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Effect of Perturbations in the Coriolis and Centrifugal Forces on the Stability of L_4 in the Relativistic R3BP

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Abstract. The centrifugal and Coriolis forces do not appear as a result of physically imposed forces, but are due to a special property of a rotation. Thus, these forces are called pseudo-forces or 'fictitious forces'. They are merely an artifact of the rotation of the reference frame adopted. This paper studies the motion of a test particle in the neighbourhood of the triangular point L_4 in the framework of the perturbed relativistic restricted three-body problem (R3BP) when small perturbations are conferred to the centrifugal and Coriolis forces. It is found that the position and stability of the triangular point are affected by both the relativistic factor and small perturbations in the Coriolis and centrifugal forces. As an application, the Sun–Earth system is considered.

Key words. Celestial mechanics—perturbations—relativity—R3BP.

1. Introduction

The general three-body problem is the problem of motion of three celestial bodies under their mutual gravitational attraction. The restricted three-body problem (R3BP) is a simplified form of the general three-body problem, in which one of the bodies is of infinitesimal mass, and therefore does not influence the motion of the remaining two massive bodies called the primaries (Bruno 1994; Valtonen and Kartunen 2006).

The circular restricted three-body problem (CR3BP) possesses five stationary solutions called Lagrangian points. Three are collinear with the primaries and the other two are in equilateral triangular configuration with the primaries. The three collinear points $L_{1,2,3}$ are unstable, while the triangular points $L_{4,5}$ are stable for the mass ratio $\mu = \frac{m_2}{m_1+m_2} < 0.03852\ldots, m_1 \geq m_2$ being the masses of the primaries (Szebehely 1967a). Wintner (1941) and Contopoulos (2002) have shown that the stability of the triangular equilibrium points is due to the existence of the Coriolis terms in the equations of motion when these equations are recorded in rotating coordinate system. In the classical problem, the effects of the gravitational attraction of the infinitesimal body and other perturbations have been ignored. Perturbations can be caused due to the lack of sphericity or triaxiality, oblateness and radiation forces of the bodies, variation of masses, atmospheric drag, solar wind, Poynting–Robertson effect and the action of the other bodies. The most striking example of perturbations owing to oblateness in the solar system is the orbit of the fifth satellite of Jupiter, Amalthea. This planet is very oblate and the satellite's orbit is too small that its line of apsides advances about 900° in one year (Moulton 1914). Such oblateness driven effects have competing disturbing effects on qualitatively similar general relativistic effects (Iorio 2006; Iorio 2009; Iorio *et al.* 2013; Renzetti 2012b). The Kirkwood gaps in the ring of the asteroid's orbits lying between the orbits of the Mars and Jupiter are examples of the perturbations produced by Jupiter on an asteroid. This enables many researchers to study the restricted problem by taking into account the effects of small perturbations in the Coriolis and centrifugal forces radiation, oblateness and triaxiality of the bodies (Sharma and Ishwar 1995; Szebehely 1967a; SubbaRao and Sharma 1975; Bhatnagar and Hallan 1978; Singh 2011; Singh and Begha 2011; Singh 2013; AbdulRaheem and Singh 2006; AbdulRaheem and Singh 2008). Szebehely (1967b) investigated the stability of triangular points by keeping the centrifugal force constant and found that the Coriolis force is a stabilizing force, whereas SubbaRao and Sharma (1975) observed that the oblateness of the primary resulted in an increase in both the Coriolis and the centrifugal forces, thereby concluding that the Coriolis force is not always a stabilizing force. This was confirmed by AbdulRaheem and Singh (2006).

The theory of general relativity is currently the most successful gravitational theory describing the nature of space and time, and well confirmed by observations (Will 2014). Regarding the three-body relativistic effects we may also cite that: The geodesic precession of the orbit of two-body system is about one-third the mass in general relativity (Renzetti 2012a) and also for the post-Newtonian tidal effects (Iorio 2014). Indeed, the Newtonian effects of the planet's gravitational fields are in order of magnitude greater than the first corrections due to general relativity, and completely swamp the higher corrections that in principle was provided by the exact Schwarzschild solution.

Krefetz (1967) computed the post-Newtonian deviations of the triangular Lagrangian points from their classical positions in a fixed frame of reference for the first time, but without explicitly stating the equations of motion. After a decade, Contopoulos (1976) dealt with the relativistic R3BP in rotating coordinates. He derived the Lagrangian of the system along with the deviations of the triangular points.

Brumberg (1972, 1991) studied the relativistic n-body problem of three bodies in more detail and gathered most of the important results on relativistic celestial mechanics. He not only obtained the equations of motion for the general problem of three bodies but also deduced the equations of motion for the restricted problem of three bodies. Maindl and Dvorak (1994) derived the equations of motion for the relativistic R3BP using the post-Newtonian approximation of relativity. They applied this model to the computation of the advance of Mercury's perihelion in the solar system and found that they were compatible with the published data.

