J. Astrophys. Astr. (2013) **34,** 341–348

Satellite Orbital Precessions Caused by the Octupolar Mass Moment of a Non-Spherical Body Arbitrarily Oriented in Space

G. Renzetti
Pescara, Italy
e-mail: renzetti.g@libero.it

Received 8 September 2013; accepted 19 September 2013

Abstract. I consider a satellite moving around a non-spherical body of mass M and equatorial radius R, and calculate its orbital precessions caused by the body's octupolar mass moment J_4. I consider only the effects averaged over one orbital period T of the satellite. I give exact formulas, not restricted to any special values of either the eccentricity e or the inclination i of the satellite's orbit. I do not assume any preferential orientation for the body's spin axis $\hat{\mathbf{k}}$ because in many cases of potential interest (exoplanets, neutron stars, black holes) it is poorly known or unknown at all.

Key words. Experimental studies of gravity—satellite orbits—harmonics of the gravity potential field.

1. Introduction

Since all astronomical bodies like planets (Null *et al.* 1975; Konopliv *et al.* 1999; Lemoine *et al.* 1998; Tapley *et al.* 2004; Reigber *et al.* 2005; Jacobson *et al.* 2006; Hirt *et al.* 2012; Smith *et al.* 2012), satellites (Anderson *et al.* 2001a, b; Iess *et al.* 2007, 2010; Yan *et al.* 2013), our Sun (Johnson *et al.* 1980; Ulrich & Hawkins 1981; Lydon & Sofia 1996; Pijpers 1998; Godier & Rozelot 1999; Roxburgh 2001; Pireaux & Rozelot 2003; Rozelot *et al.* 2004; Rozelot & Damiani 2011; Damiani *et al.* 2011; Rozelot & Fazel 2013), other main sequence stars (Hirata 1995; van Belle *et al.* 2001; Domiciano de Souza *et al.* 2003; McAlister *et al.* 2005; Meilland *et al.* 2009; Yoon *et al.* 2010; Nemravová *et al.* 2010; Dufton *et al.* 2011; van Belle 2012; Granada *et al.* 2013), white dwarfs (Anand 1965; Hartle 1967; Arutyunyan *et al.* 1971b; Boshkayev *et al.* 2012, 2013a, b), neutron stars (Hartle 1967; Arutyunyan et al. 1971a; Shibata 1998; Laarakkers & Poisson 1999; Stergioulas 2003; Boshkayev *et al.* 2012; Pappas & Apostolatos 2012), black holes (Geroch 1970; Hansen 1974; Simon & Beig 1983; Chruściel 1994; Heusler 1998; Vigeland 2010) rotate more or less rapidly, they are not perfectly spherical because of the centrifugal forces flattening them at the poles. Accordingly, the motion of a test object about them deviates from the standard Keplerian picture at the Newtonian level. Since the departures from spherical symmetry usually cause small additional accelerations with respect to the inverse-square one, the resulting orbital effects can be treated as perturbations (Kaula 2000; Capderou 2005) of the standard Keplerian elliptical motion in terms of the osculating Keplerian orbital elements.

A class of particularly important effects consist in the satellite's secular perturbations due to the even zonal multipoles $J_\ell, \ell = 2, 4, 6, \ldots$ of the aspherical gravity field of the centrifugally distorted primary of mass M and equatorial radius R. Indeed, they grow cumulatively over time, orbit after orbit. These shifts are quite important in a lot of practical applications, ranging from global gravity field determination (O'Keefe *et al.* 1959; King-Hele 1961; King-Hele *et al.* 1963; Cook 1963; Giacaglia 1964; Kozai 1966; Wagner 1973; King-Hele & Cook 1973; King-Hele 1983; Konopliv *et al.* 2006) to fundamental physics in space (Iorio 2001, 2002, 2003; Lucchesi 2003; Iorio *et al.* 2004; Lucchesi 2005; Iorio 2005a, b, 2006a, b, 2007a; Lucchesi 2007; Iorio 2007b, 2008a, b, 2009a, b, c, 2010b; Lucchesi & Peron 2010; Iorio 2010a; Xu *et al.* 2011; Iorio 2011e; Iorio *et al.* 2011; Iorio 2011a, b, c; Lucchesi 2011; Iorio 2012a, b; Iorio *et al.* 2013; Singh & Umar 2013). Thus, it is important to calculate them quite accurately.

