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On the Stability of $L_{4,5}$ in the Relativistic R3BP with Radiating Secondary

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Abstract. This paper discusses the motion of a test particle in the neighbourhood of the triangular points $L_{4,5}$ by considering the less massive primary (secondary) as a source of radiation in the framework of the relativistic restricted three-body problem (R3BP). It is found that the positions and stability of the triangular point are affected by both relativistic and electromagnetic radiation factors. It turns out that both the coordinates of the infinitesimal mass are affected, contrary to the classical where this happens only for one coordinate. A practical application of this model could be the study of dynamical evolution of dust particles in orbits around a binary system with a dark degenerate first primary and a secondary stellar companion.

Key words. Celestial mechanics—radiating secondary—relativity—R3BP.

1. Introduction

The most important of all dynamical problem is the problem of three bodies. The three-body problem is concerned with the motion of three particles attracting each other according to the Newtonian law, so that between each pair of particles there is an attractive force which is proportional to the product of the masses of the particles and the inverse of the square of the distance that separates them: They are free to move in space, and are initially supposed to be moving in any given manner. However, the complete solution of the general three-body problem still remains a formidable challenge. Regarding the general three-body problem, one can refer to Chenciner (2007); Bruno (1994); Gutzwiller (1998); Valtonen & Kartunen (2006). There are also various forms of three-body problems in general relativity (see Renzetti 2012a; Nordtvedt 1968; Iorio 2014a).

The restricted three-body problem is a simplified form of the general three-body problem which describes the motion of an infinitesimal mass moving under the gravitational effect of the two finite masses m_1 and m_2 with $m_1 \geq m_2$, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and their motion not influence by infinitesimal mass. It was originally formulated depending on the approximate circular motion of the planets around the sun, and the small masses of the asteroids and satellites of the planets.

This classical restricted three-body problem is not valid when at least one of the interacting bodies is an intense emitter of radiation. In this regard, it is reasonable to modify the model by superimposing a light repulsion field whose source coincides with the source of the gravitational field provided by the radiating body on the gravitational field of the main bodies. According to Radzievskii (1950, 1953), the problem in such a theory is called the photogravitational problem. He discussed it for three specific bodies: the sun, a planet, and a dust particle. It was found that allowing the direct solar electromagnetic radiation results in a change in the positions of the librations points. An investigation of the positions of the libration points, when the more massive primary is a source of radiation and the smaller primary (secondary) one is an oblate spheroid, was carried out by Sharma (1987). He showed that the triangular points are linearly stable for the mass parameter $0 < \mu < \mu_{\text{crit}}$, where $\mu = \frac{m_2}{m_1+m_2}$ and the critical mass value μ_{crit} decreases with the increase in oblateness and radiation force.

The effect of oblateness and electromagnetic radiation force of the primaries on the location and the linear stability of the triangular points in the restricted three-body problem were analysed by Singh and Ishwar (1999). They considered both primaries as a source of radiation as well as oblate spheroids, and observed that these points are stable for $0 < \mu < \mu_{\text{c0}}$ and unstable for $\mu_{\text{c0}} < \mu < \frac{1}{2}$, where μ_{c0} is the critical mass value of the mass parameter which depends on the radiating and oblateness coefficients. Similar problem under the influence of small perturbations in the Coriolis and centrifugal forces was studied by AbdulRaheem and Singh (2006). Considering the overall effect they observed that range of stability of triangular points decreases.

Perdiou *et al.* (2012) considered the modification of Hill's problem where the primary is radiating and the secondary is an oblate spheroid. They studied the evolution of the network of the basic families of planar periodic orbits for various values of the parameters of the problem.

In his paper, Singh (2013) has investigated the effects of small perturbations in the Coriolis and centrifugal forces, electromagnetic radiation pressure and triaxiality of the two stars (primaries) on the positions and stability of an infinitesimal mass (third body) in the framework of the planar circular R3BP. He observed that the positions of the three collinear and two triangular equilibrium points are affected by the radiation, triaxiality and a small perturbation in the centrifugal force, but are unaffected by that of the Coriolis force. The collinear points are found to remain unstable, while the triangular points are seen to be stable for $0 < \mu < \mu_{\text{c}}$ but unstable for $\mu_{\text{c}} \leq \mu \leq \frac{1}{2}$, where μ_{c} is the critical mass ratio influenced by small perturbations in the Coriolis and centrifugal forces, radiation and triaxiality. He also noticed that the Coriolis force exhibits a stabilizing behaviour, whereas the centrifugal force shows destabilizing behaviour. Therefore, in general the size of the region of stability decreases with increase in the values of the parameters involved.

In considering the primary as an oblate spheroid and secondary as a source of radiation, a different version of the problem was realized for study: Douskos *et al.* (2006) considered a R3BP which includes the effects of oblateness of the primary body and radiation of the secondary body. They determined the equilibrium points and their stability and discussed the zero-velocities curves. They found that both oblateness of the primary and radiation of the secondary reduce the stability region of the isosceles triangular equilibrium points in the parameter space. These effects also reduce the Roche lobe and thus the sphere of influence of the secondary.

