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Collinear Equilibrium Solutions of Four-body Problem

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Abstract. We discuss the equilibrium solutions of four different types of collinear four-body problems having two pairs of equal masses. Two of these four-body models are symmetric about the center-of-mass while the other two are non-symmetric. We define two mass ratios as $\mu_1 = m_1/M_{\mathrm{T}}$ and $\mu_2 = m_2/M_{\mathrm{T}}$, where m_1 and m_2 are the two unequal masses and M_{T} is the total mass of the system. We discuss the existence of continuous family of equilibrium solutions for all the four types of four-body problems.

Key words. Equilibrium solutions—celestial mechanics—four-body problem—few-body problem.

1. Introduction

The motion of systems of n-bodies under their mutual gravitational attraction has always fascinated mathematicians and astronomers which dates back to the times of Isaac Newton. Because of the complicated nature of the solutions, few-body orbits in the most general cases could not be determined before the age of computers and the development of appropriate numerical tools. Today the few-body problem is recognized as a standard tool in astronomy and astrophysics, from solar system dynamics to galactic dynamics (Murray and Dermot 1999).

During the past century, celestial mechanics has principally been devoted to the study of the three-body problem. Due to the difficulty in handling additional parameters in the four-body problems, very little analytical work has been carried out for greater than three bodies.

In this paper we discuss collinear equilibrium configurations of four-body problem. *An equilibrium configuration of four-bodies is a geometric configuration of four bodies in which the gravitational forces are balanced in such a way that the four bodies rotate together about their center-of-mass and thus the geometric configuration is maintained for all time.* Some very interesting results have been obtained by different authors on this subject. Here we will give a brief review of some of them mainly on collinear equilibrium configurations.

The straight line solutions of the n-body problem were first published by Moulton (1910). He arranged n masses on a straight line and solved the problem of the values of the masses at n arbitrary collinear points so that they remained collinear under suitable initial projections. This paper is not particularly concerned about the specific case of the four-body problem, but it can be deduced that there are 12 collinear solutions in the equal mass four-body problem. These solutions are sometimes referred to as Moulton solutions. The two papers on the classification of relative equilibria by Palmore (1975, 1982) presented several theorems on the classification of equilibrium points in the planar n-body problem.

The finiteness of central configurations for the general four-body problem has been investigated by Zhiming and Yisui (1988). They showed that for the collinear four-body problem there are twelve central configurations for each set of masses. Using algebraic and geometric methods, Arenstrof (1982) investigated the number of equivalence classes of central configurations in the planar four-body problem with three arbitrary sized masses and a fourth small mass m_4. His results showed that each three-body collinear central configuration generated exactly two non-collinear central configurations (besides four collinear ones) of four bodies with small $m_4 \geq 0$; and that the three-body equilateral triangle central configuration generated exactly eight, nine or ten planar four-body central configurations with $m_4 = 0$. Glass (1997) studied the central configurations of the classical N-body problem and the asymptotic properties of a system of repelling particles. An asymmetric configuration obtained in the eight-particle system is described and a bifurcation in the four-particle system is investigated.

Gomatan *et al.* (1999), Kozak and Oniszk (1998) and Majorana (1981) derived equilibrium solutions and analyzed their stability for different types of four-body problems. Majorana (1981) studied the linear stability of the equilibrium points in the restricted four-body problem, where three bodies of masses μ, μ and $1 - 2\mu$ rotate in an equilateral triangular configuration (the Lagrange solution), whilst the fourth body of negligible mass moves in the same plane. The equations of motion of the particle under the influence of the other three bodies were derived which led to the determination of eight equilibrium points. Kozak and Oniszk (1998) studied the motion of a negligible mass in the gravitational field generated by a collinear configuration of three bodies (of masses $m_0 \neq m_1 = m_2 = m$). They proved that the straight line (collinear) configurations are linearly unstable for any value m as in the Lagrange case of the three-body problem and obtained the stable and unstable regions around the triangular points.

More recent works on the collinear problem include those of Douskos (2010), and Ouyang and Xie (2005). Douskos discussed the existence and stability of the collinear equilibrium points of a generalized Hill problem and showed the existence of two equilibrium points for a positive oblateness co-efficient. Ouyang and Xie (2005) found regions on the configuration space where it is possible to choose masses for collinear configuration of four bodies which will make it central.

Roy and Steves (1998) discussed some special analytical solutions of the four-body problems. They showed that these solutions reduce to the Lagrange solutions of the Copenhagen problem when two of the masses are equally reduced. In this paper, we will derive the collinear solutions of Roy and Steves (1998). We will discuss the existence of continuous family of equilibrium solutions for the above-mentioned collinear four-body problems which include two symmetric arrangements of two pairs of masses (Section 2) and two non-symmetric arrangement of two pairs of masses (Section 3).

2. Symmetric collinear equilibrium configurations for two pairs of masses

In this section we have derived the two symmetric collinear equilibrium solutions of Roy and Steves (1998) and gave a very simple proof of the existence of infinite family of equilibrium solutions.

For symmetrical arrangement of two pairs of different masses, there are two possibilities:

(1) The pair of larger masses M lie in the middle and the pair of smaller masses m lie at the corners apiece, as shown in Fig. 1.
(2) The pair of smaller masses m lie in the middle and the pair of larger masses M lie at the corners apiece, as shown in Fig. 2.

