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Analytical Treatment of the Two-Body Problem with Slowly Varying Mass

W. A. Rahoma*, F. A. Abd El-Salam & M. K. Ahmed
Cairo University, Faculty of Science, Department of Astronomy, Cairo, Egypt.
**e-mail: walid_rahoma@yahoo.com*

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Abstract. The present work is concerned with the two-body problem with varying mass in case of isotropic mass loss from both components of the binary systems. The law of mass variation used gives rise to a perturbed Keplerian problem depending on two small parameters. The problem is treated analytically in the Hamiltonian frame-work and the equations of motion are integrated using the Lie series developed and applied, separately by Delva (1984) and Hanslmeier (1984). A second order theory of the two bodies eject mass is constructed, returning the terms of the rate of change of mass up to second order in the small parameters of the problem.

Key words. Mass loss—binary systems—two-body problem—perturbations.

1. Introduction

The two-body problem with variable mass is one of long-standing; its origin going back to the middle of 19th century. However, some confusion has persisted as to the dynamical equations which have to be used. Solutions of celestial mechanics problems for several variable mass bodies have been analytically tried and solved for two basic situations: the general two-body problem and the restricted three-body problem, with many modifications. The mathematical tool applied to these two basic problems usually relies upon the classical equations of motion in gravitational field with additional formal terms due to the variability of the gravitating masses. The motion is studied with the already classical procedure that interprets these terms as distinct fictitious perturbations applied to the unperturbed motion of stationary masses. There exists a related inverse procedure: interpretation of the actual perturbed motion as a formally unperturbed one, in the gravitational field generated by a variable effective mass. In this paper, we limit ourselves to the first basic problem (that of two variable-mass bodies).

2. History of the problem

The literature is full of research dealing with the problem of two-body with varying mass, and it will be beneficial to sketch some of these most important works.

Jeans (1924) was the first to pose this as an astrophysical problem basing his studies on the relationship between luminosity and star mass developed by Eddington (1924).

MacMillan (1925) introduced the detailed solution for the equation of the relative motion obtained by Mestscherskii.

Martin (1934) concluded that the eccentricity behaviour increases secularly if the mass loss of the binary star is inversely proportional to a distance power between its components.

Hadjidemetriou (1963) used Duboshin's idea of a formal comparison between the unperturbed equations in the Gylden–Meshcherskii problem and the perturbed Keplerian equations of motion in the gravitational field of a stationary mass with permanent tangential perturbation to calculate the perturbation arising from the isotropic variation of the mass of the system.

Mukhametkalieva (1987, 1988) investigated the behaviour of the eccentricity in this problem based on a representation of the eccentricity as a function of time and a periodic function of the true anomaly. Also, he obtained the Laplace integral for the Gylden problem.

Mioc *et al.* (1988a, 1988b, 1988c, 1988d), in a series of papers, covered the energy characteristics of motion in a two-body problem with variable mass.

Verhulst (1972, 1975), Hut & Verhulst (1976), Vinti (1977), Omarov & Omarkulov (1982), Omarov (1991), Minglibaev (1988), Demchenko & Omarov (1984), and Idlis & Omarov (1960) constructed analogous solutions to the Gylden problem for stationary masses in the presence of environmental resistance and quasi-elastic forces with various methods, including the Hamilton–Jacobi method. These works dealt with modeling various systems of osculating elements, discussing the structures of the intermediate motions, discussing the energy dissipation regimes combined with the influence of the elastic force, searching for the integrable cases, and discussing the feasibility of canonical transformation and Hamiltonian formalism including the case of nonconservative binary systems.

Dommangent (1963, 1964, 1981, 1982, 1997) published useful papers in which he stated that a correlation between eccentricities and orbital periods exists, such that on the average, a bigger eccentricity corresponds to a bigger period. Also, he suggested that this correlation is related to a substantial mass loss in the binary star components.

Prieto & Docobo (1997a, 1997b) and Docobo *et al.* (1999) published a series of papers in which they presented two approximate analytic solutions of the two-body problem with slowly decreasing mass, using Deprit's method of perturbations. They used Jeans law which give rise to a perturbed Keplerian problem dependent on one and two small parameters.