Bhatnagar and Hallan (1998) studied the existence and linear stability of the triangular points $L_{4,5}$ in the relativistic R3BP, and found that $L_{4,5}$ are always unstable in the whole range $0 \leq \mu \leq \frac{1}{2}$ in comparison to the classical R3BP which was stable for $\mu < \mu_0$, where μ is the mass ratio and $\mu_0 = 0.03852\ldots$ is the Routh's value.

In the beginning of the 21st century, Ragos *et al.* (2001) have investigated numerically, the linear stability of the collinear libration points $L_{1,2,3}$ in the relativistic R3BP for several cases in solar system, and found that the points $L_{1,2,3}$ were unstable.

Douskos and Perdios (2002) examined the stability of the triangular points in the relativistic R3BP and contrary to the result of Bhatnagar and Hallan (1998), they obtained a region of linear stability in the parameter space as $0 \leq \mu < \mu_0 - \frac{17(69)^{\frac{1}{2}}}{486c^2}$, where $\mu_0 = 0.03852\ldots$ is the Routh's value. They also determined the positions of the collinear points and showed that they were always unstable.

Lucas (2003) studied the chaotic amplification in the relativistic R3BP and noticed that the difference between Newtonian and post-Newtonian trajectories for the R3BP is greater for chaotic trajectories than it is for trajectories that are not chaotic. He also discussed the possibility of using this chaotic amplification effect as a novel test of general relativity. Rodica and Vasile (2006) investigated the existence and corresponding positions of the equilibrium points in orbital plane in the framework of the R3BP in Schwarzchild's gravitational field. They observed that there are three collinear libration points, and if they exist, then only two points are triangular libration points (situated in the orbital plane of the primaries). If triangular points exist, they may not form equilateral triangles; the triangles are isosceles for equal masses of the primaries, otherwise scalene.

Ahmed *et al.* (2006) also investigated the stability of the triangular points in the relativistic R3BP. In contrast to the previous result of Bhatnagar and Hallan (1998), they obtained a region of linear stability as $0 \leq \mu < \mu_0 + \frac{11387}{119232c^2}$, where μ_0 is the Routh's value.

Abd El-Salam and Abd El-Bar (2011) derived the equations of motion of the relativistic three-body in the post-Newtonian formalism.

Yamada and Asada (2012) discussed the post-Newtonian effects on Lagrange's equilateral triangular solution for the three-body problem. For three finite masses, it was found that a triangular configuration satisfies the post-Newtonian equation of motion in general relativity, if and only if, it has the relativistic corrections to each length side. This post-Newtonian configuration for three finite masses was not always equilateral and it reclaims the previous results for the restricted-problem when one mass approches zero. They also found that for the same masses and angular velocity, the post-Newtonian triangular is always smaller than the Newtonian. It is worth highlighting that so far all the 'Schwarzschild' terms have been used in studies, but gravitomagnetic Lense-Thirring (Renzetti 2013; Iorio 2001; Linsen 1991) associated with the rotation of the bodies have been neglected. From the previous works and also to the best of our knowledge, no work has been carried out on the combined effects of perturbations on the relativistic R3BP, so this increased the curiosity to study the effects of small perturbations in the centrifugal and Coriolis forces on the R3BP.

The focus of this paper is to study the difference between Newtonian and general relativistic positions and stability of triangular points in the presence of small perturbations in the centrifugal and Coriolis forces.

This paper is organized as follows. In section 2, the equations of motions are presented, section 3 describes the positions of equilibrium points, while their linear stability is analysed in section 4; the discussion is given in section 5, the numerical applications are given in section 6, finally section 7 conveys the main findings of this paper.

2. Equations of motion

The system of coordinates (ξ, η) such that the $\xi - \eta$ plane rotates in the positive direction with angular velocity equal to that of the common velocity of one primary with respect to the other keeping the origin fixed, then that coordinate system is known as synodic system.

The primaries appear at rest in the synodic or rotating frame (ξ, η) having its origin at the centre of mass and rotating along with them and are placed on the ξ-axis. The plane $\xi - \eta$ is the plane of the motion of the primaries. The coordinates (ξ, η) are sometimes called synodical. In this system,the primaries m_1 and m_2 are located at $(-\mu, 0)$, $(1-\mu, 0)$, respectively and have zero velocity. The advantage of this system is that m_1 and m_2 have fixed positions, so that the equations of motion are time-independent and therefore it is the easiest way to obtain the stationary solutions.