These arrangements are symmetric about the center of mass C.

Let $r_2 = \alpha r_1$, $\mu_1 = m/M_T$ and $\mu_2 = M/M_T$ and $\rho_{\mathbf{ij}} = \mathbf{r_{ij}}/\mathbf{r_{ij}^3}$. By symmetry $\mathbf{r_4} = -\mathbf{r_1}$ and $\mathbf{r_3} = -\mathbf{r_2}$. $M_T = 2(m + M)$ is the total mass of the system.

The classical equation of motion for the N-body problem has the form

$$m_i \frac{d^2\mathbf{r_i}}{dt^2} = \frac{\partial U}{\partial r_i} = \sum_{j \neq i} \frac{m_i m_j \left(\mathbf{r_j} - \mathbf{r_i}\right)}{|\mathbf{r_i} - \mathbf{r_j}|^3}, \qquad i = 1, 2, \ldots, n, \tag{1}$$

where the units are so chosen that the gravitational constant is equal to one, $\mathbf{r_i}$ is a vector in three space,

$$U = \sum_{1 \preceq i < j \preceq n} \frac{m_i m_j}{|\mathbf{r_i} - \mathbf{r_j}|} \tag{2}$$

is the self-potential, $\vec{r}_i$ is the location vector of the ith body and m_i is the mass of the ith body. Now using the general equations of motion given above in conjunction with the symmetry conditions, we obtain the following final form of the equations of motion

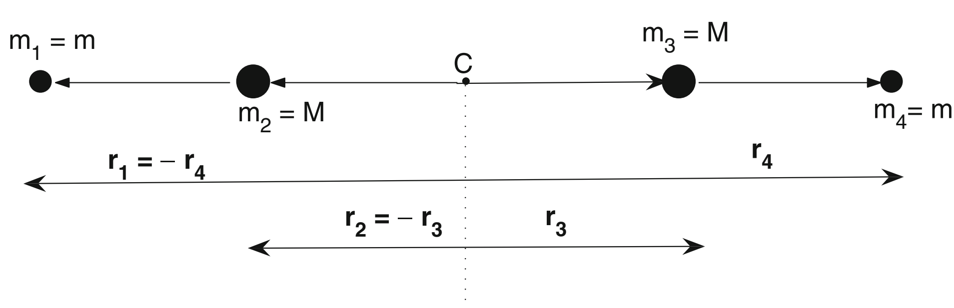

Figure 1. Symmetric collinear equilibrium configuration (1).

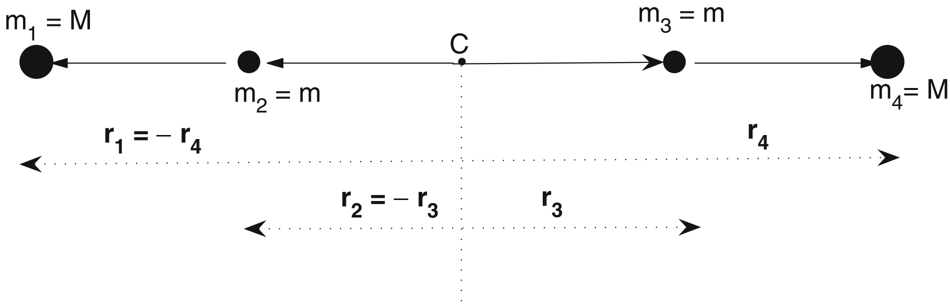

Figure 2. Symmetric collinear equilibrium configuration (2).

$$\ddot{\mathbf{r}}_1 = -\frac{M_{\mathrm{T}}}{r_1^3} R_1 \mathbf{r_1},$$

$$\ddot{\mathbf{r}}_2 = -\frac{M_{\mathrm{T}}}{r_1^3} R_2 \mathbf{r_2},$$

where

$$R_1 = \mu_2 \left(\frac{1}{(1-\alpha)^2} + \frac{1}{(1+\alpha)^2} \right) + \frac{\mu_1}{4},$$

$$R_2 = \frac{\mu_2}{4\alpha^3} + \frac{\mu_1}{\alpha} \left(\frac{1}{(1+\alpha)^2} - \frac{1}{(1-\alpha)^2} \right).$$

For a rigid rotating geometry, we must have $R_1 - R_2 = 0$. Therefore

$$\mu_2 \left(\frac{1}{(1-\alpha)^2} + \frac{1}{(1+\alpha)^2} \right) - \frac{\mu_2}{4\alpha^3} - \frac{\mu_1}{\alpha} \left(\frac{1}{(1+\alpha)^2} - \frac{1}{(1-\alpha)^2} \right) + \frac{\mu_1}{4} = 0.$$

After further simplification, we get

$$H(\alpha, \mu_1, \mu_2) = \frac{2\mu_2\left(1+\alpha^2\right) + 4\mu_1}{\left(1-\alpha^2\right)^2} + \frac{\mu_1\alpha^3 - \mu_2}{4\alpha^3} = 0. \tag{3}$$