Andrade & Docobo (2002, 2003) analyzed the dynamics of binary systems with time-dependent mass loss and periastron effect, i.e., a supposed enhanced mass loss during periastron passage, by means of analytical and numerical techniques.

3. Different models of the mass loss

Since both the relative rate of mass change and the time intervals for this change must be included into the equations of motion, here below some interesting models of the rate of mass changes are addressed.

3.1 *Mestscherskii models*

Mestscherskii was the first to point out a specific case of the two-body problem with varying mass which is integrable by introducing special space–time variables in which the problem is reduced to the classical problem of two bodies (Polyakhova 1994; Prieto & Docobo 1997a, 1997b).

These integrable cases correspond to three celebrated models by Mestscherskii for the change in the total mass of the system.

$$\mu_1(t) = \frac{1}{a + \alpha t} \tag{1}$$

$$\mu_2(t) = \frac{1}{\sqrt{a + \alpha t}} \tag{2}$$

$$\mu_3(t) = \frac{1}{\sqrt{a + \alpha t + \beta t^2}} \tag{3}$$

where a, α, β are certain constants and $\mu_i(t)(= m_1(t) + m_2(t))$, $i = 1, 2, 3$ are different models for the mass change.

3.2 *Martin model*

Martin (1934) from his work on double star systems with varying mass, reached to this statement,

$$\dot{m} = -\frac{\alpha m^n}{r^2} \tag{4}$$

where r is the value of radius vector between the two components of the system.

Hadjidemetriou (1966) addressed useful comments on the Martin model of varying mass. These are:

- The dependence of the rate of mass loss on the distance between the two components must be due to a tidal interaction, but since μ is the total mass of the system, this law states that the tidal interaction is independent of the ratio of the masses of the two components which does not seem realistic.
- The effect of one star on its mass-losing companion would most probably result in a non-isotropic loss of mass; and consequently, the treatment of the problem by usual methods is not valid.
- The tidal interaction is not likely to produce large velocities of ejection of mass, so that the ejected particles may not escape from the system instead, full on the other star.

For this reasons, such laws must be treated with great care and in close connection with the mechanism the mass loss takes place.

3.3 *Jeans models*

Jeans (1924) was the first to pose law of varying mass as an astrophysical problem. He based his studies on the theory developed by Eddington (mass-luminosity relation) by generalized law of mass loss:

$$\dot{m} = -\alpha m^n \tag{5}$$

where α, n are real numbers, the first one is $+ve$ approximate to zero and n varying between 1.4 and 4.4. This law is called Eddington–Jeans law.

Taking $n = 2$, we have Mestscherskii first integrable case, while taking $n = 3$, we have the last Mestscherskii case.

3.4 *Andrade and Docobo models*

Analyzing the dynamics of binary systems with time-dependent mass loss, the interaction between the two components must be taken into account. Andrade & Docobo (2003) could suppose that, close to periastron there is an appreciable enhancement of mass loss. This phenomenon will be called the periastron effect, and it will be more noticeable the greater the eccentricity and the smaller the minimum distance between the two stars.

Of the whole set of laws that take into account periastron effect by means of its dependence on distance, only some of them give rise to new behaviour in the evolution of the orbital elements, such as secular variations of eccentricity.

In these models they studied the following time- and distance-dependent mass-loss law:

$$\dot{\mu}(t; r; p_\theta) = \dot{\mu}(t) - \beta \frac{p_\theta}{r^2} \tag{6}$$

where the first term represent time-dependent mass loss, and the last one introduces the periastron effect, where r is the distance between the two components, p_θ is the total angular momentum and β is another small parameter close to zero.

4. Perturbation technique

In many cases in celestial mechanics, the series development of the disturbing function is not easily treated and is complicated. To avoid this difficulty we use an alternative approach (Delva 1984; Hanslmeier 1984) in which the procedure can be performed with an operator. A special linear differential operator, the Lie operator, produces a Lie series. The convergence of the series is the same as for Taylor series, since the series is only another analytical form of the Taylor series. In addition, we can change the step size easily (if necessary).