Singh and Umar (2012) studied the motion of an infinitesimal mass around seven equilibrium points in the framework of the elliptical R3BP under the assumption that primary of the system is non-luminous, oblate spheroid and the secondary is luminous. They found that a practical application of this case could be the study of the dynamical evolution of dust particles in orbits around a binary system with a dark degenerate primary and a secondary stellar companion. They found the conditional stability of motion around the triangular points for $0 < \mu < \mu_c$, where μ is the mass ratio. The critical mass ratio μ_c depends on the combined effect of electromagnetic radiation, oblateness, eccentricity, and the semi-major axis of the elliptic orbits; an increase in any of these parameters has destabilizing results on the orbits of the test particles. Therefore, overall effect is that the size of the region of stability decreases when the value of these parameters increases. The collinear points and the out-of-plane equilibrium points are found to be unstable for any combination of the parameters considered. Further, they carried out a numerical analysis by computing the positions of the triangular points and the critical mass ratio of two binaries RXJ 0450.1-5836 and Nova Cen 1969 (Cen X-4).

Singh and Amuda (2014) have recently investigated the motion of a test particle around the triangular equilibrium points under the influence of secondary radiation and its Poynting–Robertson (P–R) effect when the first primary is an oblate spheroid. It is seen that the triangular points are influenced by the presence of the following parameters: electromagnetic radiation from the secondary and the incidental P–R effect and the oblateness of the first primary. They also studied the linear stability of the problem and applied it to the binary system RXJ 0450.1-5836. They found that triangular points are unstable due to positive roots in Lyapunov sense when P–R effect is considered against their conditional stability in the absence of P–R drag effect. Regarding the relevance of asphericity of the primaries, in association with general relativity, the following examples can be cited: Idrisi & Taqvi (2013, 2014); Sharma *et al*. (2001); Katour *et al*. (2014); Bhavneet & Aggarwal (2013, 2014); Iorio (2007a, b, 2009, 2011, 2013a, b, 2014b); Iorio *et al*. (2011); Renzetti (2012b, 2013, 2014); Hallan *et al*. (2000).

Further, the classical model assumes that the masses of the bodies are constant, but there are numerous practical problems where the mass does not remain constant. There is a decrease in stellar mass, on account of light emission. A satellite moving around a radiating star surrounded by cloud varies its mass owing to the particles of the cloud. Regarding the restricted problem dealing with variable mass of one, two or three bodies under different aspects, one may refer to Singh *et al*. (2010) and Singh & Leke (2010).

Bhatnagar and Hallan (1998) studied the existence and linear stability of the triangular points $L_{4,5}$ in the relativistic R3BP, and found that $L_{4,5}$ are always unstable in the whole range $0 \leq \mu \leq \frac{1}{2}$, comparison to the classical R3BP where they are stable for $\mu < \mu_0$, here μ is the mass ratio and $\mu_0 = 0.03852\ldots$ is the Routh's value.

Douskos and Perdios (2002) examined the stability of the triangular points in the relativistic R3BP and contrary to the results of Bhatnagar and Hallan (1998), they obtained a region of linear stability in the parameters space $0 \leq \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2}$, where $\mu_0 = 0.03852\ldots$ is Routh's value. They also determined the positions of the collinear points and showed that they are always unstable.

Abd El-Salam and Abd El-Bar (2014) have studied the photogravitational restricted three-body within the framework of the post-Newtonian approximation. The mass of the primaries are assumed to change under the effect of continuous radiation processes. They computed the locations of the triangular points. They also obtained series forms of these locations and recorded new analytical results. It is also important to note that compared to the existing works by other researchers till date, in this paper, we do not consider the rotation of the primaries, i.e., the Lense–Thirring is completely neglected (Josef & Hans 1918; Iorio 2001b; Iorio *et al*. 2004; Ashby & Allison 1993; Snellen *et al*. 2014) which is one of the cause of rotation in general relativity. The effects of Lense–Thirring on the position of the libration points might be not negligible in view of the fact that fast spinning primaries are well known. For example, the recent discovery of the past rotating planet Beta Pictoris b (Snellen *et al*. 2014).

The aim of this paper is to investigate the triangular points and their linear stability under the relativistic treatment of R3BP when the smaller primary is luminous.

This paper is organized as follows: In section 2, the equations the motion are presented; section 3 describes the positions of equilibrium points, while their linear stability is analyzed in section 4; the discussion is given in section 5. Section 6 gives the numerical results. Finally, section 7 presents the main findings of this paper.

2. Equations of motion

The system of coordinates (ξ, η) such that the $\xi - \eta$ plane rotates in the positive direction with angular velocity equal to the common velocity of one primary with respect to the other keeping the origin fixed, such a coordinate system is known as synodic system. The direction of the ξ-axis is chosen such that the two masses always lie along it with bigger and smaller primary placed at $(-\mu, 0)$, $(1-\mu, 0)$, respectively. The primaries appear at rest in a synodic system with the frame rotating along them. This implies that such a system has zero velocity.