The pertinent equations of motion of an infinitesimal mass in the relativistic R3BP in a barycentric synodic coordinate system $(\xi,\ \eta)$ with origin at the centre of mass of the primaries with dimlensionless variables can be denoted as (Brumberg 1972; Bhatnagar and Hallan 1998):

$$\ddot{\xi} - 2n\dot{\eta} = \frac{\partial W}{\partial \xi} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \dot{\xi}}\right), \quad \ddot{\eta} + 2n\dot{\xi} = \frac{\partial W}{\partial \eta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \dot{\eta}}\right) \tag{1}$$

with

$$\begin{aligned} W = {} & \frac{1}{2}(\xi^2+\eta^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \\ & + \frac{1}{c^2}\left[\begin{array}{l} \dfrac{-3}{2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)(\xi^2+\eta^2) + \dfrac{1}{8}\{\dot{\xi}^2 + \dot{\eta}^2 + 2(\xi\dot{\eta} - \eta\dot{\xi}) \\ + (\xi^2+\eta^2)\}^2 + \dfrac{3}{2}\left(\dfrac{1-\mu}{\rho_1} + \dfrac{\mu}{\rho_2}\right)(\dot{\xi}^2 + \dot{\eta}^2 \\ + 2(\xi\dot{\eta} - \eta\dot{\xi}) + (\xi^2+\eta^2)) - \dfrac{1}{2}\left(\dfrac{(1-\mu)^2}{\rho_1^2} + \dfrac{\mu^2}{\rho_2^2}\right) \\ + \mu(1-\mu)\left\{\left(4\dot{\eta} + \dfrac{7}{2}\xi\right)\left(\dfrac{1}{\rho_1} - \dfrac{1}{\rho_2}\right) - \dfrac{\eta^2}{2}\left(\dfrac{\mu}{\rho_1^3} + \dfrac{1-\mu}{\rho_2^3}\right)\right\} \\ + \left(\dfrac{-1}{\rho_1\rho_2} + \dfrac{3\mu-2}{2\rho_1} + \dfrac{1-3\mu}{2\rho_2}\right) \end{array}\right], \end{aligned} \tag{2}$$

$$n = 1 - \frac{3}{2c^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right), \tag{3}$$

$$\begin{aligned} \rho_1^2 &= (\xi+\mu)^2 + \eta^2, \\ \rho_2^2 &= (\xi+\mu-1)^2 + \eta^2, \end{aligned} \tag{4}$$

where $0 < \mu \leq \frac{1}{2}$ is the ratio of mass of the smaller primary to the total mass of the primaries, ρ_1 and ρ_2 are distances of the infinitesimal mass from the bigger and smaller primary, respectively, n the mean motion of the primaries and c is the velocity of light.

We now introduce small perturbations in the centrifugal and Coriolis forces with the help of the parameters $\psi = 1+\varepsilon_1$; $|\varepsilon_1| << 1$, $\varphi = 1+\varepsilon_2$; $|\varepsilon_2| << 1$. The unperturbed value of each is unity. Consequently, the equations of motion can be taken as

$$\ddot{\xi} - 2\varphi n\dot{\eta} = \frac{\partial W}{\partial \xi} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \dot{\xi}}\right),$$

$$\ddot{\eta} + 2\varphi n\dot{\xi} \;=\; \frac{\partial W}{\partial \eta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \dot{\eta}}\right), \tag{5}$$

where

$$\begin{aligned} W = {} & \frac{1}{2}\psi(\xi^2+\eta^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \\ & + \frac{1}{c^2}\left[\begin{array}{l} \dfrac{-3}{2}\left(1 - \dfrac{1}{3}\mu\,(1-\mu)\right)(\xi^2+\eta^2)\,\psi + \dfrac{1}{8}\{\varphi\,\dot{\xi}^2 + \varphi\dot{\eta}^2 \\ + 2\varphi(\xi\dot{\eta} - \eta\dot{\xi}) + \psi\,(\xi^2+\eta^2)\}^2 \\ + \dfrac{3}{2}\left(\dfrac{1-\mu}{\rho_1} + \dfrac{\mu}{\rho_2}\right)\{\varphi\,\dot{\xi}^2 + \varphi\dot{\eta}^2 + 2\varphi(\xi\dot{\eta} - \eta\dot{\xi}) \\ + \psi(\xi^2+\eta^2)\} - \dfrac{1}{2}\left(\dfrac{(1-\mu)^2}{\rho_1^2} + \dfrac{\mu^2}{\rho_2^2}\right) \\ + \mu(1-\mu)\left\{\left(4\varphi\dot{\eta} + \dfrac{7}{2}\psi\xi\right)\left(\dfrac{1}{\rho_1} - \dfrac{1}{\rho_2}\right)\right. \\ \left. - \dfrac{\psi\eta^2}{2}\left(\dfrac{\mu}{\rho_1^3} + \dfrac{1-\mu}{\rho_2^3}\right)\right\} + \psi\left(\dfrac{-1}{\rho_1\rho_2} + \dfrac{3\mu-2}{2\rho_1} + \dfrac{1-3\mu}{2\rho_2}\right) \end{array}\right]. \end{aligned} \tag{6}$$