Let $\mathcal{H}(x, y, p_x, p_y, t)$ be the Hamiltonian function, x, y be the co-ordinates, p_x, p_y be the momenta, and t be the time. Then the equations of motion are:

$$\dot{x} = \frac{dx}{dt} = \frac{\partial \mathcal{H}}{\partial p_x} \qquad \dot{p}_x = \frac{dp_x}{dt} = -\frac{\partial \mathcal{H}}{\partial x},$$
$$\dot{y} = \frac{dy}{dt} = \frac{\partial \mathcal{H}}{\partial p_y} \qquad \dot{p}_y = \frac{dp_y}{dt} = -\frac{\partial \mathcal{H}}{\partial y}.$$

The linear Lie operator has the general form:

$$D = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dp_x}{dt}\frac{\partial}{\partial p_x} + \frac{dp_y}{dt}\frac{\partial}{\partial p_y} + \frac{\partial}{\partial t} \tag{7}$$

and the solution $\vec{x}(x, y, p_x, p_y, t)$, $\vec{y}(x, y, p_x, p_y, t)$, $\overrightarrow{p_x}(x, y, p_x, p_y, t)$, and $\overrightarrow{p_y}(x, y, p_x, p_y, t)$ are then given by the Lie series:

$$\begin{aligned} \vec{x}(x, y, p_x, p_y, t) &= [\{\exp(t - t_0)D\}x]_{\vec{x}=\overrightarrow{x_0}} \\ &= \sum_{j=0} [D^j \vec{x}]_{\overrightarrow{x_0}} \frac{(t - t_0)^j}{j!} \end{aligned} \tag{8}$$

$$\begin{aligned} \vec{y}(x, y, p_x, p_y, t) &= [\{\exp(t - t_0)D\}y]_{\vec{y}=\overrightarrow{y_0}} \\ &= \sum_{j=0} [D^j \vec{y}]_{\overrightarrow{y_0}} \frac{(t - t_0)^j}{j!} \end{aligned} \tag{9}$$

$$\begin{aligned} \overrightarrow{p_x}(x, y, p_x, p_y,) &= [\{\exp(t - t_0)D\}p_x]_{\overrightarrow{p_x}=\overrightarrow{p_{x_0}}} \\ &= \sum_{j=0} [D^j \overrightarrow{p_x}]_{\overrightarrow{p_{x_0}}} \frac{(t - t_0)^j}{j!} \end{aligned} \tag{10}$$

$$\begin{aligned} \overrightarrow{p_y}(x, y, p_x, p_y, t) &= [\{\exp(t - t_0)D\}p_y]_{\overrightarrow{p_y}=\overrightarrow{p_{y_0}}} \\ &= \sum_{j=0} [D^j \overrightarrow{p_y}]_{\overrightarrow{p_{y_0}}} \frac{(t - t_0)^j}{j!} \end{aligned} \tag{11}$$

where $D^j\vec{x}$, $D^j\vec{y}$, $D^j\overrightarrow{p_x}$, and $D^j\overrightarrow{p_y}$ are to be evaluated for the initial conditions $\overrightarrow{x_0}(x_0, y_0, p_{x_0}, p_{y_0}, t_0)$, $\overrightarrow{y_0}(x_0, y_0, p_{x_0}, p_{y_0}, t_0)$, $\overrightarrow{p_{x0}}(x_0, y_0, p_{x_0}, p_{y_0}, t_0)$, and $\overrightarrow{p_{y_0}}(x_0, y_0, p_{x_0}, p_{y_0}, t_0)$.

5. Hamiltonian

The Hamiltonian is constructed in terms of Delaunay's variables as:

$$\mathcal{K} \equiv \mathcal{K}(l, L, G; t) = -\frac{1}{2}\frac{\mu^2}{L^2} + \frac{\dot{\mu}}{\mu} Le \sin E \tag{12}$$

Equation (12) was already obtained by Deprit (1983), where $\mathcal{H}$ is the Hamiltonian of the two-body problem.