The pertinent equations of motion of an infinitesimal mass in the relativistic R3BP in a barycentric synodic coordinate system (ξ, η) with origin at the centre of mass of the primaries having dimensionless variables can be written as (Brumberg 1972; Bhatnagar & Hallan 1998)

$$\begin{aligned}\ddot{\xi} - 2n\dot{\eta} &= \frac{\partial W}{\partial \xi} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \dot{\xi}}\right),\\ \ddot{\eta} + 2n\dot{\xi} &= \frac{\partial W}{\partial \eta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial W}{\partial \dot{\eta}}\right),\end{aligned} \tag{1}$$

with

$$\begin{aligned} W = {} & \frac{1}{2}(\xi^2+\eta^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} + \frac{1}{c^2}\Bigg[-\frac{3}{2}\left(1-\frac{1}{3}\mu(1-\mu)\right)(\xi^2+\eta^2) \\ & + \frac{1}{8}\{\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})+(\xi^2+\eta^2)\}^2 \\ & + \frac{3}{2}\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)(\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})+(\xi^2+\eta^2)) \end{aligned}$$

$$-\frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}+\frac{\mu^2}{\rho_2^2}\right)+\mu(1-\mu)\left\{\left(4\dot{\eta}+\frac{7}{2}\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\right.$$

$$\left.\left.-\frac{\eta^2}{2}\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\left(\frac{-1}{\rho_1\rho_2}+\frac{3\mu-2}{2\rho_1}+\frac{1-3\mu}{2\rho_2}\right)\right\}\right], \tag{2}$$

$$n=1-\frac{3}{2c^2}\left(1-\frac{1}{3}\mu(1-\mu)\right), \tag{3}$$

$$\begin{aligned}\rho_1^2&=(\xi+\mu)^2+\eta^2,\\ \rho_2^2&=(\xi+\mu-1)^2+\eta^2,\end{aligned} \tag{4}$$

where $0<\mu\leq\frac{1}{2}$ is the ratio of mass of the smaller primary to the total mass of the primaries; ρ_1 and ρ_2 are distances of the infinitesimal mass from the bigger and smaller primary. With the introduction of a radiation factor q_2 for the smaller primary (less massive), the equations of motion can be expressed as follows:

$$\begin{aligned}\ddot{\xi}-2n\dot{\eta}&=\frac{\partial W}{\partial\xi}-\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial W}{\partial\dot{\xi}}\right),\\ \ddot{\eta}+2n\dot{\xi}&=\frac{\partial W}{\partial\eta}-\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial W}{\partial\dot{\eta}}\right),\end{aligned} \tag{5}$$

where

$$W=\frac{1}{2}(\xi^2+\eta^2)+\frac{1-\mu}{\rho_1}+\frac{q_2\mu}{\rho_2}+\frac{1}{c^2}\left[-\frac{3}{2}\left(1-\frac{1}{3}\mu(1-\mu)\right)(\xi^2+\eta^2)\right.$$

$$+\frac{1}{8}\{\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})+(\xi^2+\eta^2)\}^2$$

$$+\frac{3}{2}\left(\frac{1-\mu}{\rho_1}+\frac{q_2\mu}{\rho_2}\right)(\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})+(\xi^2+\eta^2))$$

$$-\frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}+\frac{q_2^2\mu^2}{\rho_2^2}\right)+q_2\mu(1-\mu)\left\{\left(4\dot{\eta}+\frac{7}{2}\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\right.$$

$$\left.\left.-\frac{\eta^2}{2}\left(\frac{q_2\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\left(\frac{-1}{\rho_1\rho_2}+\frac{q_2\mu-2(1-\mu)}{2\rho_1}+\frac{(1-\mu)-2q_2\mu}{2\rho_2}\right)\right\}\right], \tag{6}$$

The radiation factor q_2 is given by $F_{p2}=F_{g2}(1-q_2)$ such that $0<1-q_2<<1$ Radzievskii (1950), where F_{g2} and F_{p2} are, respectively the gravitational and electromagnetic radiation forces.

3. Location of triangular points

The libration points can be obtained from equation (5) after formulating $\dot{\xi}=\dot{\eta}=\ddot{\xi}=\ddot{\eta}=0$.

These points are the solutions of the equations

$$\frac{\partial W}{\partial \xi} = 0 = \frac{\partial W}{\partial \eta} \quad \text{with} \;\; \dot{\xi} = \dot{\eta} = 0$$

that is,

$$\begin{aligned}
&\xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{q_2\mu(\xi-1+\mu)}{\rho_2^3} + \frac{1}{c^2}\Bigg[-3\xi\left(1-\frac{1}{3}\mu(1-\mu)\right) + \frac{1}{2}\xi(\xi^2+\eta^2) \\
&\quad - \frac{3}{2}(\xi^2+\eta^2)\left(\frac{(1-\mu)(\xi+\mu)}{\rho_1^3} + \frac{q_2\mu(\xi-1+\mu)}{\rho_2^3}\right) + 3\left(\frac{1-\mu}{\rho_1} + \frac{q_2\mu}{\rho_2}\right)\xi \\
&\quad + \frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4} + \frac{q_2^2\mu^2(\xi-1+\mu)}{\rho_2^4} + q_2\mu(1-\mu)\Bigg\{\frac{7}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \\
&\quad + \frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_1^3}\right) + \frac{3}{2}\eta^2\left(\frac{q_2\mu(\xi+\mu)}{\rho_1^5} + \frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right) \\
&\quad + \frac{(\xi+\mu)}{\rho_1^3\rho_2} + \frac{(\xi-1+\mu)}{\rho_1\rho_2^3} - \frac{(q_2\mu-2(1-\mu))\;(\xi+\mu)}{2\rho_1^3} \\
&\quad - \frac{((1-\mu)-2q_2\mu)\;(\xi-1+\mu)}{2\rho_2^3}\Bigg\}\Bigg],
\end{aligned} \tag{7}$$