3. Locations of triangular points

The libration points are obtained from equations (5) on changing $\dot{\xi} = \dot{\eta} = \ddot{\xi} = \ddot{\eta} = 0$. These points are the solutions of the equations

$$\frac{\partial W}{\partial \xi} = 0 = \frac{\partial W}{\partial \eta}, \quad \text{with } \dot{\xi} = \dot{\eta} = 0,$$

$$\begin{aligned} & \psi\xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{\mu(\xi-1+\mu)}{\rho_2^3} \\ & + \frac{1}{c^2}\left[\begin{array}{l} -3\psi\xi\left(1 - \dfrac{1}{3}\mu(1-\mu)\right) + \dfrac{1}{2}\psi^2\xi\,(\xi^2+\eta^2) \\ -\dfrac{3}{2}\psi\,(\xi^2+\eta^2)\left(\dfrac{(1-\mu)(\xi+\mu)}{\rho_1^3} + \dfrac{\mu(\xi-1+\mu)}{\rho_2^3}\right) \\ +3\psi\left(\dfrac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right)\xi + \dfrac{(1-\mu)^2(\xi+\mu)}{\rho_1^4} + \dfrac{\mu^2(\xi-1+\mu)}{\rho_2^4} \\ +\mu(1-\mu)\left\{\begin{array}{l} \dfrac{7}{2}\psi\left(\dfrac{1}{\rho_1} - \dfrac{1}{\rho_2}\right) + \dfrac{7}{2}\psi\xi\left(\dfrac{-(\xi+\mu)}{\rho_1^3} + \dfrac{(\xi-1+\mu)}{\rho_2^3}\right) \\ + \dfrac{3}{2}\psi\eta^2\left(\dfrac{\mu(\xi+\mu)}{\rho_1^5} + \dfrac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right) \\ + \psi\left(\dfrac{\xi+\mu}{\rho_1^3\,\rho_2} + \dfrac{\xi-1+\mu}{\rho_1\,\rho_2^3} - \dfrac{(3\mu-2)(\xi+\mu)}{2\rho_1^3}\right. \\ \left. - \dfrac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\right) \end{array}\right\} \end{array}\right] = 0, \\ & \eta F = 0, \end{aligned} \tag{7}$$

with

$$F=\left(\psi-\frac{(1-\mu)}{\rho_1^3}-\frac{\mu}{\rho_2^3}\right)$$
$$+\frac{1}{c^2}\left[\begin{array}{l}-3\psi\left(1-\frac{1}{3}\mu\,(1-\mu)\right)+\frac{1}{2}\psi^2(\xi^2+\eta^2)+3\,\psi\,\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\\ -\frac{3}{2}\psi(\xi^2+\eta^2)\left(\frac{(1-\mu)}{\rho_1^3}+\frac{\mu}{\rho_2^3}\right)+\frac{(1-\mu)^2}{\rho_1^4}+\frac{\mu^2}{\rho_2^4}\\ +\mu(1-\mu)\left\{\frac{7}{2}\psi\,\xi\,\left(\frac{-1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\psi\,\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu)}{\rho_2^3}\right)\right\}\\ +\frac{3}{2}\psi\,\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{(1-\mu)}{\rho_2^5}\right)+\psi\left(\frac{1}{\rho_1^3\,\rho_2}+\frac{1}{\rho_1\,\rho_2^3}-\frac{(3\mu-2)}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}\right)\end{array}\right].$$

The triangular points are the solutions of equations (7) with $\eta \neq 0$. Since $\frac{1}{c^2} << 1$ and in the case $\frac{1}{c^2} \to 0$, one can obtain $\rho_1 = \rho_2 = \frac{1}{\psi^{\frac{1}{3}}}$; we assume in the relativistic R3BP that $\rho_1 = \frac{1}{\psi^{\frac{1}{3}}} + x$ and $\rho_2 = \frac{1}{\psi^{\frac{1}{3}}} + y$, where $x, y << 1$. Substituting these values in equations (4), solving them for ξ , η and ignoring terms of second and higher orders of x and y, we get

$$\xi = \frac{x-y}{\psi^{\frac{1}{3}}} + \frac{1-2\mu}{2},$$

$$\eta = \pm\left(\frac{\sqrt{\frac{4}{\psi^{\frac{2}{3}}}-1}}{2} + \frac{x+y}{2\psi^{\frac{1}{3}}\left(\frac{1}{\psi^{\frac{2}{3}}}-\frac{1}{4}\right)^{\frac{1}{2}}}\right).$$