Here, I offer a complete calculation of the orbit perturbations caused by the octupolar mass moment J_4 of an axisymmetric body whose spin axis is arbitrarily oriented in space. In other words, contrary to what was made so far in the literature, I do not align the unit vector $\hat{\mathbf{k}}$ of the body's spin axis with the z axis of the coordinate system adopted. Indeed, in several cases of potential interest, the orientation of $\hat{\mathbf{k}}$ in space is not known. The orbital perturbations due to the quadrupolar mass moment J_2 for an arbitrary $\hat{\mathbf{k}}$ were computed in Iorio (2011d), while those up to $\ell = 20$ for $\hat{\mathbf{k}} \parallel \hat{\mathbf{z}}$ can be found in Iorio (2003).

Finally, I wish to mention that the quadrupole J_2 of an aspherical body disturbs the motion of a satellite also at the post-Newtonian level (Soffel *et al.* 1988; Heimberger *et al.* 1990; Brumberg 1991). These effects could be measured with the Juno mission to Jupiter (Iorio 2013).

2. My calculation

In the following, a, e, i, Ω, ω are the standard Keplerian orbital elements parametrizing the Keplerian ellipse in the two-body problem: the semimajor axis, the eccentricity, the inclination, the longitude of the ascending node, and the argument of pericenter.

I assume the following perturbing potential (Capderou 2005)

$$\Delta U_4 = -\frac{GM}{r} J_4 \left(\frac{R}{a}\right)^4 P_4(\hat{\mathbf{k}} \cdot \hat{\mathbf{r}}), \tag{1}$$

where P_4 is the Legendre polynomial of degree 4, G is the gravitational constant, $\hat{\mathbf{k}}$ is the unit vector of the spin axis of the body, and $\hat{\mathbf{r}}$ is the unit vector instantaneously pointing from the body to the satellite. In my calculation, I do not align $\hat{\mathbf{k}}$ along with the z axis.

Now, I compute the average $\langle \Delta U_4 \rangle_T$ of eq. (1) over one orbital period $T = 2\pi/n = 2\pi\sqrt{a^3/GM}$ of the satellite. I do not show the cumbersome result since it has not a direct, explicit connection with the observations.

Then, I apply the standard Lagrange planetary equations to $\langle \Delta U_4 \rangle_T$ and get the rates of change of the orbital elements averaged over the particle's orbital period T. Below, I show the exact expressions for a, e, i, Ω.

$$\left\langle \frac{\mathrm{d}a}{\mathrm{d}t} \right\rangle_T = 0, \tag{2}$$

$$\left\langle \frac{\mathrm{d}e}{\mathrm{d}t} \right\rangle_T = -\frac{15 n J_4 e}{512(1-e^2)^3} \left(\frac{R}{a}\right)^4 \mathcal{E}(\hat{\mathbf{k}}, i, \Omega, \omega), \tag{3}$$

$$\left\langle \frac{\mathrm{d}i}{\mathrm{d}t} \right\rangle_T = \frac{15 n J_4}{2048(1-e^2)^4} \left(\frac{R}{a}\right)^4 \mathcal{I}(\hat{\mathbf{k}}, i, \Omega, \omega), \tag{4}$$

$$\left\langle \frac{\mathrm{d}\Omega}{\mathrm{d}t} \right\rangle_T = -\frac{15 n J_4 \csc i}{64(1-e^2)^4} \left(\frac{R}{a}\right)^4 \mathcal{N}(\hat{\mathbf{k}}, i, \Omega, \omega). \tag{5}$$

The quantities $\mathcal{E}, \mathcal{I}, \mathcal{P}$ in eqs (3)–(5) are complicated functions of the body's spin axis $\hat{\mathbf{k}}$ and of the orbital parameters i, Ω, ω. They are

$$\begin{aligned}
\mathcal{E} = {} & \{-8\hat{k}_z \sin i \cos 2\omega (\hat{k}_x \cos\Omega + \hat{k}_y \sin\Omega) \\
& + 4\cos i \cos 2\omega[-2\hat{k}_x\hat{k}_y \cos 2\Omega + (\hat{k}_x^2 - \hat{k}_y^2)\sin 2\Omega] \\
& + \sin 2\omega[(\hat{k}_x^2 - \hat{k}_y^2)(3+\cos 2i)\cos 2\Omega + 2(\hat{k}_x^2 + \hat{k}_y^2 - 2\hat{k}_z^2)\sin^2 i \\
& - 4\hat{k}_z \sin 2i (\hat{k}_y \cos\Omega - \hat{k}_x \sin\Omega) + 2\hat{k}_x\hat{k}_y(3+\cos 2i)\sin 2\Omega]\} \\
& \cdot \{-24 + 21\hat{k}_x^2 + 21\hat{k}_y^2 + 14\hat{k}_z^2 + 7[(\hat{k}_x^2 + \hat{k}_y^2 - 2\hat{k}_z^2)\cos 2i \\
& + 2(\hat{k}_x^2 - \hat{k}_y^2)\sin^2 i \cos 2\Omega + 4\hat{k}_z \sin 2i (\hat{k}_y \cos\Omega - \hat{k}_x \sin\Omega) \\
& + 4\hat{k}_x\hat{k}_y \sin^2 i \sin 2\Omega]\},
\end{aligned} \tag{6}$$