and

$$\eta F = 0$$

with

$$\begin{aligned}
F = &\left(1 - \frac{(1-\mu)}{\rho_1^3} - \frac{q_2\mu}{\rho_2^3}\right) + \frac{1}{c^2}\Bigg[-3\left(1-\frac{1}{3}\mu(1-\mu)\right) + \frac{1}{2}(\xi^2+\eta^2) \\
&+3\left(\frac{(1-\mu)}{\rho_1} + \frac{q_2\mu}{\rho_2}\right) - \frac{3}{2}(\xi^2+\eta^2)\left(\frac{(1-\mu)}{\rho_1^3} + \frac{q_2\mu}{\rho_2^3}\right) + \left(\frac{(1-\mu)^2}{\rho_1^4} + \frac{q_2^2\mu^2}{\rho_2^4}\right) \\
&+q_2\mu(1-\mu)\Bigg\{\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3} + \frac{1}{\rho_2^3}\right) - \left(\frac{q_2\mu}{\rho_1^3} + \frac{(1-\mu)}{\rho_2^3}\right) + \frac{3}{2}\eta^2\left(\frac{q_2\mu}{\rho_1^5} + \frac{(1-\mu)}{\rho_2^5}\right) \\
&+\frac{1}{\rho_1^3\rho_2} + \frac{1}{\rho_1\rho_2^3} - \left(\frac{(q_2\mu-2(1-\mu))}{2\rho_1^3} + \frac{((1-\mu)-2q_2\mu)}{2\rho_1^3}\right)\Bigg\}\Bigg].
\end{aligned}$$

The triangular points are the solutions of equations (7) with $\eta \neq 0$.

Since in the absence of both the electromagnetic radiation and relativistic effect $\left(q_2 = 1, \frac{1}{c^2} \to 0\right)$, one can obtain $\rho_1 = \rho_2 = 1$,

We, therefore assume in the relativistic R3BP with electromagnetic radiation that $\rho_1 = 1 + x$ and $\rho_2 = 1 + y$, where $x, y << 1$ depending upon the radiation and relativistic terms. Substituting these values in equations (4), solving them for ξ, η and ignoring terms of second and higher powers of x and y, we get

$$\xi = x - y + \frac{1-2\mu}{2}, \quad \eta = \pm\left(\frac{\sqrt{3}}{2} + \frac{x+y}{\sqrt{3}}\right) \tag{8}$$

which give the positions of triangular points in terms of x and y. In order to determine x and y, we substitute the values of $\rho_1, \rho_2, \xi, \eta$ from the equations (7) with $\eta \neq 0$ and neglecting second and higher order terms in x, y, $\frac{1}{c^2}$ and $(1-q_2)$ since they are very small quantities, we have

$$\left(\frac{3}{2}-\frac{\mu}{2}-q_2\mu\right)x-\left(\mu+\frac{q_2\mu}{2}\right)y-\frac{\mu}{2}+\frac{q_2\mu}{2}$$
$$+\frac{1}{c^2}\left\{\frac{(-20+11q_2)\,\mu}{16}+\frac{\left(20+18q_2-11q_2^2\right)\mu^2}{16}+\frac{\left(20-41q_2+3q_2^2\right)\mu^3}{16}\right\}=0$$

and

$$3\,(1-\mu)\,x+3q_2\mu y+(1-q_2)\,\mu$$
$$+\frac{1}{c^2}\left\{\frac{(-12+33q_2)\,\mu}{8}+\frac{\left(-20-14q_2+13q_2^2\right)\mu^2}{8}+\frac{\left(12-7q_2-5q_2^2\right)\mu^3}{8}\right\}=0. \tag{9}$$

Solving these equations for x and y, we get

$$x=\frac{\mu\left(q_2^2-2q_2+1\right)}{3\left(-1-2q_2+\left(q_2^2+q_2+1\right)\mu\right)}+\frac{\mu}{24\left(-1-2q_2+\left(q_2^2+q_2+1\right)\mu\right)\,c^2}$$
$$\{(6q_2+33q_2^2-10q_2^3-20)\mu+(12+29q_2-70q_2^2+2q_2^3)\mu^2$$
$$+(-12-3q_2+33q_2^2)\},$$

$$y=\frac{\left(q_2+q_2^2-2\right)\mu+(-3q_2+3)}{3\left(-1-2q_2+\left(q_2^2+q_2+1\right)\mu\right)}+\frac{1}{24\left(-1-2q_2+\left(q_2^2+q_2+1\right)\mu\right)\,c^2}$$
$$\{(12+33q_2)+(-84-36q_2+3q_2^2)\mu+(28+105q_2-21q_2^2-13q_2^3)\mu^2$$
$$+(24-70q_2+14q_2^2+5q_2^3)\mu^3\}. \tag{10}$$

Thus, the coordinates of the triangular points $(\xi, \pm\eta)$ denoted by L_4 and L_5, respectively, can be obtained by substituting values of x and y in equation (8).