The equations of motion can be analytically integrated up to the second order of perturbation using the Deprit's method (1969) or Kamel method (1969).

We need first to expand the Hamiltonian function (12) in a small parameter ε using Jeans law of mass:

$$\dot{m}_k = -\alpha_k m_k^{n(k)}, \quad k = 1, 2 \tag{13}$$

For $\varepsilon = 0$, the Keplerian case would be obtained.

It seems that the parameter ε should be related to the coefficient α_k which appears in the Jeans law of mass variation. We will simply choose ε as nondimensional value of α_k. This choice also justifies the application of the method as far as the second order only, since α_k is very small and higher order would not contribute significantly. Let the total mass of the binary system be given by $\mu = m_1 + m_2$.

Expanding the function $\mu(t)$ in a Taylor series yields:

$$\mu = \mu_0 - (\alpha_1 m_{10}^{n(1)} + \alpha_2 m_{20}^{n(2)})(t - t_0) + \frac{1}{2}(\alpha_1^2 n(1) m_{10}^{2n(1)-1} + \alpha_2^2 n(2) m_{20}^{2n(2)-1})(t - t_0)^2 \tag{14}$$

where m_{10}, m_{20} are the values of mass for each component in certain initial instant t_0.

The Hamiltonian can now be rewritten in expandable form as:

$$\mathcal{K} = \mathcal{K}_0 + \sum_{n=1}^{2}\sum_{k=1}^{2}\frac{\alpha_k^n}{n!}\mathcal{K}_{nk} - \alpha_1\alpha_2 m_{10}^{n(1)} m_{20}^{n(2)}\left\{\frac{(t-t_0)^2}{L^2} + \frac{2}{\mu_0^2}(t-t_0)eL\sin E\right\} \tag{15}$$

with

$$\mathcal{K}_0 = -\frac{\mu_0^2}{2L^2}$$

$$\mathcal{K}_{11} = m_{10}^{n(1)}\left\{\frac{\mu_0}{L^2}(t-t_0) - \frac{1}{\mu_0}eL\sin E\right\}$$

$$\mathcal{K}_{12} = m_{20}^{n(2)}\left\{\frac{\mu_0}{L^2}(t-t_0) - \frac{1}{\mu_0}eL\sin E\right\}$$

$$\mathcal{K}_{21} = -m_{10}^{2n(1)-1}\left\{\frac{(n(1)\mu_0 + m_{10})}{L^2}(t-t_0)^2 + \frac{2m_{10}}{\mu_0^2}(t-t_0)eL\sin E\right\}$$

$$\mathcal{K}_{22} = -m_{20}^{2n(2)-1}\left\{\frac{(n(2)\mu_0 + m_{20})}{L^2}(t-t_0)^2 + \frac{2m_{20}}{\mu_0^2}(t-t_0)eL\sin E\right\}$$

where $\mathcal{K}_0$ represents the Keplerian part of the problem, $\mathcal{K}_{11}$, $\mathcal{K}_{12}$ represent the first order contributions that come from the first and second body mass loss, $\mathcal{K}_{21}$, $\mathcal{K}_{22}$ represent the second order contributions, while the term factored by $\alpha_1\alpha_2$ represents the coupling effect between the two bodies.

Introduce $\mathcal{X}_i = \alpha_i/\alpha$, $i = 1, 2$.

The Hamiltonian can now be rewritten as:

$$\mathcal{K} = -\frac{\mu_0^2}{2L^2} + \sum_{s=1}^{2}\frac{\alpha^s}{s!}\left\{\tilde{\zeta}_s(L, G)\tau^{s-1}\sin E + \tau^s\tilde{\eta}_s(L)\right\} \tag{16}$$