Now substituting the values of $\rho_1, \rho_2, \xi, \eta$, $\psi = 1+\varepsilon_1, \varphi = 1+\varepsilon_2\,(|\varepsilon_1| << 1, |\varepsilon_2| << 1)$ from the previous equations (7) with $\eta \neq 0$ and neglecting the second and higher order terms in x, y, $\frac{1}{c^2}$, $\varepsilon_1, \varepsilon_2$ and their products, we have

$$\begin{aligned}&(1-\mu)x-\mu y-\frac{3\mu(1-2\mu)(1-\mu)}{8c^2}=0,\\ &(1-\mu)x+\mu y+\frac{7(\mu-\mu^2)}{8c^2}=0.\end{aligned}\tag{8}$$

Solving these equations for x and y, we get

$$x = -\frac{\mu(2+3\mu)}{8c^2},$$

$$y = -\frac{(1-\mu)(5-3\mu)}{8c^2}.$$

Thus, the coordinates of the triangular points $(\xi, \pm\eta)$ denoted by L_4 and L_5, respectively are

$$\begin{aligned} \xi &= \frac{1-2\mu}{2}\left(1+\frac{5}{4c^2}\right), \\ \eta &= \pm\frac{\sqrt{3}}{2}\left\{1-\frac{4\varepsilon_1}{9}+\frac{1}{12c^2}(-5+6\mu-6\mu^2)\right\}. \end{aligned} \tag{9}$$

4. Stability of L_4

Let (a, b) be the coordinates of the triangular point L_4. We set $\xi = a+\alpha$, $\eta = b+\beta$, $(\alpha, \beta << 1)$ in the equations (5) of motion. First, we compute the terms on their R.H.S, neglecting higher order terms, we get

$$\left(\frac{\partial W}{\partial \xi}\right)_{\xi=a+\alpha\,,\ \eta=b+\beta} = A\alpha + B\beta + C\dot{\alpha} + D\dot{\beta},$$

where

$$A = \frac{3}{4}\left\{1+\frac{5}{3}\varepsilon_1 + \frac{1}{2c^2}(2-19\mu+19\mu^2)\right\},$$

$$B = \frac{3\sqrt{3}}{4}(1-2\mu)\left\{1+\frac{11}{9}\,\varepsilon_1-\frac{2}{3c^2}\right\},$$

$$C = \frac{\sqrt{3}}{2c^2}(1-2\mu),$$

$$D = \frac{5\mu^2-5\mu+6}{2c^2}.$$

Similarly, we obtain

$$\left(\frac{\partial W}{\partial \eta}\right)_{\xi=a+\alpha\,,\ \eta=b+\beta} = A_1\alpha + B_1\beta + C_1\dot{\alpha} + D_1\dot{\beta},$$

where

$$A_1 = \frac{3\sqrt{3}}{4}(1-2\mu)\left\{1+\frac{11}{9}\,\varepsilon_1-\frac{2}{3c^2}\right\},$$

$$B_1 = \frac{9}{4}\left\{1+\frac{7}{9}\,\varepsilon_1+\frac{7(-2+3\mu-3\mu^2)}{6c^2}\right\},$$

$$C_1 = \frac{1}{2c^2}(-4+\mu+\mu^2),$$

$$D_1 = -\frac{\sqrt{3}}{2c^2}(1-2\mu).$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \dot{\xi}}\right)_{\xi=a+\alpha\,,\ \eta=b+\beta} = A_2\dot{\alpha} + B_2\dot{\beta} + C_2\ddot{\alpha} + D_2\ddot{\beta},$$

where

$$A_2 = \frac{\sqrt{3}}{2c^2}(1-2\mu),$$
$$B_2 = -\frac{4-\mu+\mu^2}{2c^2},$$
$$C_2 = \frac{17-2\mu+2\mu^2}{4c^2},$$
$$D_2 = -\frac{\sqrt{3}}{4c^2}(1-2\mu).$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \dot{\eta}}\right)_{\xi=a+\alpha\,,\,\eta=b+\beta} = A_3\dot{\alpha} + B_3\dot{\beta} + C_3\ddot{\alpha} + D_3\ddot{\beta},$$

where

$$\begin{aligned} A_3 &= \frac{6-5\mu+5\mu^2}{2c^2}, \\ B_3 &= -\frac{\sqrt{3}}{2c^2}(1-2\mu), \\ C_3 &= -\frac{\sqrt{3}}{4c^2}(1-2\mu), \\ D_3 &= \frac{3(5-2\mu+2\mu^2)}{4c^2}. \end{aligned}$$