We re-arrange equation (3) to get

$$H(\alpha, \mu_1, \mu_2) = \frac{f(\mu_1, \mu_2, \alpha)}{g(\alpha)},$$

where

$$\begin{aligned} f(\mu_1, \mu_2, \alpha) &= \mu_1\alpha^7 + (8\mu_2 - 2\mu_1)\alpha^5 - \mu_2\alpha^4 \\ &\quad + (8\mu_2 + 17\mu_1)\alpha^3 + 2\mu_2\alpha^2 - \mu_2 \end{aligned}$$

and

$$g(\alpha) = 4\alpha^3(1-\alpha^2)^2.$$

We know that $\mu_1 + \mu_2 = 1/2$, therefore $f(\mu_1, \mu_2, \alpha)$ becomes

$$\begin{aligned} f(\mu_1, \alpha) &= \mu_1\alpha^7 + (4 - 10\mu_1)\alpha^5 - (1/2 - \mu_1)\alpha^4 + (4 + 9\mu_1)\alpha^3 \\ &\quad + (1 - 2\mu_1)\alpha^2 + \mu_1 - 1/2, \end{aligned}$$

$$\lim_{\alpha\to 0} f(\mu_1, \alpha) = \mu_1 - \frac{1}{2} < 0 \text{ and } \lim_{\alpha\to 1} f(\mu_1, \alpha) = 8 > 0.$$

As $f(\mu_1, \alpha)$ is a polynomial in α and therefore continuous, and $\lim_{\alpha\to 0} f(\mu_1, \alpha) < 0$ for $0 \le \mu_1 < 0.5$ and $\lim_{\alpha\to 1} f(\mu_1, \alpha) > 0$ for all μ_1 therefore by the intermediate value theorem of calculus, for any μ_1 in (0, 1/2), there exists at least one α in (0, 1) such that $H(\alpha, \mu_1) = 0$.

This shows that there exists a continuous family of equilibrium solutions for the two collinear and symmetric arrangements of the two pairs of equal masses as shown in Figs 1 and 2.

3. Non-symmetric collinear equilibrium solutions for two pairs of masses

In this section, we discuss the two non-symmetric collinear equilibrium solutions of Roy and Steves (1998) and in both cases prove the existence of equilibrium solutions for all mass ratios.

3.1 *Case-I*

From Fig. 3 we can see that this is a non-symmetric arrangement of the four bodies.

Let

$$\mathbf{r_2} = \alpha\mathbf{r_1};\ \mathbf{r_3} = -\beta\mathbf{r_1};\ \mathbf{r_4} = -\gamma\mathbf{r_1}. \tag{4}$$

By the center-of-mass relation

$$\sum_{i=1}^{4} m_i\mathbf{r_i} = \mathbf{0}.$$

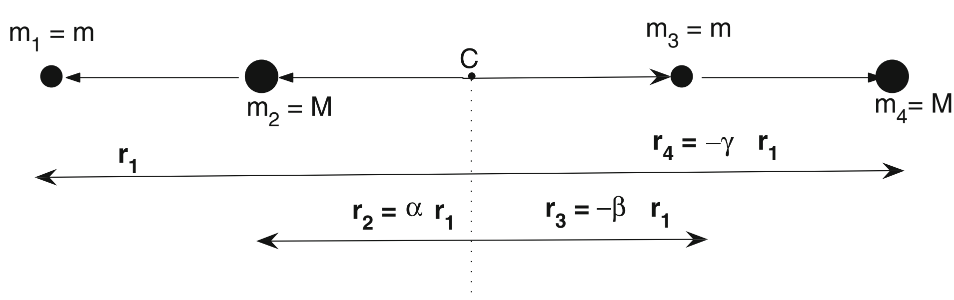

Figure 3. Non-symmetric collinear equilibrium configuration (Case-I).

Thus

$$m\mathbf{r_1} + M\mathbf{r_2} + m\mathbf{r_3} + M\mathbf{r_4} = \mathbf{0}.$$

After substituting $\mu_1 = m/M_{\mathrm{T}}$ and $\mu_2 = M/M_{\mathrm{T}}$, we get

$$\mu_1\,(\mathbf{r_1} + \mathbf{r_3}) + \mu_2(\mathbf{r_2} + \mathbf{r_4}) = \mathbf{0}. \tag{5}$$

From equations (4) and (5), we get

$$(\mu_1\,(1-\beta) + \mu_2\,(\alpha-\gamma))\,\mathbf{r_1} = \mathbf{0}.$$

As $\mathbf{r_1} \neq \mathbf{0}$, therefore

$$\mu_1\,(1-\beta) + \mu_2\,(\alpha-\gamma) = 0. \tag{6}$$

The equations of motion are given as below which are derived using equations (1) and (2).