where

$$\tilde{\zeta}_s(L, G) = (\Phi_s + \Pi_{2(s-1)})eL$$

$$\tilde{\eta}_s(L) = \frac{(\Psi_s + \Pi_{s-1})}{L^2}$$

$$\tau = t - t_0$$

with

$$\Psi_1 = \mu_0(\mathcal{X}_1 m_{10}^{n(1)} + \mathcal{X}_2 m_{20}^{n(2)})$$

$$\Psi_2 = -\mu_0(n(1)\mathcal{X}_1^2 m_{10}^{2n(1)-1} + n(2)\mathcal{X}_2^2 m_{20}^{2n(2)-1}) + \mathcal{X}_1^2 m_{10}^{2n(1)} + \mathcal{X}_2^2 m_{20}^{2n(2)}$$

$$\Phi_1 = -\frac{1}{\mu_0}(\mathcal{X}_1 m_{10}^{n(1)} + \mathcal{X}_2 m_{20}^{n(2)})$$

$$\Phi_2 = -\frac{2}{\mu_0^2}(\mathcal{X}_1^2 m_{10}^{2n(1)} + \mathcal{X}_2^2 m_{20}^{2n(2)})$$

$$\Pi_1 = -2\mathcal{X}_1\mathcal{X}_2 m_{10}^{n(1)} m_{20}^{n(2)}$$

$$\Pi_2 = -\frac{4}{\mu_0^2}\mathcal{X}_1\mathcal{X}_2 m_{10}^{n(1)} m_{20}^{n(2)}$$

$$\Pi_0 = 0.$$

6. Solution of the problem

The non-vanishing final expressions of the variation in the orbital elements can be written as:

$$\dot{l} = \frac{m_0^2}{L^3} + \sum_{s=1}^{2}\frac{\alpha^s}{s!}\tau^{s-1}\left\{\tilde{\zeta}_{s,L}\sin E + \frac{a}{2r}\frac{G^2}{eL^3}\tilde{\zeta}_s \sin 2E + \tau\tilde{\eta}_{s,L}\right\} \tag{17}$$

$$\dot{g} = \sum_{s=1}^{2}\frac{\alpha^s}{s!}\tau^{s-1}\left\{\tilde{\zeta}_{s,G}\sin E - \frac{a}{2r}\frac{G}{eL^2}\tilde{\zeta}_s \sin 2E\right\} \tag{18}$$

$$\dot{L} = -\sum_{s=1}^{2}\frac{\alpha^s}{s!}\tau^{s-1}\left\{\frac{a}{r}\tilde{\zeta}_s \cos E\right\} \tag{19}$$

where

$$\tilde{\zeta}_{s,L} = \frac{\partial\tilde{\zeta}_s}{\partial L} = \frac{1}{e}(\Phi_s + \Pi_{2(s-1)})$$

$$\tilde{\zeta}_{s,G} = \frac{\partial\tilde{\zeta}_s}{\partial G} = -\frac{G}{eL}(\Phi_s + \Pi_{2(s-1)})$$

$$\tilde{\eta}_{s,L} = \frac{\partial\tilde{\eta}_s}{\partial L} = -\frac{2}{L^3}(\Psi_s + \Pi_{s-1}).$$

Since only the mutual gravitational attraction is considered and the mass lost by one body of the system is transferred to its companion, the total mass of the system is kept constant. This turns a constant mean motion $\tilde{n}$.

The linear Lie operator D, in terms of the Delaunay elements, has the general form:

$$D = \frac{dl}{dt}\frac{\partial}{\partial l} + \frac{dg}{dt}\frac{\partial}{\partial g} + \frac{dL}{dt}\frac{\partial}{\partial L} + \frac{dG}{dt}\frac{\partial}{\partial G} + \frac{\partial}{\partial t}.$$

