the parametric resonance of oscillations of core crossing neutrinos.

As we have seen, observing the parametric resonance in oscillations of solar and atmospheric neutrinos in the earth is not an easy task. Can one create the necessary matter density profile and observe the parametric resonance in neutrino oscillations in the laboratory (i.e. short-baseline) experiments? Unfortunately, the answer to this question seems to be negative. The parametric resonance can occur when the mean oscillation length in matter approximately coincides with the matter density modulation length [3,4,6]: $l_m \simeq L$. In a matter of density N_i the oscillation length is given by $l_m = \pi/\omega_i$ where ω_i was defined in (1). Let us require $l_m \lesssim 1$ km. Assume first that $V_i \gtrsim \delta$, i.e. ω_i are dominated by matter density terms. Than for $l_m \lesssim 1$ km one would need a matter of mass density $\rho_i \geq 3.3\times 10^4$ g/cm^3, clearly not a feasible value. Conversely, for $\rho_i \leq 10$ g/cm^3, one finds $l_m \gtrsim 3300$ km, a distance comparable with the earth's radius. Consider now the opposite case, $\delta \gg V_i$. Then the oscillation length in matter essentially coincides with the vacuum oscillations length which in principle can be rather short provided that the vacuum mixing angle θ_0 is small (otherwise this would contradict reactor and accelerator data). However, in this case there is another problem. Requiring $l_m \lesssim 1$ km one finds $\delta \gtrsim 2.5 \times 10^{-10}$ eV. For $\rho_i \lesssim 10$ g/cm^3 one therefore has $V_i/\delta \lesssim 10^{-3}$. This means that the mixing angles in matter are very close to the vacuum one, $\theta_i \simeq \theta_0(1 + V_i/\delta)$, and so their difference is very small: $\Delta\theta = \theta_2 - \theta_1 \simeq (\Delta V/\delta)\,\theta_0 \lesssim 10^{-3}\theta_0$. When the difference of mixing angles in matter is small, the parametric effects can manifest themselves only if neutrinos travel over a large number of periods, $n \simeq \pi/4\Delta\theta$ [see (14)]. Therefore in this case the necessary baseline is $\sim \pi^2/(4\theta_0\Delta V) \gtrsim 3 \times 10^3$ km, again too large. One can conclude that the sole presently known object where the parametric resonance in neutrino oscillations can take place is the earth, as was first pointed out in [7,8].

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