$$\ddot{\mathbf{r}}_1 = -\frac{M_{\mathrm{T}}}{r_1^3} R_1 \mathbf{r_1},$$

$$\ddot{\mathbf{r}}_2 = -\frac{M_{\mathrm{T}}}{r_1^3} R_2 \mathbf{r_2},$$

$$\ddot{\mathbf{r}}_3 = -\frac{M_{\mathrm{T}}}{r_1^3} R_3 \mathbf{r_3},$$

$$\ddot{\mathbf{r}}_4 = -\frac{M_{\mathrm{T}}}{r_1^3} R_4 \mathbf{r_4},$$

where

$$\begin{aligned}
R_1 &= \frac{\mu_1}{(1+\beta)^2} + \mu_2\left(\frac{1}{(1-\alpha)^2} + \frac{1}{(1+\gamma)^2}\right),\\
R_2 &= \frac{\mu_1}{\alpha}\left(\frac{1}{(\alpha+\beta)^2} + \frac{1}{(1-\alpha)^2}\right) + \frac{\mu_2}{\alpha\,(\alpha+\gamma)^2},\\
R_3 &= \frac{\mu_1}{\beta\,(1+\beta)^2} + \frac{\mu_2}{\beta}\left(\frac{1}{(\alpha+\beta)^2} - \frac{1}{(\gamma-\beta)^2}\right),\\
R_4 &= \frac{\mu_1}{\gamma}\left(\frac{1}{(1+\gamma)^2} + \frac{1}{(\gamma-\beta)^2}\right) + \frac{\mu_2}{(\gamma+\alpha)^2}.
\end{aligned}$$

For a rigid rotating solution, we must have $R_1 = R_2 = R_3 = R_4 > 0$ i.e. $R_i = R > 0$. We therefore require $\rho_i = R_i - R_1 = 0$, $i = 2, 3, 4$ for suitable values of α, β and γ.

If $R_1 = R_2 = R_3$, then necessarily $R_4 = R_3 = R_2 = R_1$ (Roy and Steves 1998). To show that the collinear non-symmetric arrangement of the two pairs of equal masses as shown in Fig. 3 has an equilibrium solution for any mass ratio, it is sufficient to show that $R_2 - R_1 = 0$, $R_3 - R_1 = 0$ has a solution.

Let $\mu := \frac{\mu_1}{\mu_2}$. Since, $\mu_1 = \frac{m}{M_{\mathrm{T}}} < \mu_2 = \frac{M}{M_{\mathrm{T}}}$, μ is in $(0, 1)$.

We first express γ in terms of α and β using equation (6). Then, dividing both $R_2 - R_1$ and $R_3 - R_1$ by μ_2, we only need to solve the system $R_{21} = 0$, $R_{31} = 0$, where

$$\begin{aligned} R_{21} &= \frac{\mu\left(\frac{1}{(1-\alpha)^2} + \frac{1}{(\alpha+\beta)^2}\right)}{\alpha} + \frac{1}{\alpha(2\alpha+\mu(1-\beta))^2} - \frac{1}{(1-\alpha)^2} \\ &\quad - \frac{\mu}{(1+\beta)^2} - \frac{1}{(1+\alpha+\mu(1-\beta))^2}, \end{aligned}$$

$$\begin{aligned} R_{31} &= \frac{\frac{\mu}{(1+\beta)^2} + \frac{1}{(\alpha+\beta)^2} - \frac{1}{(\alpha-\beta+\mu(1-\beta))^2}}{\beta} - \frac{1}{(1-\alpha)^2} - \frac{\mu}{(1+\beta)^2} \\ &\quad - \frac{1}{(1+\alpha+\mu(1-\beta))^2}. \end{aligned}$$

Let T be the open triangle limited by the lines $\alpha = \beta$, $\alpha = 1$ and $\beta = 0$. The following inequalities hold on T: $0 < \alpha < 1$, $0 < \beta < 1$, $\beta < \alpha$. We deduce that $1-\alpha > 0$, $1-\beta > 0$, $\alpha - \beta > 0$ and $0 < \mu \le 1$. We also have $2\alpha + \mu(1-\beta) > 0$, $1+\alpha+\mu(1-\beta) > 0$, $\alpha-\beta+\mu(1-\beta) > 0$. All denominators involved in the rational functions R_{21} and R_{31} are strictly positive. This shows that for any $\mu \in [0, 1]$, R_{21} and R_{31} are continuous on T.

Let $I_1 := (0, 0.8)$ and $I_2 := [0.8, 1]$. For a given μ, let Rec1 be the rectangle limited by the lines $\alpha = \frac{41}{100}$, $\alpha = \frac{90}{100}$, $\beta = \frac{\mu}{100}$ and $\beta = \frac{40}{100}$. Let Rec2 be the rectangle limited by $\alpha = \frac{80}{100}$, $\alpha = \frac{99.99}{100}$, $\beta = 10^{-10}$ and $\beta = 0.1$. Rec1 and Rec2 are inside T.

Using the continuity of R_{21} and R_{31} on T, the existence of a continuous family of solutions can be shown by proving that for any $\mu \in I_1$, the graph of R_{21} intersects Rec1 at the lines $\beta = \frac{\mu}{100}$, $\beta = \frac{40}{100}$ and the graph of R_{31} intersects Rec1 at the lines $\alpha = \frac{41}{100}$, $\alpha = \frac{90}{100}$.

To show, for example, that R_{21} intersects Rec1 on the lowest horizontal side $\left(\beta = \frac{\mu}{100}\right)$ we will show that $R_{21}\left(\frac{41}{100}, \frac{\mu}{100}\right)$ is positive and $R_{31}\left(\frac{41}{100}, \frac{\mu}{100}\right)$ is negative. Since R_{21} is continuous on T, using the intermediate value theorem, there exists a value $\alpha \in \left(\frac{41}{100}, \frac{90}{100}\right)$ such that $R_{21}\left(\alpha, \frac{\mu}{100}\right) = 0$, which means R_{21} intersects Rec1 on the side $\beta = \frac{\mu}{100}$.