$$\begin{aligned}
\mathcal{I} = {} & 112\hat{k}_z \cos 4i \sin 2\omega (\hat{k}_y \cos\Omega - \hat{k}_x \sin\Omega)[-1 + 2\hat{k}_y \sin 2\Omega \hat{k}_x + 3\hat{k}_z^2 \\
& + (\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega]e^2 + 16\hat{k}_z \cos 2i \sin 2\omega(\hat{k}_x \sin\Omega - \hat{k}_y \cos\Omega) \\
& \cdot [-35\hat{k}_z^2 + 70\hat{k}_y \sin 2\Omega \hat{k}_x + 35(\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega + 25]e^2 \\
& + 2\sin 2i \sin 2\omega[-56\hat{k}_z^4 - 24(3 - 7\hat{k}_z^2)\hat{k}_z^2 + 3(1-\hat{k}_z^2)(5 - 21\hat{k}_z^2) \\
& - 35(\hat{k}_x^4 - 6\hat{k}_y^2\hat{k}_x^2 + \hat{k}_y^4)\cos 4\Omega - 8\hat{k}_x\hat{k}_y(35\hat{k}_z^2 - 5)\sin 2\Omega \\
& - 4(\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega(70\hat{k}_y \sin 2\Omega \hat{k}_x + 35\hat{k}_z^2 - 5)]e^2 \\
& + 7\sin 4i \sin 2\omega\{8\hat{k}_z^4 - 24(1-\hat{k}_z^2)\hat{k}_z^2 + 3(1-\hat{k}_z^2)^2 \\
& - 4(\hat{k}_x^2 - \hat{k}_y^2)(1 - 7\hat{k}_z^2)\cos 2\Omega + (\hat{k}_x^4 - 6\hat{k}_y^2\hat{k}_x^2 + \hat{k}_y^4)\cos 4\Omega \\
& + 4\hat{k}_x\hat{k}_y[(\hat{k}_x^2 - \hat{k}_y^2)\sin 4\Omega - 2(1 - 7\hat{k}_z^2)\sin 2\Omega]\}e^2
\end{aligned}$$