They are,

$$\xi=-\frac{2\left(q_2^2+q_2+1\right)\mu^2+\left(-3q_2-q_2^2-5\right)\mu+3}{2\left(-1-q_2+\left(q_2^2+q_2+1\right)\mu\right)}$$
$$-\frac{\begin{array}{c}(4+11q_2)+\left(-11q_2-10q_2^2-24\right)\mu+(16+33q_2\\-18q_2^2-q_2^3)\,\mu^2+\left(4-33q_2+28q_2^2+q_2^3\right)\mu^3\end{array}}{8\left(-1-2q_2+\left(q_2^2+q_2+1\right)\mu\right)\,c^2},$$

$$\eta = \pm\frac{\sqrt{3}}{72}\left\{\frac{(-12-96q_2)+\left(28+28q_2+52q_2^2\right)\mu}{\left(-1-2q_2+\left(q_2^2+q_2+1\right)\mu\right)}\right.$$
$$\left.+\frac{(12+33q_2)+\left(-96-39q_2+36q_2^2\right)\mu+\left(8+111q_2+12q_2^2-23q_2^3\right)\mu^2+\left(36-41q_2-56q_2^2+7q_2^3\right)\mu^3}{\left(-1-2q_2+\left(q_2^2+q_2+1\right)\mu\right)\ c^2}\right\}. \quad (11)$$

Inserting $q_2 = 1-(1-q_2) = 1-\delta$, where $\delta = 1-q_2$ and neglecting second and higher powers of δ as $0 \le \delta << 1$, (10) and (11) can be formulated, respectively as

$$x = -\frac{\mu\,(2+3\mu)}{8c^2} + \left\{\frac{\mu\left(-21-14\mu+35\mu^2\right)+\mu\,(3\mu+2)\ (2-3\mu)}{24\,(\mu-1)\ c^2}\right\}\delta,$$
$$y = -\frac{(1-\mu)\,(5-3\mu)}{8c^2} - \frac{\delta}{3} + \left\{\frac{\left(11+\mu+9\mu^2\right)+(3\mu-5)\ (2-3\mu)}{24c^2}\right\}\delta, \quad (12)$$
$$\xi = \frac{1-2\mu}{2}\left(1+\frac{5}{4c^2}\right)+\frac{1}{3}\delta-\left\{\frac{\left(-1-4\mu+36\mu^2-26\mu^3\right)}{24\,(\mu-1)\ c^2}\right\}\delta,$$
$$\eta = \pm\left\{\frac{\sqrt{3}}{2}\left(1+\frac{1}{12c^2}(-5+6\mu-6\mu^2)\right)\right.$$
$$\left.-\frac{\sqrt{3}}{9}\delta+\frac{\sqrt{3}}{72\,(\mu-1)\ c^2}(-1-38\mu+8\mu^2+26\mu^3)\delta\right\}. \quad (13)$$

4. Stability of $L_{4,5}$

Since the nature of linear stability about the point L_5 will be similar to that of L_4 it will be sufficient to consider only the stability near L_4.

Let (a, b) be the coordinates of the triangular point L_4.

We set $\xi = a+\alpha$, $\eta = b+\beta$, $(\alpha\ , \beta\ << 1)$ in equation (5).

First, we compute the terms of R.H.S. using $q_2 = 1-\delta, 0 \le \delta = 1-q_2 << 1$ and neglecting second and higher order terms of small quantities, we get

$$\left(\frac{\partial W}{\partial \xi}\right)_{\xi=a+\alpha\ ,\ \eta=b+\beta} = A\alpha + B\beta + C\dot{\alpha} + D\dot{\beta},$$

where

$$A = \frac{3}{4}\left\{1+\frac{1}{2c^2}(2-19\mu+19\mu^2)\right\}$$
$$-\left\{\frac{1}{48\,(\mu-1)\ c^2}(-28+82\mu+3\mu^2-321\mu^3+234\mu^4)+\left(\frac{3\mu}{2}-1\right)\right\}\delta,$$

$$
\begin{aligned}
B &= \frac{3\sqrt{3}}{4}(1-2\mu)\left(1-\frac{2}{3c^2}\right)\\
&\quad+\frac{\sqrt{3}}{6}\left\{-(\mu-2)+\frac{1}{48\,(\mu-1)\;c^2}(96-309\mu+314\mu^2-151\mu^3+40\mu^4)\right\}\delta,
\end{aligned}
$$

$$
C=\frac{\sqrt{3}}{2c^2}(1-2\mu)+\left\{\frac{2\sqrt{3}}{9c^2}(\mu+1)\right\}\delta,
$$

$$
D=\frac{1}{2c^2}(6-5\mu+5\mu^2)+\left\{\frac{1}{3c^2}(-2+3\mu-6\mu^2)\right\}\delta.
$$