Since the denominator of R_{21} is positive, we just need to know the sign of its numerator. The numerator of $R_{21}\left(\frac{41}{100}, \frac{\mu}{100}\right)$ can be written as

$$\begin{aligned} f(\mu) &= 1.0 \times 10^6\mu^{13} - 1.184 \times 10^8\mu^{12} - 2.30 \times 10^{10}\mu^{11} + 2.62 \times 10^{12}\mu^{10} \\ &\quad + 1.45 \times 10^{14}\mu^9 - 8.06 \times 10^{15}\mu^8 - 1.03 \times 10^{18}\mu^7 - 2.62 \times 10^{19}\mu^6 \\ &\quad + 5.02 \times 10^{21}\mu^5 + 2.17 \times 10^{22}\mu^4 + 3.32 \times 10^{22}\mu^3 + 2.08 \times 10^{22}\mu^2 \\ &\quad + 4.29 \times 10^{21}\mu + 8.07 \times 10^{19}. \end{aligned}$$

Using an optimization assistant (Maple), we obtain that the above function is positive on I_1.

The numerator of $R_{21}\left(\frac{90}{100}, \frac{\mu}{100}\right)$ can be written as

$$\begin{aligned} f(\mu) \;=\; & 1.0 \times 10^{3}\mu^{13} - 2.09 \times 10^{4}\mu^{12} - 3.8 \times 10^{7}\mu^{11} + 5.56 \times 10^{8}\mu^{10} \\ & + 5.6 \times 10^{11}\mu^{9} - 4.3 \times 10^{12}\mu^{8} - 3.7 \times 10^{15}\mu^{7} + 1.9 \times 10^{15}\mu^{6} \\ & + 8.9 \times 10^{18}\mu^{5} + 5.6 \times 10^{19}\mu^{4} + 1.2 \times 10^{20}\mu^{3} \\ & + 5.5 \times 10^{19}\mu^{2} - 9.3 \times 10^{19}\mu - 8.5 \times 10^{19}. \end{aligned}$$

It is a negative function on I_1. We apply the same method on the side $\beta = \frac{40}{100}$. The numerator of $R_{21}\left(\frac{41}{100}, \frac{40}{100}\right)$ becomes

$$\begin{aligned} f(\mu) \;=\; & 1.60 \times 10^{17}\mu^{5} + 1.2 \times 10^{18}\mu^{4} + 3.06 \times 10^{18}\mu^{3} + 3.13 \times 10^{18}\mu^{2} \\ & + 1.0 \times 10^{18}\mu + 4.28 \times 10^{16}. \end{aligned}$$

The function is positive on I_1. For $R_{21}\left(\frac{90}{100}, \frac{40}{100}\right)$, we obtain

$$\begin{aligned} f(\mu) \;=\; & 1.80 \times 10^{16}\mu^{5} + 7.97 \times 10^{17}\mu^{4} + 1.28 \times 10^{19}\mu^{3} + 8.85 \times 10^{19}\mu^{2} \\ & + 1.96 \times 10^{19}\mu + 7.43 \times 10^{18}. \end{aligned}$$

The function is negative on I_1. This proves that the graph of R_{12} intersects Rec1 on the side $\beta = \frac{40}{100}$. The numerator of $R_{31}\left(\frac{41}{100}, \frac{\mu}{100}\right)$ becomes

$$f(\mu) = 10000\mu \begin{pmatrix} \mu^{12} + 3.36 \times 10^{4}\mu^{11} - 1.47 \times 10^{6}\mu^{10} + 2.10 \times 10^{8}\mu^{9} \\ -8.30 \times 10^{9}\mu^{8} - 5.80 \times 10^{11}\mu^{7} + 4.58 \times 10^{13}\mu^{6} \\ +3.18 \times 10^{14}\mu^{5} - 4.9979 \times 10^{16}\mu^{4} - 5.35 \times 10^{17}\mu^{3} \\ -1.49 \times 10^{18}\mu^{2} - 1.59 \times 10^{19}\mu - 5.75 \times 10^{17} \end{pmatrix}.$$

It is positive on I_1. The numerator of $R_{31}\left(\frac{41}{100}, \frac{40}{100}\right)$ becomes

$$\begin{aligned} f(\mu) \;=\; & 2.46 \times 10^{15}\mu^{5} + 1.46 \times 10^{16}\mu^{4} + 2.83 \times 10^{16}\mu^{3} - 1.37 \times 10^{16}\mu^{2} \\ & - 1.05 \times 10^{17}\mu - 1.24 \times 10^{17}. \end{aligned}$$

It is negative on I_2. Using again the intermediate value theorem, we obtain that the graph of R_{31} intersects Rec1 at the side $\alpha = \frac{41}{100}$. The numerator of $R_{31}\left(\frac{90}{100}, \frac{\mu}{100}\right)$ becomes

$$f(\mu) = 100\mu \begin{pmatrix} -1.0\mu^{12} - 82.0\mu^{11} + 7.0 \times 10^{4}\mu^{10} - 2.17 \times 10^{6}\mu^{9} \\ -9.59 \times 10^{8}\mu^{8} + 5.15 \times 10^{10}\mu^{7} + 3.89 \times 10^{12}\mu^{6} \\ -2.32 \times 10^{14}\mu^{5} - 1.56 \times 10^{15}\mu^{4} + 6.47 \times 10^{15}\mu^{3} \\ +5.24 \times 10^{16}\mu^{2} + 1.021 \times 10^{17}\mu + 6.36 \times 10^{16} \end{pmatrix}.$$