Thus, the variational equations of motion corresponding to equations (5), on utilizing equation (3), can be obtained as

$$\begin{aligned} p_1\ddot{\alpha} + p_2\ddot{\beta} + p_3\dot{\alpha} + p_4\dot{\beta} + p_5\alpha + p_6\beta = 0, \\ q_1\ddot{\alpha} + q_2\ddot{\beta} + q_3\dot{\alpha} + q_4\dot{\beta} + q_5\alpha + q_6\beta = 0, \end{aligned} \tag{10}$$

where

$$\begin{aligned} &p_1 = 1 + C_2, \quad p_2 = D_2, \quad p_3 = A_2 - C, \\ &p_4 = \left\{B_2 - 2\,(1+\varepsilon_2)\left(1 - \tfrac{1}{2c^2}(3-\mu+\mu^2)\right) - D\right\}, \quad p_5 = -A, \quad p_6 = -B \\ &q_1 = C_3, \quad q_2 = 1 + D_3, \quad q_3 = 2\,(1+\varepsilon_2)\left(1 - \tfrac{1}{2c^2}(3-\mu+\mu^2)\right) - C_1 + A_3, \\ &q_4 = B_3 - D_1, \quad q_5 = -A_1, \quad q_6 = -B_1. \end{aligned}$$

Then, the characteristic equation is

$$\begin{aligned} (p_1q_2 \;-\; p_2q_1)\lambda^4 + (p_1q_6 + p_5q_2 + p_3q_4 - p_6q_1 - p_2q_5 - p_4q_3)\lambda^2 \\ +\; p_5q_6 - p_6q_5 = 0. \end{aligned} \tag{11}$$

Substituting the values of $p_i, q_i, i = 1, 2, \ldots, 6$ in (11), the characteristic equation (11) becomes

$$\begin{aligned} \lambda^4 + \left(1 - 3\varepsilon_1 - \frac{9}{c^2} + 8\varepsilon_2\right)\lambda^2 - \frac{3}{8}\Bigg(\frac{195\mu - 231\mu^2 + 72\mu^3 - 36\mu^4}{c^2} \\ -18\mu - 44\mu\varepsilon_1 + 18\mu^2 + 44\mu^2\varepsilon_1\Bigg) = 0. \end{aligned} \tag{12}$$

For $\frac{1}{c^2} \rightarrow 0$ and in the absence of small perturbations in the centrifugal Coriolis forces (i.e., $\varepsilon_1 = 0, \varepsilon_2 = 0$) this reduces to its well-known classical restricted problem form (see Szebehely 1967a):

$$\lambda^4 + \lambda^2 + \frac{27\mu(1-\mu)}{4} = 0.$$

The discriminant of (12) is

$$\Delta = \frac{-54\mu^4}{c^2} + \frac{108\mu^3}{c^2} + \left(\frac{-693}{2c^2} + 27 + 66\varepsilon_1\right)\mu^2 + \left(\frac{-585}{2c^2} - 27 - 66\varepsilon_1\right)\mu + \left(\frac{-36}{2c^2} + 1 - 6\varepsilon_1 + 16\varepsilon_2\right) \tag{13}$$

Its roots are

$$\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2}, \tag{14}$$

where

$$b = 1 - 3\varepsilon_1 - \frac{9}{c^2} + 8\varepsilon_2.$$

The discriminant of the original form of the characteristic equation (11), before normalizing to obtain (12) is

$$\Delta = \frac{54}{c^2}\mu^4 - \frac{108}{c^2}\mu^3 + \left(66\varepsilon_1 + 27 + \frac{395}{2c^2}\right)\mu^2 + \left(-66\varepsilon_1 - 27 - \frac{287}{2c^2}\right)\mu + 1 - \frac{2}{c^2} - 6\varepsilon_1 + 16\varepsilon_2. \tag{15}$$

This is mathematically equivalent to (13)

Now we have from (15),

$$\frac{d\Delta}{d\mu} = \frac{216}{c^2}\mu^3 - \frac{324}{c^2}\mu^2 + 2\left(66\varepsilon_1 + 27 + \frac{395}{2c^2}\right)\mu + \left(-66\varepsilon_1 - 27 - \frac{287}{2c^2}\right). \tag{16}$$

From (16), we get

$$\frac{d^2\Delta}{d\mu^2} = \frac{648}{c^2}\mu^2 - \frac{648}{c^2}\mu + 2\left(66\varepsilon_1 + 27 + \frac{395}{2c^2}\right)\mu + \left(66\varepsilon_1 + 27 + \frac{395}{2c^2}\right). \tag{17}$$