Applying the operator D to l, g, L, G, and t yields:

$$Dl = \frac{dl}{dt} + \frac{\partial l}{\partial t} = \dot{l} + \tilde{n} \tag{20}$$

$$Dg = \frac{dg}{dt} = \dot{g} \tag{21}$$

$$DL = \frac{dL}{dt} = \dot{L} \tag{22}$$

$$DG = \frac{dG}{dt} = 0 \tag{23}$$

$$Dt = 1. \tag{24}$$

The solutions $\vec{l}(l,\ g,\ L,\ G,\ t)$, $\vec{g}(l, g, L, G, t)$, $\vec{L}(l, g, L, G, t)$, and $\vec{G}(l, g, L, G, t)$ are then given in terms of the Lie series as:

$$\vec{l}(l, g, L, G, t) = \sum_{j=0}[D^j\vec{l}]_{\overrightarrow{l_0}}\frac{(t-t_0)^j}{j!} \tag{25}$$

$$\vec{g}(l, g, L, G, t) = \sum_{j=0}[D^j\vec{g}]_{\overrightarrow{g_0}}\frac{(t-t_0)^j}{j!} \tag{26}$$

$$\vec{L}(l, g, L, G, t) = \sum_{j=0}[D^j\vec{L}]_{\overrightarrow{L_0}}\frac{(t-t_0)^j}{j!} \tag{27}$$

$$\vec{G}(l, g, L, G, t) = \sum_{j=0}[D^j\vec{G}]_{\overrightarrow{G_0}}\frac{(t-t_0)^j}{j!} \tag{28}$$

where $D^j\vec{l}$, $D^j\vec{g}$, $D^j\vec{L}$, and $D^j\vec{G}$ are to be evaluated for the initial condition $\overrightarrow{l_0}(l_0,\ g_0,\ L_0,\ G_0,\ t_0)$, $\overrightarrow{g_0}(l_0,\ g_0,\ L_0,\ G_0,\ t_0)$, $\overrightarrow{L_0}(l_0,\ g_0,\ L_0,\ G_0,\ t_0)$, and $\overrightarrow{G_0}(l_0,\ g_0,\ L_0,\ G_0,\ t_0)$. To find the terms of the series, it will be necessary to calculate the multiple action of D to the variables l, g, L, G, and t. The single action to l, g, L, and G produces:

$$Dx = x', \quad x = (l, g, L, G) \tag{29}$$

and hence, the multiple action gives:

$$D^j x = D^{j-1}x', \quad j \geq 1.$$

6.1 *The series for l*

The double action of the Lie operator, D on the mean anomaly l can be computed as:

$$D^2 l = \left(\frac{dl}{dt}\frac{\partial}{\partial l} + \frac{dg}{dt}\frac{\partial}{\partial g} + \frac{dL}{dt}\frac{\partial}{\partial L} + \frac{dG}{dt}\frac{\partial}{\partial G} + \frac{\partial}{\partial t}\right)[\dot{l} + \tilde{n}] \tag{30}$$

Setting

$$\tilde{\zeta}_{s,LL} = -(\Phi_s + \Pi_{2(s-1)})\frac{G^2}{e^3 L^3}$$

$$\tilde{\eta}_{s,LL} = \frac{6}{L^4}(\Psi_s + \Pi_{s-1})$$

equation (30) can be written as:

$$D^2 l = \sum_{n=0}^{5}\sum_{m=0}^{5}\left[\alpha\left(\frac{a}{r}\right)^m \mathcal{W}^l_{nm}\cos nE + \alpha^2\left\{\left[\left(\frac{a}{r}\right)^m \mathcal{Y}^l_{nm} + \mathcal{S}^l_n\right]\sin nE\right.\right.$$
$$\left.\left. + \ \tau\left(\frac{a}{r}\right)^m \mathcal{Z}^l_{nm}\cos nE\right\}\right] \tag{31}$$

where the non-vanishing coefficients are:

$$\mathcal{W}^l_{11} = \left(\frac{m_0^2}{L^3}\right)\left[\tilde{\zeta}_{1,L} + \frac{3\tilde{\zeta}_1}{L}\right]$$

$$\mathcal{W}^l_{13} = -\frac{m_0^2 G^2}{4L^6}\tilde{\zeta}_1$$

$$\mathcal{W}^l_{22} = \frac{m_0^2 G^2}{eL^6}\tilde{\zeta}_1$$

$$\mathcal{W}^l_{33} = \frac{m_0^2 G^2}{4L^6}\tilde{\zeta}_1$$

$$\mathcal{S}^l_1 = \frac{1}{2}\tilde{\zeta}_{2,L}$$

$$\mathcal{Y}^l_{12} = -\frac{3G^2}{4eL^3}\tilde{\zeta}_1\tilde{\zeta}_{1,L} + \frac{G^2}{4e^3L^4}(1+2e^2)\tilde{\zeta}_1^2$$