$$
\begin{aligned}
& + 56(3e^2+2)\hat{k}_z \cos 3i\,(\hat{k}_x \cos\Omega + \hat{k}_y \sin\Omega)[2\hat{k}_z^2 - 3(1-\hat{k}_z^2) \\
& + 3(\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega + 6\hat{k}_x\hat{k}_y \sin 2\Omega] \\
& + 28(3e^2+2)\sin 3i[-1 + 2\hat{k}_y \sin 2\Omega\hat{k}_x + 7\hat{k}_z^2 + (\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega] \\
& \cdot [(\hat{k}_x^2 - \hat{k}_y^2)\sin 2\Omega - 2\hat{k}_x\hat{k}_y \cos 2\Omega] \\
& + 8\hat{k}_z \cos i\{-4\cos 2\omega[3\hat{k}_x(3-7\hat{k}_z^2)\cos\Omega + 7\hat{k}_x(\hat{k}_x^2 - 3\hat{k}_y^2)\cos 3\Omega \\
& + 3\hat{k}_y(3-7\hat{k}_z^2)\sin\Omega - 7\hat{k}_y(\hat{k}_y^2 - 3\hat{k}_x^2)\sin 3\Omega]e^2 \\
& - (3e^2+2)(\hat{k}_x \cos\Omega + \hat{k}_y \sin\Omega)[3 + 42\hat{k}_y \sin 2\Omega\hat{k}_x - 21\hat{k}_z^2 \\
& + 21(\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega]\} + 4\sin i\{4e^2 \cos 2\omega[4\hat{k}_x\hat{k}_y(1-7\hat{k}_z^2)\cos 2\Omega \\
& - 2(\hat{k}_x^2 - \hat{k}_y^2)((1-7\hat{k}_z^2)\sin 2\Omega - 14\hat{k}_x\hat{k}_y \cos 4\Omega) \\
& - 7(\hat{k}_x^4 - 6\hat{k}_y^2\hat{k}_x^2 + \hat{k}_y^4)\sin 4\Omega] - (3e^2+2)[3 + 42\hat{k}_y \sin 2\Omega\hat{k}_x - 21\hat{k}_z^2 \\
& + 21(\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega][(\hat{k}_x^2 - \hat{k}_y^2)\sin 2\Omega - 2\hat{k}_x\hat{k}_y \cos 2\Omega]\}, \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N} = {} & [\hat{k}_z \cos i + \sin i\,(\hat{k}_x \sin\Omega - \hat{k}_y \cos\Omega)] \\
& \cdot \{-14\cos^3 i\,(2\cos 2\omega e^2 - 3e^2 - 2)(\hat{k}_y \cos\Omega - \hat{k}_x \sin\Omega)^3 \\
& + 42\cos^2 i[e^2 \sin 2\omega(\hat{k}_x \cos\Omega + \hat{k}_y \sin\Omega) \\
& - \hat{k}_z(2\cos 2\omega e^2 - 3e^2 - 2)\sin i](\hat{k}_y \cos\Omega - \hat{k}_x \sin\Omega)^2 \\
& + \cos i[6(7\cos 2i\hat{k}_z^2 - 7\hat{k}_z^2 + 2)\cos 2\omega e^2 \\
& + (3e^2+2)(14\hat{k}_y \sin 2\Omega\hat{k}_x + 14\hat{k}_z^2 - 21\hat{k}_z^2 \cos 2i \\
& + 7(\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega - 1)](\hat{k}_y \cos\Omega - \hat{k}_x \sin\Omega) \\
& + e^2 \sin 2\omega(\hat{k}_x \cos\Omega + \hat{k}_y \sin\Omega)[14\hat{k}_y \sin 2\Omega\hat{k}_x + 14\hat{k}_z^2 \\
& + 7(\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega - 21\hat{k}_z(\hat{k}_z \cos 2i + 2\sin 2i\,(\hat{k}_x \sin\Omega - \hat{k}_y \cos\Omega)) - 5] \\
& + \hat{k}_z \sin i[2(6-7\hat{k}_z^2)\cos 2\omega e^2 + 7\hat{k}_z^2 \cos 2i\,(2\cos 2\omega e^2 - 3e^2 - 2) \\
& + (3e^2+2)(14\hat{k}_y \sin 2\Omega\hat{k}_x + 7(\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega - 1)]\}. \qquad (8)
\end{aligned}
$$

There are no restrictions on the values which the orientation of $\hat{\mathbf{k}}$, the eccentricity e and the inclination i can assume in eqs (6)–(8).

The exact expression of the precession of ω is too cumbersome to be explicitly displayed here. Thus, I expand it in powers of e, and show only the leading term of order zero. It is

$$
\left\langle \frac{\mathrm{d}\omega}{\mathrm{d}t} \right\rangle_T = \frac{15nJ_4}{256}\left(\frac{R}{a}\right)^4 \mathcal{P}(\hat{\mathbf{k}}, i, \Omega, \omega) + \mathcal{O}(e^2), \qquad (9)
$$