The discriminant of the equation $\frac{d^2\Delta}{d\mu^2} = 0$ is

$$\Delta_1 = -5184\,(66\varepsilon_1 + 27) < 0,$$

indicating that $\frac{d^2\Delta}{d\mu^2} > 0$ for $\mu \in [0, 1/2]$, hence $\frac{d\Delta}{d\mu}$ is a monotone increasing function in $[0, 1/2]$

But

$$\left(\frac{d\Delta}{d\mu}\right)_{\mu=0} = -66\varepsilon_1 - 27 - \frac{287}{2c^2} < 0, \tag{18}$$

$$\left(\frac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} = 0. \tag{19}$$

The equations (18) and (19) imply that $\frac{\mathrm{d}\Delta}{\mathrm{d}\mu} \le 0$ for $\mu \in [0, 1/2]$

Hence Δ is a monotone decreasing function in $[0, 1/2]$. Also,

$$(\Delta)_{\mu=0} = 1 - \tfrac{2}{c^2} - 6\varepsilon_1 + 16\varepsilon_2 > 0,$$

$$(\Delta)_{\mu=\frac{1}{2}} = -\tfrac{34}{c^2} - \tfrac{33\varepsilon_1}{2} + 16\varepsilon_2 - \tfrac{13}{2} < 0.$$

Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\frac{1}{2}}$ are of opposite signs, and Δ is monotone decreasing and continuous, there is only one value of μ, e.g., μ_c in the interval $[0, 1/2]$ for which Δ vanishes.

Solving the equation $\Delta = 0$, using (13) or (15), we obtain the critical value of the mass parameter as

$$\mu_c = \frac{1}{2} - \frac{(69)^{\frac{1}{2}}}{18}\left(1 + \frac{17}{27c^2}\right) + \frac{16\varepsilon_2}{3\,(69)^{\frac{1}{2}}} - \frac{76\varepsilon_1}{27\,(69)^{\frac{1}{2}}}, \tag{20}$$

$\mu_c = \mu_0 - \frac{17(69)^{\frac{1}{2}}}{486c^2} + \frac{4(36\varepsilon_2 - 19\varepsilon_1)}{27(69)^{\frac{1}{2}}}$, where $\mu_0 = 0.0385\ldots$ is the Routh's value.

We consider the following three scenarios of the values of μ separately:

(1) When $0 \le \mu < \mu_c$, $\Delta > 0$, the values of λ^2 given by (14) are negative and therefore all the four characteristic roots are distinctly pure imaginary numbers. Hence, the triangular points are stable.
(2) When $\mu_c < \mu \le \frac{1}{2}$, $\Delta < 0$, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.
(3) When $\mu = \mu_c$, $\Delta = 0$; the values of λ^2 given by (14) are the same. This induces instability of the triangular points.

Hence the stability region is

$$0 \le \mu < \mu_0 - \frac{17\,(69)^{\frac{1}{2}}}{486c^2} + \frac{4\,(36\varepsilon_2 - 19\varepsilon_1)}{27\,(69)^{\frac{1}{2}}}. \tag{21}$$

With the absence of small perturbations in the centrifugal and Coriolis forces (i.e., $\varepsilon_1 = 0, \varepsilon_2 = 0$), μ_c reduces to the critical mass value of the relativistic R3BP. This confirms the result of Douskos and Perdios (2002), but disagrees with that of Ahmed *et al.* (2006). In the presence of the perturbations ε_1 and ε_2 and in the absence of relativistic term $\frac{1}{c^2}$, μ_c verifies the result of Bhatnagar and Hallan (1978).

Hence, it can be noted that $\mu_c < \mu_0$ for $\varepsilon_1 > 0$ and $\mu_c > \mu_0$ for $\varepsilon_2 > 0$, showing that the centrifugal force is a destabilizing force when the Coriolis force is kept constant and vice versa. This agrees with the results of Singh (2013) and AbdulRaheem and Singh (2006).

5. Discussion

Equations (5)–(6) describe the motion of a test particle in the relativistic R3BP with small perturbations ε_1 andε_2 in the centrifugal and Coriolis forces, respectively.

Equations (9) determine the positions of triangular points which are affected by the perturbations in the centrifugal force and the relativistic factor. In the absence of the perturbation ε_1, these positions correspond to those of Bhatnagar and Hallan (1998), Douskos and Perdios (2002) and Ahmed *et al.* (2006). Equation (20) gives the critical value of the mass parameter which depends upon the small perturbations in the centrifugal, Coriolis forces and relativistic factor. This critical value is used to determine the size of the region of stability and also helps in analysing the behaviour of the parameters involved therein. It is obvious from (20) that the centrifugal force and the relativistic factor have destabilizing effects, while the Coriolis force shows stabilizing effect and thus when there is no perturbation in the Coriolis force, the region of stability decreases in size with the increase in the values of the parameters involved and increases with the increase in the values of the parameters when there is no perturbation in the centrifugal force. Equation (21) describes the region of stability. In the absence of perturbations (i.e., $\varepsilon_1 = 0, \varepsilon_2 = 0$), the stability results obtained in this study are in agreement with those of Douskos and Perdios (2002) but disagrees with Ahmed *et al.* (2006) and Bhatnagar & Hallan (1998). In the absence of relativistic terms, the results compatible with those of AbdulRaheem and Singh (2006) when the primaries are spherical darkbodies.