Similarly, we obtain

$$
\left(\frac{\partial W}{\partial \eta}\right)_{\xi=a+\alpha\,,\;\eta=b+\beta}=A_1\alpha+B_1\beta+C_1\dot{\alpha}+D_1\dot{\beta},
$$

where

$$
\begin{aligned}
A_1 &= \frac{3\sqrt{3}}{4}(1-2\mu)\left(1-\frac{2}{3c^2}\right)\\
&\quad+\frac{\sqrt{3}}{6}\left\{-(\mu-2)+\frac{1}{48\,(\mu-1)\;c^2}(96-309\mu+314\mu^2-151\mu^3+40\mu^4)\right\}\delta,\\
B_1 &= \frac{9}{4}\left\{1+\frac{7\left(-2+3\mu-3\mu^2\right)}{6c^2}\right\}\\
&\quad+\left\{\left(\frac{3\mu}{2}-1\right)+\frac{1}{16\,(\mu-1)\;c^2}(4-26\mu+173\mu^2-279\mu^3+118\mu^4)\right\}\delta,\\
C_1 &= \frac{1}{2c^2}\left(-4+\mu-\mu^2\right)+\left\{\frac{1}{3c^2}(7\mu-2)\right\}\delta,\\
D_1 &= -\frac{\sqrt{3}}{2c^2}(1-2\mu)-\left\{\frac{2\sqrt{3}}{9c^2}(1-8\mu+9\mu^2)\right\}\delta.\\
\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial W}{\partial \dot{\eta}}\right)_{\xi=a+\alpha\,,\;\eta=b+\beta} &= A_2\dot{\alpha}+B_2\dot{\beta}+C_2\ddot{\alpha}+D_2\ddot{\beta},
\end{aligned}
$$

where

$$
\begin{aligned}
A_2 &= \frac{\sqrt{3}}{2c^2}(1-2\mu)+\left\{\frac{2\sqrt{3}}{9c^2}(\mu+1)\right\}\delta,\\
B_2 &= -\frac{1}{2c^2}\left(4-\mu+\mu^2\right)+\left\{\frac{1}{3c^2}(7\mu-2)\right\}\delta,
\end{aligned}
$$

$$C_2 = \frac{1}{4c^2}\left(17 - 2\mu + 2\mu^2\right) - \left\{\frac{1}{3c^2}(7\mu + 1)\right\}\delta,$$

$$D_2 = -\frac{\sqrt{3}}{4c^2}(1 - 2\mu) - \left\{\frac{\sqrt{3}}{9c^2}(\mu + 1)\right\}\delta.$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial W}{\partial \dot{\eta}}\right)_{\xi=a+\alpha\,,\,\eta=b+\beta} = A_3\dot{\alpha} + B_3\dot{\beta} + C_3\ddot{\alpha} + D_3\ddot{\beta},$$

where

$$A_3 = \frac{1}{2c^2}(6 - 5\mu + 5\mu^2) + \left\{\frac{1}{3c^2}(-2 + 3\mu - 6\mu^2)\right\}\delta,$$

$$B_3 = -\frac{\sqrt{3}}{2c^2}(1 - 2\mu) - \left\{\frac{2\sqrt{3}}{9c^2}(1 - 8\mu + 9\mu^2)\right\}\delta,$$

$$C_3 = -\frac{\sqrt{3}}{4c^2}(1 - 2\mu) - \left\{\frac{\sqrt{3}}{9c^2}(\mu + 1)\right\}\delta,$$

$$D_3 = \frac{3\left(5 - 2\mu + 2\mu^2\right)}{4c^2} - \left\{\frac{1}{3c^2}(9\mu - 1)\right\}\delta.$$

Thus, the variational equations of motion corresponding to equations (5), on utilizing equation (3), can be shown as

$$\begin{aligned} &P_1\ddot{\alpha} + P_2\ddot{\beta} + P_3\dot{\alpha} + P_4\dot{\beta} + P_5\alpha + P_6\beta = 0,\\ &Q_1\ddot{\alpha} + Q_2\ddot{\beta} + Q_3\dot{\alpha} + Q_4\dot{\beta} + Q_5\alpha + Q_6\beta = 0, \end{aligned} \tag{14}$$

where

$$P_1 = 1 + C_2,\ P_2 = D_2,\ P_3 = A_2 - C,$$
$$P_4 = \left\{B_2 - 2\left(1 - \frac{1}{2c^2}(3 - \mu + \mu^2)\right) - D\right\},\ P_5 = -A,\ P_6 = -B,$$
$$Q_1 = C_3,\ Q_2 = 1 + D_3,\ Q_3 = 2\left(1 - \frac{1}{2c^2}(3 - \mu + \mu^2)\right) - C_1 + A_3\,,$$
$$Q_4 = B_3 - D_1,\ Q_5 = -A_1,\ Q_6 = -B_1,$$

Then the associated characteristic equation is

$$\begin{aligned}(P_1Q_2 - P_2Q_1)\lambda^4 + (P_1Q_6 \;+\; P_5Q_2 + P_3Q_4 - P_6Q_1&\\ - P_2Q_5 - P_4Q_3)\lambda^2 + P_5Q_6 - P_6Q_5 = 0.&\end{aligned} \tag{15}$$