with

$$
\begin{aligned}
\mathcal{P} = & -35\hat{k}_z^4\cos 4i + 4\cos 2i[-5 + (6 - 7\hat{k}_z^2)\cos 2\omega]\hat{k}_z^2 \\
& -28\cos 3\Omega\sin i[\hat{k}_x(\hat{k}_x^2 - 3\hat{k}_y^2)\sin 2\omega - 11\hat{k}_y(\hat{k}_y^2 - 3\hat{k}_x^2)\cos i]\hat{k}_z \\
& +2\cos\Omega\{\hat{k}_y[154\cos 2i\hat{k}_z^2 - 11(7\hat{k}_z^2 - 9) \\
& +8\cos 2\omega(7\hat{k}_z^2\sin^2 i - 3)]\sin 2i + 2\hat{k}_x(-21\hat{k}_x^2 - 21\hat{k}_y^2 - 14\hat{k}_z^2 \\
& +14\hat{k}_z^2\cos 2i + 24)\sin i\sin 2\omega\}\hat{k}_z + 4\sin i\{\hat{k}_x\cos i[-154\cos 2i\hat{k}_z^2 \\
& +11(7\hat{k}_z^2 - 9) + 8\cos 2\omega(3 - 7\hat{k}_z^2\sin^2 i)] \\
& +\hat{k}_y[7\hat{k}_z^2(1 + 2\cos 2i) + 3]\sin 2\omega\}\sin\Omega\hat{k}_z \\
& +28\sin i[11\hat{k}_x(\hat{k}_x^2 - 3\hat{k}_y^2)\cos i + \hat{k}_y(\hat{k}_y^2 - 3\hat{k}_x^2)\sin 2\omega]\sin 3\Omega\hat{k}_z \\
& +32[5(1 - \hat{k}_x^2) - 2] - 5[21\hat{k}_x^4 + (42\hat{k}_y^2 + 28\hat{k}_z^2 - 32)\hat{k}_x^2 \\
& +7(3\hat{k}_y^4 + 4\hat{k}_z^2\hat{k}_y^2 + 3\hat{k}_z^4)] + [7\hat{k}_z^4\cos 4i + 3(1 - 2\hat{k}_z^2)]\cos 2\omega \\
& -4(\hat{k}_x^2 - \hat{k}_y^2)\{5[7(1 - \hat{k}_z^2\cos 2i) - 8] + (1 - 7\hat{k}_z^2)\cos 2\omega\}\cos 2\Omega \\
& -7(\hat{k}_x^4 - 6\hat{k}_y^2\hat{k}_x^2 + \hat{k}_y^4)(\cos 2\omega + 5)\cos 4\Omega \\
& -8\hat{k}_x\hat{k}_y\{5[7(1 - \hat{k}_z^2\cos 2i) - 8] + (1 - 7\hat{k}_z^2)\cos 2\omega\}\sin 2\Omega \\
& +28\cos^4 i(\hat{k}_y\cos\Omega - \hat{k}_x\sin\Omega)^2[12\hat{k}_z^2 + 4\cot i(\hat{k}_y\cos\Omega - \hat{k}_x\sin\Omega)\hat{k}_z \\
& +2\cos 2\omega(\hat{k}_y\cos\Omega - \hat{k}_x\sin\Omega)^2 - 7(1 - \hat{k}_z^2) \\
& +7(\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega + 14\hat{k}_x\hat{k}_y\sin 2\Omega] \\
& +56\cos^3 i(\hat{k}_y\cos\Omega - \hat{k}_x\sin\Omega)\{\hat{k}_z\sin i[6\hat{k}_z^2 \\
& +4\cos 2\omega(\hat{k}_y\cos\Omega - \hat{k}_x\sin\Omega)^2 \\
& -13(1 - \hat{k}_z^2) + 13(\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega + 26\hat{k}_x\hat{k}_y\sin 2\Omega] \\
& -2\sin 2\omega(\hat{k}_y\cos\Omega - \hat{k}_x\sin\Omega)^2(\hat{k}_x\cos\Omega + \hat{k}_y\sin\Omega)\} \\
& +2\cos i\{\sin 2\omega[-42\cos 2i((\hat{k}_x^2 - \hat{k}_y^2)\sin 2\Omega - 2\hat{k}_x\hat{k}_y\cos 2\Omega)\hat{k}_z^2 \\
& +21\hat{k}_x(\hat{k}_x^2 - 3\hat{k}_y^2)\cos 3\Omega\sin 2i\hat{k}_z + 2[7\hat{k}_x^2 + 14\hat{k}_y\sin 2\Omega\hat{k}_x + 7\hat{k}_y^2 + 21\hat{k}_z^2 \\
& +7(\hat{k}_x^2 - \hat{k}_y^2)\cos 2\Omega - 12]((\hat{k}_x^2 - \hat{k}_y^2)\sin 2\Omega - 2\hat{k}_x\hat{k}_y\cos 2\Omega)] \\
& -14\hat{k}_z^2[\hat{k}_z^2 - 9(1 - \hat{k}_z^2)]\cos 3i\} + 28\hat{k}_x\hat{k}_y(\hat{k}_y^2 - \hat{k}_x^2)(\cos 2\omega + 5)\sin 4\Omega \\
& +4\cos^2 i\{21\cos 4\Omega\hat{k}_x^4 - 21\hat{k}_z\cos\Omega\sin i\sin 2\omega\hat{k}_x^3 - 7\hat{k}_z\cot i\sin\Omega\hat{k}_x^3 \\
& -7\hat{k}_z\cot i\sin 3\Omega\hat{k}_x^3 - 126\hat{k}_y^2\cos 4\Omega\hat{k}_x^2 + 7\hat{k}_y\hat{k}_z\cos\Omega\cot i\hat{k}_x^2 \\
& +21\hat{k}_y\hat{k}_z\cos 3\Omega\cot i\hat{k}_x^2 - 21\hat{k}_y\hat{k}_z\sin i\sin 2\omega\sin\Omega\hat{k}_x^2 \\
& +63\hat{k}_y\hat{k}_z\sin i\sin 2\omega\sin 3\Omega\hat{k}_x^2 - 21\hat{k}_y^2\hat{k}_z\cos\Omega\sin i\sin 2\omega\hat{k}_x
\end{aligned}
$$