6. Numerical results

In the following, the previous results are applied to the Sun–Earth system.

For the Sun–Earth system: $\mu = 30035 \times 10^{-10}$ and the corresponding dimensionless speed of light is $c = 10064.84$. Using equation (20) for arbitrary values of small perturbation parameters in centrifugal and Coriolis forces, we show their effects on the critical mass μ_c as shown in Tables 1 and 2 respectively.

Table 1. Effect of the centrifugal force.

Perturbation parameter (ε_1)	μ_0	Critical mass equation (20) with $\varepsilon_1 = \varepsilon_2 = 0$	Critical mass equation (20) with $\varepsilon_2 = 0$
5×10^{-2}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$215776994 \times 10^{-10}$
-5×10^{-2}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$554640877 \times 10^{-10}$
1×10^{-2}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$351322547 \times 10^{-10}$
-1×10^{-2}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$419095324 \times 10^{-10}$
1×10^{-3}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$381820297 \times 10^{-10}$
-1×10^{-3}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$388597575 \times 10^{-10}$
5×10^{-4}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$383514616 \times 10^{-10}$
-5×10^{-4}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$386903255 \times 10^{-10}$
1×10^{-4}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$384870072 \times 10^{-10}$
-1×10^{-4}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$385547799 \times 10^{-10}$
5×10^{-5}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$385039504 \times 10^{-10}$
-5×10^{-5}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$385378367 \times 10^{-10}$
1×10^{-5}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$385175050 \times 10^{-10}$
-1×10^{-5}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$385242822 \times 10^{-10}$
5×10^{-6}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$385191993 \times 10^{-10}$
-5×10^{-6}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$385225879 \times 10^{-10}$

Table 2. Effect of the Coriolis force.

Perturbation parameter (ε_2)	μ_0	Critical mass equation (20) with $\varepsilon_1 = \varepsilon_2 = 0$	Critical mass equation (20) with $\varepsilon_1 = 0$
1×10^{-2}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$449414724 \times 10^{-10}$
-1×10^{-2}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$321003147 \times 10^{-10}$
2×10^{-2}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$513620512 \times 10^{-10}$
-2×10^{-2}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$256797359 \times 10^{-10}$
1×10^{-3}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$391629515 \times 10^{-10}$
-1×10^{-3}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$378788356 \times 10^{-10}$
2×10^{-3}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$398050093 \times 10^{-10}$
-2×10^{-3}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$372367778 \times 10^{-10}$
1×10^{-4}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$385850994 \times 10^{-10}$
-1×10^{-4}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$384566878 \times 10^{-10}$
2×10^{-4}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$386493052 \times 10^{-10}$
-2×10^{-4}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$383924820 \times 10^{-10}$
1×10^{-5}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$385273142 \times 10^{-10}$
-1×10^{-5}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$385144730 \times 10^{-10}$
2×10^{-5}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$385337347 \times 10^{-10}$
-2×10^{-5}	$385208965 \times 10^{-10}$	$385208946 \times 10^{-10}$	$385080524 \times 10^{-10}$

7. Conclusion

By considering small perturbations in the centrifugal and Coriolis forces in the relativistic CR3BP, we have determined the positions of the triangular points and have examined their linear stability. It is found that their positions are sligthly affected by a small change in the centrifugal force. It is also observed that the general destabilizing and stabilizing characteristics of the centrifugal and Coriolis forces, respectively remains unaltered, resulting in either decrease or increase of the region of stability, respectively. This is confirmed by Tables 1 and 2.

We have noticed that the expressions for A, D, A_2, C_2 in Bhatnagar and Hallan (1998) differ from the present unperturbed study. Consequently, the expression for p_1, p_3, p_4, p_5 and the characteristic equations are also different. This led Bhatnagar and Hallan (1998) to infer that triangular points are unstable, contrary to Douskos and Perdios (2002) and the present results. In future studies, it might be important to look into the combined effects of the relativistic and perturbations factors i.e., $\frac{\varepsilon_1}{c^2}$, $\frac{\varepsilon_2}{c^2}$ and also the Lense–Thirring effects connected with the rotation of the primaries.

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