Substituting the values of P_i, Q_i, $i = 1, 2, ..., 6$ in (15) and neglecting second and higher power of small quantities, the characteristic equation (15) becomes

$$\lambda^4 + \lambda^2 + d = 0, \tag{16}$$

where

$$\begin{aligned}
b &= 1-\frac{9}{c^2}+\left(\frac{10+84\mu-21\mu^2-12\mu^3}{12c^2}\right)\delta,\\
d &= \frac{27}{4}\mu(1-\mu)+\frac{-585\mu+693\mu^2-216\mu^3+108\mu^4}{8c^2}\\
&\quad+\left(\frac{3}{2}\mu(1-\mu)+\frac{960-2253\mu-2226\mu^2+9243\mu^3-6390\mu^4+816\mu^5}{192(\mu-1)c^2}\right)\delta.
\end{aligned}$$

While $\frac{1}{c^2}\to 0$ and when the smaller primary is non-luminous (i.e., $\delta=0$), (16) reduces to its well-known classical restricted problem form (see Szebehely 1967):

$$\lambda^4+\lambda^2+\frac{27}{4}\mu(1-\mu)=0.$$

The discriminant of (16) is

$$\begin{aligned}
\Delta &= \frac{1}{(\mu-1)}\left\{\frac{(-54-17\delta)}{c^2}\,\mu^5+\frac{(1296+1049\delta)}{8c^2}\mu^4\right.\\
&\quad+\left(27+6\delta+\frac{(-7272-3105\delta)}{16c^2}\right)\mu^3+\left(-54-12\delta+\frac{(5112+511\delta)}{8c^2}\right)\mu^2\\
&\quad\left.+\left(28+6\delta+\frac{(-14904+1661\delta)}{48c^2}\right)\mu+\left(-1+\frac{18}{c^2}-\frac{65}{3c^2}\delta\right)\right\}.
\end{aligned}\tag{17}$$

The roots are given as

$$\lambda^2=\frac{-b\pm\sqrt{\Delta}}{2},\tag{18}$$

where

$$b=1-\frac{9}{c^2}+\left(\frac{10+84\mu-21\mu^2-12\mu^3}{12c^2}\right)\delta.$$

Now we have from (17),

$$\begin{aligned}
\frac{\mathrm{d}\Delta}{\mathrm{d}\mu} &= \frac{1}{(\mu-1)^2}\{(54+12\delta)\,\mu^3-81\mu^2-18\mu\delta-6\delta-27\}\\
&\quad+\frac{1}{16(\mu-1)^2c^2}\{(-3456-1088\delta)\,\mu^5+(12096+7654\delta)\,\mu^4\\
&\quad+(-24912-14602\delta)\,\mu^3+(21816+9315\delta)\,\mu^2+(4680-207\delta)\}<0.
\end{aligned}\tag{19}$$

From (17) and (19), it can be seen that Δ is a decreasing and continuous function, hence monotone in $(0,\frac{1}{2}]$.

But

$$
\begin{aligned}
(\Delta)_{\mu=0} &= 1 - \frac{18}{c^2} + \frac{65\delta}{3c^2} > 0, \\
(\Delta)_{\mu=\frac{1}{2}} &= -\frac{17}{4} - \frac{9\delta}{2} + \frac{2491\delta}{96c^2} + \frac{423}{2c^2} < 0.
\end{aligned} \tag{20}
$$

Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\frac{1}{2}}$ are of opposite signs, and Δ is monotone and continuous, there is only one value of μ, e.g., μ_c in the interval $(0, \frac{1}{2}]$ for which Δ vanishes. Solving the equation $\Delta = 0$, using (17), we obtain the critical value of the mass parameter as

$$
\begin{aligned}
\mu_c &= \frac{1}{2} - \frac{\sqrt{69}}{18} - \frac{17\sqrt{69}}{486c^2} - \frac{2\delta}{27\sqrt{69}} + \left(\frac{-3690189 + 682606\sqrt{69}}{2414448c^2}\right)\delta, \\
\mu_c &= \mu_0 - \frac{17\sqrt{69}}{486c^2} - \frac{2\delta}{27\sqrt{69}} + \left(\frac{-3690189 + 682606\sqrt{69}}{2414448c^2}\right)\delta,
\end{aligned} \tag{21}
$$

where $\mu_0 = 0.03852\ldots$ is the Routh's value.

We consider the following three regions of the values of μ separately.

(1) When $0 < \mu < \mu_c$, the values of λ^2 given by (18) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.
(2) When $\mu_c < \mu \leq \frac{1}{2}$, $\Delta < 0$, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.
(3) When $\mu = \mu_c$, $\Delta = 0$, the values of λ^2 given by (18) are the same. This induces instability of the triangular points.

Hence the stability region is

$$
0 < \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2} - \frac{2\delta}{27\sqrt{69}} + \left(\frac{-3690189 + 682606\sqrt{69}}{2414448c^2}\right)\delta. \tag{22}
$$

When the mass reduction factor q_2 is unity (i.e., $\delta = 0$), μ_c reduces to the critical mass value of the relativistic R3BP. This confirms the result of Douskos and Perdios (2002).