$$
\begin{aligned}
&- 7\hat{k}_y^2\hat{k}_z \cot i \sin \Omega\hat{k}_x + 16\hat{k}_z \cot i \sin \Omega\hat{k}_x + 280\hat{k}_y\hat{k}_z^2 \sin 2\Omega\hat{k}_x \\
&- 96\hat{k}_y \sin 2\Omega\hat{k}_x - 252\hat{k}_y\hat{k}_z^2 \cos 2i \sin 2\Omega\hat{k}_x + 21\hat{k}_y^2\hat{k}_z \cot i \sin 3\Omega\hat{k}_x \\
&+ 84\hat{k}_y(\hat{k}_x^2 - \hat{k}_y^2) \sin 4\Omega\hat{k}_x + 7\hat{k}_z^4 - [49(1 - \hat{k}_z^2) + 16]\hat{k}_z^2 \\
&- 6[7\hat{k}_z^2(\cos 2i - 1) + 2] \cos 2\omega(\hat{k}_y \cos \Omega - \hat{k}_x \sin \Omega)^2 \\
&- 3(1 - \hat{k}_z^2)[7(1 - \hat{k}_z^2) - 16] - 2(\hat{k}_x^2 - \hat{k}_y^2)(63 \cos 2i\hat{k}_z^2 - 70\hat{k}_z^2 + 24) \cos 2\Omega \\
&+ 21\hat{k}_y^4 \cos 4\Omega + 7\hat{k}_y^3\hat{k}_z \cos \Omega \cot i - 16\hat{k}_y\hat{k}_z \cos \Omega \cot i \\
&- 7\hat{k}_y^3\hat{k}_z \cos 3\Omega \cot i - 21\hat{k}_y^3\hat{k}_z \sin i \sin 2\omega \sin \Omega - 21\hat{k}_y^3\hat{k}_z \sin i \sin 2\omega \sin 3\Omega\}.
\end{aligned}
\tag{10}
$$

3. Conclusions

Thanks to their generality about the orientation of the central body's spin axis, my results can be applied to those systems in which it is poorly known, or even unknown at all.

I plan to extend my calculation to the even zonals of higher degree and to the odd zonals as well.

References

Anand, S. P. S. 1965, *Proc. Nat. Acad. Sci.* **54**, 23.

Anderson, J. D., Jacobson, R. A., Lau, E. L., Moore, W. B., Schubert, G. 2001a, *Geophys. Res.* **106**, 32963.

Anderson, J. D., Jacobson, R. A., McElrath, T. P. *et al.* 2001b, *Icarus*, **153**, 157.

Arutyunyan, G. G., Sedrakyan, D. M., Chubaryan, É. V. 1971a, *Sov. Astron.*, **15**, 390.

Arutyunyan, G. G., Sedrakyan, D. M., Chubaryan, É. V. 1971b, *Astrophys.*, **7**, 274.

Boshkayev, K., Quevedo, H., Ruffini, R. 2012, *Phys. Rev. D*, **86**, 064043.

Boshkayev, K., Izzo, L., Rueda Hernandez, J. A., Ruffini, R. 2013a, *AA*, **555**, A151.

Boshkayev, K., Rueda, J. A., Ruffini, R., Siutsou, I. 2013b, *AJ*, **762**, 117.

Brumberg, V. A. 1991, Essential Relativistic Celestial Mechanics (Adam Hilger, Bristol).

Capderou, M. 2005, Satellites. Orbits and Missions (Springer-Verlag France, Paris).

Chruściel, P. 1994, in Differential Geometry and Mathematical Physics, *Contemporary Mathematics*, vol. 170, edited by J. Beem & K. Duggal, pp. 23–49 (Providence: AMS).

Cook, A. H. 1963, *Space Sci. Rev.*, **2**, 355.

Damiani, C., Rozelot, J. P., Lefebvre, S., Kilcik, A., Kosovichev, A. G. 2011, *J. Atmos. Solar-Terrestrial Phys.*, **73**, 241.

Domiciano de Souza, A., Kervella, P., Jankov, S. *et al.* 2003, *AA*, **407**, L47.