It is positive on I_1. The numerator of $R_{31}\left(\frac{90}{100}, \frac{40}{100}\right)$ becomes

$$\begin{aligned} f(\mu) \;=\; & 1.64 \times 10^{7}\mu^{5} - 1.98 \times 10^{9}\mu^{4} - 1.65 \times 10^{10}\mu^{3} - 4.48 \times 10^{10}\mu^{2} \\ & - 4.55 \times 10^{10}\mu - 1.62 \times 10^{10}. \end{aligned}$$

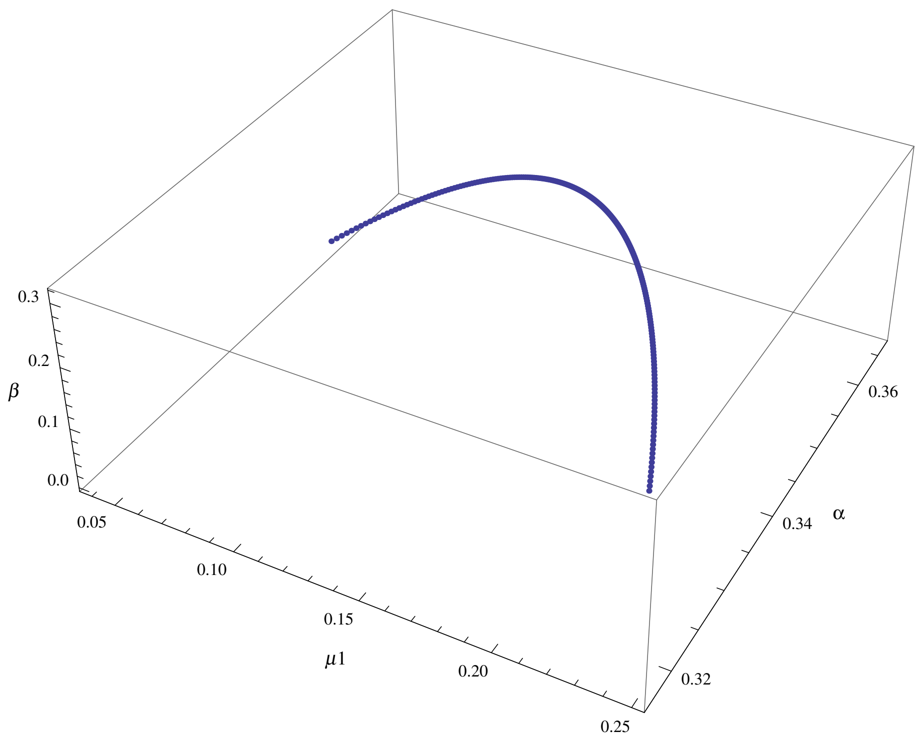

Figure 4. Family of equilibrium solutions in the first non-symmetric arrangement of two pairs of masses.

It is negative on I_2. Using again the intermediate value theorem, we obtain that the graph of R_{31} intersects Rec1 also at the side $\alpha = \frac{90}{100}$.

The graph of R_{21} intersects Rec1 on its two horizontal sides; and the graph of R_{31} intersects Rec1 on its two vertical sides. As they are both continuous on Rec1, they intersect in Rec1.

The continuous family of solutions can also be seen in Fig. 4 where the values of α and β are given for values of μ_1 and μ_2.

3.2 *Case-II*

This is the second non-symmetric collinear arrangement of the four masses as shown in Fig. 5. This will be treated the same way as the first case.

Let

$$\mathbf{r_2} = \alpha \mathbf{r_1}; \quad \mathbf{r_3} = -\beta \mathbf{r_1}; \quad \mathbf{r_4} = -\gamma \mathbf{r_1}. \tag{7}$$

From the center-of-mass relation

$$\mu_1(\mathbf{r_1} + \mathbf{r_2}) + \mu_2 (\mathbf{r_3} + \mathbf{r_4}) = \mathbf{0}, \tag{8}$$

where $\mu_1 = m/M_{\mathrm{T}}$ and $\mu_2 = M/M_{\mathrm{T}}$. Now from equations (7) and (8), we get

$$(\mu_1 (1 + \alpha) - \mu_2 (\beta + \gamma)) \mathbf{r_1} = \mathbf{0}.$$

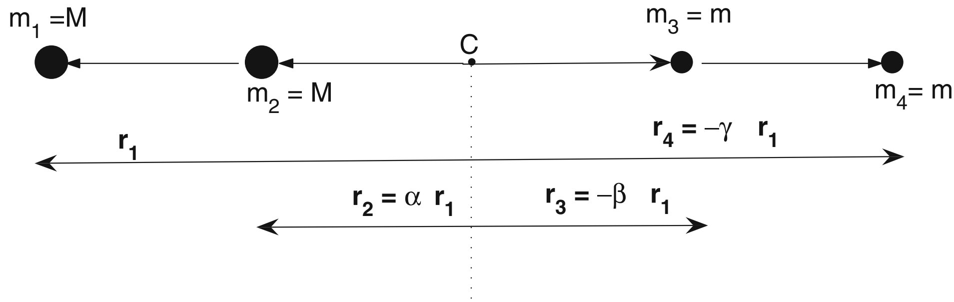

Figure 5. Non-symmetric collinear equilibrium configuration (Case-II).