5. Discussion

Equations (5)–(6) describe the motion of a test particle in the relativistic R3BP with the less massive primary radiating. Equations (13) determine the positions of triangular points which are slightly affected by both the mass reduction and relativistic factors due to the presence of small quantities coupling terms. When the value of the mass reduction factor is unity, these positions correspond to those of Douskos and Perdios (2002). It is noticed from equation (12) that the distances ρ_1 and ρ_2 of the infinitesimal mass from the primaries are different which implies that the triangular points $L_{4,5}$ form scalene triangles with primaries contrary to the classical case in which they form equilateral triangles. From (13) it is observed that the ordinate is also affected by the mass ratio μ, in comparison to the classical case where only the abscissa is affected by the mass ratio.

Equations (21) give the critical value of the mass parameter which depends upon the mass reduction and relativistic factors. This critical value is used to determine the size of the region of stability and also helps in analyzing the behaviour of the parameters involved therein. It is noticed from (21) that the mass reduction and the relativistic factors have destabilizing effects. Equation (22) describes the region of stability. From equation (22) it is seen that the region of stability is affected by both the mass reduction and the relativistic factors combined together due to the presence of a coupling term. When the mass reduction factor is unity (i.e., $\delta = 0$) the stability results obtained in this study are in agreement with the results of Douskos and Perdios (2002) but disagree with that of Bhatnagar and Hallan (1998).

In the absence of relativistic terms it is noticed from equations (13) and (21) that our results coincide with those of (i) Singh (2013) when the primaries are spherical in the absence of Coriolis and centrifugal forces and the smaller primary is only luminous; (ii) AbdulRaheem & Singh (2006) when the primaries are spherical in the absence of small perturbations in the Coriolis and centrifugal forces and the smaller primary is only luminous; (iii) Singh & Umar (2012) when the orbit is circular in the absence of oblateness of the first primary (i.e., $A = 0$); (iv) Singh & Amuda (2014) when the first primary is spherical in the absence of the P–R drag of the second primary; (v) Douskos *et al.* (2006) when the secondary is radiating and the oblateness of the primary is neglected (i.e., $A_1 = 0$).

6. Numerical results

In the following, we will apply the earlier results to the binary systems, CenX-4 and RXJ0450.15856 to compute the value of the critical mass parameter and locations of the triangular equilibrium points. The necessary data has been adopted from Singh and Umar (2012).

(1) In the binary system CenX-4, we consider

$m_1 = 1.9996 M_{\text{sun}}$, velocity of light $= 299792.458$km/s,

$m_2 = 0.0801 M_{\text{sun}}$, constant of gravity $\gamma = 6.67259 \times 10^{-8}$ /cm^3/g/s^2,

$\mu = 0.038515$

The dimensionless velocity of light of this system is $c = 1.307039186 \times 10^{10}$. The radiation factor $q_2 = 0.993$ and distance between primaries a =1.7 kpc

(2) In the binary system RX0450.1-5856, we consider

$m_1 = 1.4 M_{\text{sun}}$, the radiation factor $q_2 = 0.9965$,

$m_2 = 0.15 M_{\text{sun}}$, distance between primaries $a = 205$ pc,

$\mu = 0.0967$, velocity of light $= 299792.458$ km/s,

Constant of gravity $\gamma = 6.67259 \times 10^{-8}$ cm^3/g/s^2.

The dimensionless velocity of light of this system is $c = 1.662513320 \times 10^{11}$.

Table 1. Locations and critical mass.

	Binary system	
	CenX-4	RXJ0450.1-5856
ξ		
Classical	0.4614850000	0.4033000000
Relativistic	0.4614850000	0.4033000000
Equation (13)	0.4638183333	0.4044666667
$\pm\eta$		
Classical	0.8660254040	0.8660254040
Relativistic	0.8660254040	0.8660254040
Equation (13)	0.86467825333	0.8653518287
μ_c		
Classical	0.0385208965	0.0385208965
Relativistic	0.0385208965	0.0385208965
Equation (21)	0.0384584742	0.0384896853

7. Conclusion

By considering a second primary radiation in the relativistic CR3BP, we have determined the positions of the triangular points and have examined their stability. It is found that their positions are slightly affected by both the radiation and relativistic factors. It can be noticed from Table 1 that the relativistic factor has no independent effect on the locations and stability region whereas the combined effect of relativistic and radiation factors play a significant role. This supports the fact that when the speed of light is large, the results tend to be in agreement with the classical ones. It can also be noticed from equation (12) that both the factors have destabilizing tendency. We have observed that the expressions for A, D, A_2, C_2 in Bhatnagar and Hallan (1998) differ from those of the present study when the radiation factor is unity (i.e., $\delta = 0$). Consequently, the expression for P_1,P_3, P_4, P_5 and the characteristic equation are also different. This led Bhatnagar and Hallan (1998) to infer that triangular points are unstable, contrary to Douskos and Perdios (2002) and the present results. The aim is to consider the mass loss or variation of the masses in a future work.

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