Dufton, P. L., Dunstall, P. R., Evans, C. J. *et al.* 2011, *AJl*, **743**, L22.

Geroch, R. 1970, *J. Math. Phys.*, **11**, 2580.

Giacaglia, G. E. O. 1964, *AJ*, **69**, 303.

Godier, S., Rozelot, J.-P. 1999, *AA*, **350**, 310.

Granada, A., Ekström, S., Georgy, C. *et al.* 2013, *AA*, **553**, A25.

Hansen, R. O. 1974, *J. Math. Phys.*, **15**, 46.

Hartle, J. B. 1967, *AJ*, **150**, 1005.

Heimberger, J., Soffel, M., Ruder, H. 1990, *Celest. Mech. Dyn. Astron.*, **47**, 205.

Heusler, M. 1998, *Living Rev. Relativ.*, **1**, 6.

Hirata, R. 1995, *Publ. Astron. Soc. Japan*, **47**, 195.

Hirt, C., Claessens, S. J., Kuhn, M., Featherstone, W. E. 2012, *Planet. Space Sci.*, **67**, 147.
Iess, L., Rappaport, N. J., Jacobson, R. A. *et al.* 2010, *Science*, **327**, 1367.
Iess, L., Rappaport, N. J., Tortora, P. *et al.* 2007, *Icarus*, **190**, 585.
Iorio, L. 2001, *Il Nuovo Cimento B*, **116**, 777.
Iorio, L. 2002, *Phys. Lett. A*, **298**, 315.
Iorio, L. 2003, *Celest. Mech. Dyn. Astron.*, **86**, 277.
Iorio, L. 2005a, *AA*, **431**, 385.
Iorio, L. 2005b, *AA*, **433**, 385.
Iorio, L. 2006a, *J. Geodesy*, **80**, 128.
Iorio, L. 2006b, *Classical Quantum Gravity*, **23**, 5451.
Iorio, L. 2007a, *Astrophys. Space Sci.*, **312**, 337.
Iorio, L. 2007b, *Astrophys. Space Sci.*, **310**, 73.
Iorio, L. 2008a, *Int. J. Modern Phys. D*, **17**, 815.
Iorio, L. 2008b, *Astrophys. Space Sci.*, **318**, 51.
Iorio, L. 2009a, *Space Sci. Rev.*, **148**, 363.
Iorio, L. 2009b, *AJ*, **137**, 3615.
Iorio, L. 2009c, *Gen. Relativ. Gravit.*, **41**, 1717.
Iorio, L. 2010a, *Central European J. Phys.*, **8**, 25.
Iorio, L. 2010b, *Central European J. Phys.*, **8**, 509.
Iorio, L. 2011a, *Astrophys. Space Sci.*, **331**, 485.
Iorio, L. 2011b, *Int. J. Modern Phys. D*, **20**, 181.
Iorio, L. 2011c, *Gen. Relativ. Gravit.*, **43**, 1697.
Iorio, L. 2011d, *Phys. Rev. D*, **84**, 124001.
Iorio, L. 2011e, *Europhys. Lett.*, **96**, 30001.
Iorio, L. 2012a, *JHEP*, **5**, 73.
Iorio, L. 2012b, *Adv. Space Res.*, **50**, 334.
Iorio, L. 2013, *Classical Quantum Gravity*, **30**, 195011.
Iorio, L., Ciufolini, I., Pavlis, E. C. *et al.* 2004, *Classical Quantum Gravity*, **21**, 2139.
Iorio, L., Lichtenegger, H. I. M., Ruggiero, M. L., Corda, C. 2011, *Astrophys. Space Sci.*, **331**, 351.
Iorio, L., Ruggiero, M., Corda, C. 2013, *Acta Astronautica*, **91**, 141.
Jacobson, R. A., Antreasian, P. G., Bordi, J. J. *et al.* 2006, *Astron. J.*, **132**, 2520.
Johnson, W., Winget, D., Douglass, D., Horn, H. 1980, in Nonradial and Nonlinear Stellar Pulsation, Lecture Notes in Physics, vol. 125, edited by H. Hill & W. Dziembowski, pp. 357–368 (Springer Berlin Heidelberg).
Kaula, W. 2000, Theory of Satellite Geodesy: Applications of Satellites to Geodesy (Dover, New York).
King-Hele, D. G. 1961, *Geophys. J. Int.*, **4**, 3.
King-Hele, D. G. 1983, *Geophys. J. Int.*, **74**, 7.
King-Hele, D. G., Cook, G. E. 1973, *Nature*, **246**, 86.
King-Hele, D. G., Cook, G. E., Rees, J. M. 1963, *Nature*, **197**, 785.
Konopliv, A. S., Banerdt, W. B., & Sjogren, W. L. 1999, *Icarus*, **139**, 3.
Konopliv, A. S., Yoder, C. F., Standish, E. M., Yuan, D.-N., Sjogren, W. L. 2006, *Icarus*, **182**, 23.
Kozai, Y. 1966, *Space Sci. Rev.*, **5**, 818.
Laarakkers, W. G., & Poisson, E. 1999, *Astrophys. J.*, **512**, 282.
Lemoine, F., Kenyon, S., Factor, J. *et al.* 1998, The Development of the Joint NASA GSFC and the National Imagery and Mapping Agency (NIMA) Geopotential ModelEGM96, NASA Technical Paper NASA/TP1998206861, Goddard Space Flight Center, Greenbelt, USA.
Lucchesi, D. M. 2003, *Phys. Lett. A*, **318**, 234.
Lucchesi, D. M. 2005, *Int. J. Modern Phys. D*, **14**, 1989.