As $\mathbf{r_1} \neq \mathbf{0}$, therefore

$$\mu_1 (1+\alpha) - \mu_2 (\beta + \gamma) = 0.$$

The equations of motion are given below which are derived in the same way as Roy and Steves (1998) but in our own notations.

$$\begin{aligned}
\ddot{\mathbf{r}}_1 &= -\frac{M_{\mathrm{T}}}{r_1^3} R_1 \mathbf{r_1},\\
\ddot{\mathbf{r}}_2 &= -\frac{M_{\mathrm{T}}}{r_1^3} R_2 \mathbf{r_2},\\
\ddot{\mathbf{r}}_3 &= -\frac{M_{\mathrm{T}}}{r_1^3} R_3 \mathbf{r_3},\\
\ddot{\mathbf{r}}_4 &= -\frac{M_{\mathrm{T}}}{r_1^3} R_4 \mathbf{r_4},
\end{aligned}$$

where

$$\begin{aligned}
R_1 &= \frac{\mu_1}{(1-\alpha)^2} + \mu_2 \left(\frac{1}{(1+\beta)^2} + \frac{1}{(1+\gamma)^2} \right),\\
R_2 &= \frac{\mu_2}{\alpha} \left(\frac{1}{(\alpha+\beta)^2} + \frac{1}{(\alpha+\gamma)^2} \right) - \frac{\mu_1}{\alpha(1-\alpha)^2},\\
R_3 &= \frac{\mu_1}{\beta} \left(\frac{1}{(1+\beta)^2} + \frac{1}{(\alpha+\beta)^2} \right) - \frac{\mu_2}{\beta\,(\gamma-\beta)^2},\\
R_4 &= \frac{\mu_1}{\gamma} \left(\frac{1}{(1+\gamma)^2} + \frac{1}{(\gamma+\alpha)^2} \right) + \frac{\mu_2}{\gamma\,(\gamma-\beta)^2}.
\end{aligned}$$

For a rigid rotating solution, we must have $R_1 = R_2 = R_3 = R_4 > 0$ i.e. $R_i = R > 0$. We therefore require $\rho_i := R_i - R_1 = 0$, $i = 2, 3, 4$ for the set of values of α and β. As mentioned earlier, we just need two equations since if two are satisfied, the third will automatically be satisfied.

$$\begin{aligned}\rho_3 &= \frac{\mu_1}{(\alpha-1)^2} - \frac{\mu_1^2(-0.5+\mu_1)}{\beta(-2\mu_1\beta+(1+\alpha)(0.5-\mu_1))^2} \\ &\quad - \left(\frac{1}{(1+\beta)^2}+\frac{1}{(\alpha+\beta)^2}\right)\frac{\mu_1}{\beta} \\ &\quad + (0.5-\mu_1)\left(\frac{1}{(1+\beta)^2}+\frac{{\mu_1}^2}{(0.5(1+\alpha)-\mu_1(1+\beta))^2}\right). \end{aligned} \tag{9}$$

$$\begin{aligned}\rho_4 &= \frac{\mu_1}{(\alpha-1)^2}\left(1+\frac{1}{\alpha}\right)+(0.5-\mu_1)\left(\frac{1}{(1+\beta)^2}+\frac{{\mu_1}^2}{((0.5(1+\alpha)-\mu_1(1+\beta))^2}\right) \\ &\quad - (0.5-\mu_1)\left(\frac{1}{(\alpha+\beta)^2}+\frac{{\mu_1}^2}{(0.5(1+\alpha)-\mu_1(1+\beta))^2}\right)\alpha^{-1}. \end{aligned} \tag{10}$$

To show that the collinear non-symmetric arrangement of the two pairs of masses as shown in Fig. 5 has an equilibrium solution for any mass ratio, one is led to seek a solution (α, β) of the system of equations $\rho_3 = 0$ and $\rho_4 = 0$, given in equations (9) and (10), for any $\mu_1 \in (0, 0.25)$. The complexity of the intersection of the surfaces