Lucchesi, D. M. 2007, *Adv. Space Res.*, **39**, 324.
Lucchesi, D. M. 2011, *Adv. Space Res.*, **47**, 1232.
Lucchesi, D. M., Peron, R. 2010, *Phys. Rev. Lett.*, **105**, 231103.
Lydon, T. J., Sofia, S. 1996, *PRL*, **76**, 177.
McAlister, H. A., ten Brummelaar, T. A., Gies, D. R. *et al.* 2005, *AJ*, **628**, 439.
Meilland, A., Stee, P., Chesneau, O., Jones, C. 2009, *AA*, **505**, 687.
Nemravová, J., Harmanec, P., Kubát, J. *et al.* 2010, *AA*, **516**, A80.
Null, G. W., Anderson, J. D., Wong, S. K. 1975, *Science*, **188**, 476.
O'Keefe, J. A., Eckels, A., Squires, R. K. 1959, *Science*, **129**, 565.
Pappas, G., Apostolatos, T. A. 2012, *Phys. Rev. Lett.*, **108**, 231104.
Pijpers, F. P. 1998, *MNRAS*, **297**, L76.
Pireaux, S., Rozelot, J.-P. 2003, *Astrophys. Space Sci.*, **284**, 1159.
Reigber, C., Schmidt, R., Flechtner, F. *et al.* 2005, *J. Geodyn.*, **39**, 1.
Roxburgh, I. W. 2001, *AA*, **377**, 688.
Rozelot, J., Pireaux, S., Lefebvre, S., Corbard, T. 2004, arXiv:astro-ph/0403382.
Rozelot, J.-P., Damiani, C. 2011, *European Phys. J. H*, **36**, 407.
Rozelot, J. P., Fazel, Z. 2013, *Solar Physics*.
Shibata, M. 1998, *Prog. Theor. Phys.*, **99**, 69.
Simon, W., Beig, R. 1983, *J. Math. Phys.*, **24**, 1163.
Singh, J., Umar, A. 2013, *Astrophys. Space Sci.*, at press.
Smith, D. E., Zuber, M. T., Phillips, R. J. *et al.* 2012, *Science*, **336**, 214.
Soffel, M., Wirrer, R., Schastok, J., Ruder, H., Schneider, M. 1988, *Celest. Mech.*, **42**, 81.
Stergioulas, N. 2003, *Living Rev. Relativ.*, **6**, 3.
Tapley, B. D., Bettadpur, S., Watkins, M., Reigber, C. 2004, *Geophys. Res. Lett.*, **31**, L09607.
Ulrich, R. K., Hawkins, G. W. 1981, *Astrophysical J.*, **246**, 985.
van Belle, G. T. 2012, *Astron. Astrophys. Rev.*, **20**, 51.
van Belle, G. T., Ciardi, D. R., Thompson, R. R., Akeson, R. L., Lada, E. A. 2001, *Astrophysical J.*, **559**, 1155.
Vigeland, S. J. 2010, *Phys. Rev. D*, **82**, 104041.
Wagner, C. A. 1973, *J. Geophys. Res.*, **78**, 3271.
Xu, Y., Yang, Y., Zhang, Q., Xu, G. 2011, *MNRAS*, **415**, 3335.
Yan, J., Zhong, Z., Li, F. *et al.* 2013, *Adv. Space Res.*, **52**, 512.
Yoon, J., Peterson, D. M., Kurucz, R. L., Zagarello, R. J. 2010, *Astrophysical J.*, **708**, 71.