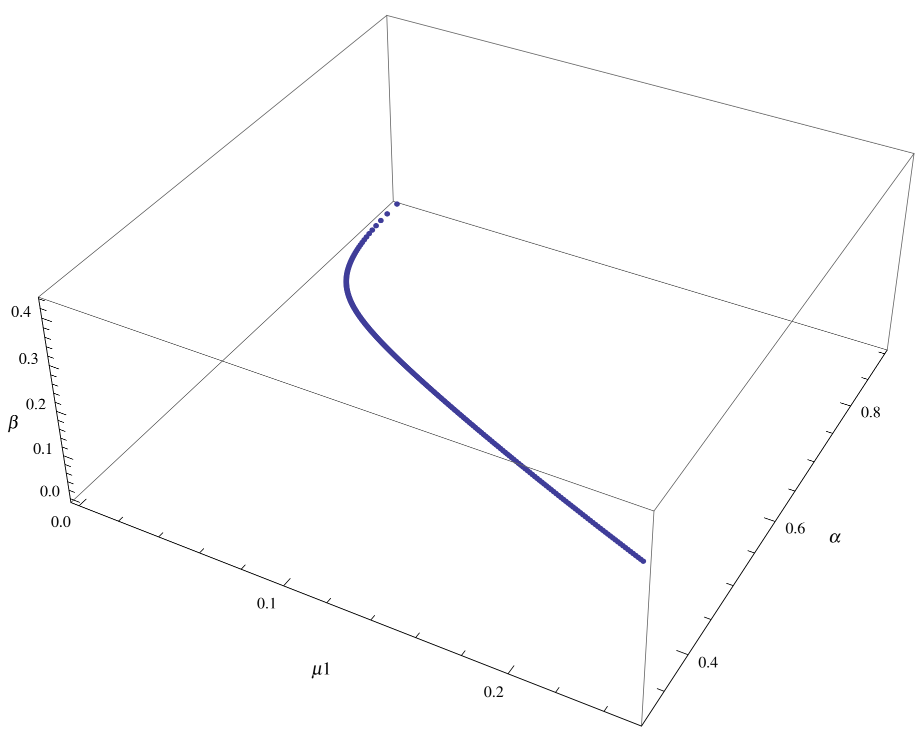

Figure 6. Family of equilibrium solutions in the second non-symmetric arrangement of two pairs of masses.

Table 1. Values of α, β and μ_1 in the second non-symmetric case.

μ_1	α	β	μ_1	α	β
0.01	0.782007	0.0362671	0.14	0.384098	0.41868
0.02	0.724575	0.0714107	0.15	0.368254	0.428895
0.03	0.681272	0.106182	0.16	0.35493	0.433821
0.04	0.644181	0.140684	0.17	0.344116	0.433518
0.05	0.610614	0.174873	0.18	0.335629	0.428377
0.06	0.579356	0.208616	0.19	0.329202	0.419041
0.07	0.5498	0.241705	0.20	0.324507	0.40629
0.08	0.521647	0.273846	0.21	0.321204	0.390919
0.09	0.494793	0.304656	0.22	0.318978	0.373661
0.10	0.469267	0.333649	0.23	0.317556	0.355141
0.11	0.445203	0.360236	0.24	0.316706	0.33587
0.12	0.422812	0.383743	0.25	0.316243	0.316243
0.13	0.402356	0.403451			

defined by ρ_3 and ρ_4 did not allow the use of the method of rectangles as in the previous case. We then solved the system numerically. We considered a subdivision of $(0, 0.25)$ with a large number of values for μ_1. For each of them we found a solution (α, β). These solutions are shown in Fig. 6. Some of these solutions are also given in Table 1 for convenience.

4. Conclusions

In this paper, we discussed four different types of four-body collinear configurations with two pairs of masses. They include (m, M, M, m), (M, m, m, M), (m, M, m, M) and (M, M, m, m). The first two of these four-body models i.e. (m, M, M, m) and (M, m, m, M) are symmetric about the centre-of-mass of the system while the other two are non-symmetric. We define two mass ratios as $\mu_1 = \frac{m_1}{M_T}$ and $\mu_2 = \frac{m_2}{M_T}$, where m_1 and m_2 are the two unequal masses and M_T is the total mass.

In the symmetric arrangements, we use the symmetry conditions to derive the equations of motion and then use the intermediate value theorem of calculus to show that there exists a continuous family of equilibrium solutions for any μ_1 in $(0, 0.5)$. The equations of motion in the non-symmetric cases (m, M, m, M) and (M, M, m, m) are derived in the same way as in the symmetric cases. The analysis of these cases were not as straight forward as the first two cases. In the first of the non-symmetric cases, we used the method of rectangles to show that there exists a continuous family of solutions for all mass ratios. These solutions are also derived numerically. In the second non-symmetric case, because of the complexity of the intersections of surfaces defined by ρ_3 and ρ_4, it was not possible to use the method of rectangles. Therefore the existence of solutions is shown numerically.

References

Arenstrof, R. F. 1982, *Celest. Mech.*, **28**, 9–15.

Douskos, C. N. 2010, *Astrophys. Space Sci.*, **326**, 263–271.

Glass, K. 1997, *Phys. Lett.*, **A235**, 591–596.

Gomatan, J., Steves, B. A., Roy, A. E. 1999, "*The Dynamics of Small Bodies in the Solar System: A Major Key to Solar System Studies*", (eds) Steves B. A., Roy A. E., pp. 373–378.

Kozak, D., Oniszk, E. 1998, *Romanian Astron. J.*, **8(1)**, 27–31.

Majorana, A. 1981, *Celest. Mech.*, **25**, 267–270.

Moulton, F. 1910, *Ann. Math.*, **12**, 1–15.

Murray, C. D., Dermot, S. F. 1999, Solar System Dynamics, Cambridge University Press.

Ouyang, T., Xie, Z. 2005, *Celest. Mech. Dynamical Astron.*, **93**, 147–166.

Palmore, J. 1975, *Bull. Amer. Math. Soc.*, **81**, 489.

Palmore, J. 1982, *Celest. Mech.*, **28**, 17–24.

Roy, A. E., Steves, B. A. 1998, *Planet. Space Sci.*, **46(11/12)**, 1475–1486.

Zhiming, Y., Yisui, S. 1988, *Scientia Sinica*, **XXXI(6)**, 724–733.