$$\mathcal{Y}^l_{13} = -\frac{G^2}{16L^3}\tilde{\zeta}_1\tilde{\zeta}_{1,L}$$

$$\mathcal{Y}^l_{15} = \frac{\mu G^4}{4eL^8}\tilde{\zeta}_1^2$$

$$\mathcal{Y}^l_{21} = \frac{1}{2}\tilde{\zeta}_{1,L}^2 - \frac{1}{2}\tilde{\zeta}_1\tilde{\zeta}_{1,LL} + \frac{G^2}{4eL^3}\tilde{\zeta}_2$$

$$\mathcal{Y}^l_{22} = \frac{1}{4}\left[\frac{G^2}{eL^3}\right]^2\tilde{\zeta}_1^2$$

$$\mathcal{Y}^l_{23} = -\frac{G^2}{4L^3}\tilde{\zeta}_1\tilde{\zeta}_{1,L} + \frac{1}{4}\left[\frac{G^2}{eL^3}\right]^2\tilde{\zeta}_1^2$$

$$\mathcal{Y}^l_{24} = \frac{\mu}{8L^2}\left[\frac{G^2}{eL^3}\right]^2\tilde{\zeta}_1^2$$

$$\mathcal{Y}^l_{25} = -\frac{\mu}{8L^2}\left[\frac{G^2}{eL^3}\right]^2\tilde{\zeta}_1^2$$

$$\mathcal{Y}^l_{32} = \frac{G^2}{eL^3}\tilde{\zeta}_1\tilde{\zeta}_{1,L}$$

$$\mathcal{Y}^l_{34} = \frac{G^4}{4eL^6}\left[\frac{1}{2e^2L}\left(1+2e^2\right) - 1\right]\tilde{\zeta}_1^2$$

$$\mathcal{Y}^l_{35} = \frac{G^4}{4eL^6}\tilde{\zeta}_1^2$$

$$\mathcal{Y}^l_{43} = \frac{G^2}{16L^3}\tilde{\zeta}_1\tilde{\zeta}_{1,L}$$

$$\mathcal{Y}^l_{45} = -\frac{\mu G^4}{8e^2L^8}\tilde{\zeta}_1^2$$

$$\mathcal{B}^l_{54} = \frac{G^4}{8eL^6}\tilde{\zeta}_1^2$$

$$\mathcal{Z}^l_{00} = \tilde{\eta}_{2,L}$$

$$\mathcal{Z}^l_{11} = \tilde{\eta}_{1,L}\tilde{\zeta}_{1,L} - \tilde{\zeta}_1\tilde{\eta}_{1,LL} + \frac{m_0^2}{2L^3}\left(\tilde{\zeta}_{2,L} + \frac{3}{L}\tilde{\zeta}_2\right)$$

$$\mathcal{Z}^l_{13} = -\left(\frac{G_1^2}{4L^3}\tilde{\eta}_{1,L}\tilde{\zeta} - \frac{m_0^2}{8L^6}\tilde{\zeta}_2\right)$$

$$\mathcal{Z}^l_{22} = \frac{G^2}{eL^3}\left[\tilde{\zeta}_1\tilde{\eta}_{1,L} + \frac{m_0^2}{2L^3}\tilde{\zeta}_2\right]$$

$$\mathcal{Z}^l_{33} = \frac{G^2}{4L^3}\left[\tilde{\zeta}_1\tilde{\eta}_{1,L} + \frac{m_0^2}{2L^3}\tilde{\zeta}_2\right]$$

which can be rewritten as:

$$D^2l = \sum_{n=0}^{5}\sum_{m=0}^{5}\Bigg[-\frac{1}{\mu_0}(\alpha_1 m_{10}^{n(1)} + \alpha_2 m_{20}^{n(2)})\mathbb{W}^l_{nm}\cos nE + \frac{1}{\mu_0^2}(\alpha_1 m_{10}^{n(1)} + \alpha_2 m_{20}^{n(2)})^2$$
$$\times \{(\mathbb{Y}^l_{nm} + \mathbb{S}^l_n)\sin nE + \tau\mathbb{Z}^l_{nm}\cos nE\} + \frac{2}{L^3}\Bigg\{\alpha_1^2[\mu_0 n(1) m_{10}^{2n(1)-1} - m_{10}^{2n(1)}]$$
$$+\alpha_2^2[\mu_0 n(2) m_{20}^{2n(2)-1} - m_{20}^{2n(2)}] + \frac{4}{\mu_0^2}\alpha_1\alpha_2 m_{10}^{n(1)} m_{20}^{n(2)}\Bigg\}\Bigg] \tag{32}$$

where the non-vanishing coefficients are:

$$\mathbb{W}^l_{11} = \left(\frac{m_0^2}{L^3}\right)\left[\frac{1}{e} + 3e\right]$$

$$\mathbb{W}^l_{13} = -\frac{m_0^2 G^2 e}{4L^5}$$

$$\mathbb{W}^l_{22} = \frac{m_0^2 G^2}{L^5}$$

$$\mathbb{W}^l_{33} = \frac{m_0^2 G^2 e}{4L^5}$$

$$\mathbb{S}^l_1 = -\frac{1}{e}$$

$$\mathbb{Y}^l_{12} = -\frac{3G^2}{4eL^2} + \frac{G^2}{4eL^2}(1+2e^2)$$

$$\mathbb{Y}^l_{13} = -\frac{G^2}{16L^2}$$

$$\mathbb{Y}^l_{15} = \frac{\mu G^4 e}{4L^6}$$

$$\mathbb{Y}^l_{21} = \frac{1}{2e^2} + \frac{1}{2}\frac{G^2}{e^2L^2} - \frac{G^2}{2L^2}$$

$$\mathbb{Y}^l_{22} = \frac{G^4}{4L^4}$$

$$\mathbb{Y}^l_{23} = -\frac{G^2 e^2}{4L^2}$$

$$\mathbb{Y}^l_{24} = \frac{\mu G^4}{8L^6}$$

$$\mathbb{Y}^l_{25} = -\frac{\mu G^4}{8L^6}$$

$$\mathbb{Y}^l_{32} = \frac{G^2}{eL^2}$$

$$\mathbb{Y}^l_{34} = \frac{G^4 e}{4L^4}\left[\frac{1}{2e^2L}(1+2e^2) - 1\right]$$

$$\mathbb{Y}^l_{35} = \frac{G^4 e}{4L^4}$$

$$\mathbb{Y}^l_{43} = \frac{G^2}{16L^2}$$

$$\mathbb{Y}^l_{45} = -\frac{\mu G^4}{8L^6}$$

$$\mathbb{Y}^l_{54} = \frac{G^4 e}{8L^4}$$

$$\mathbb{Z}^l_{11} = \frac{1}{L^2}\left[\frac{2\mu_0^2}{eL} + 6e\mu_0^2 - \frac{m_0^2}{L}\left(\frac{1}{e} + 3e\right)\right]$$

$$\mathbb{Z}^l_{13} = -\frac{e}{2L^5}\left(\mu_0^2 G^2 + \frac{m_0^2}{2}\right)$$

$$\mathbb{Z}^l_{22} = \frac{G^2}{L^5}[2\mu_0^2 - m_0^2]$$

$$\mathbb{Z}^l_{33} = \frac{G^2 e}{4L^5}[2\mu_0^2 - m_0^2].$$

Then the solution

$$\begin{aligned} l(t) &= [l]_{\overrightarrow{l_0}} + [Dl]_{\overrightarrow{l_0}}(t-t_0) + [D^2 l]_{\overrightarrow{l_0}}\frac{(t-t_0)^2}{2} \\ &= [l]_{\overrightarrow{l_0}} + [\dot{l} + \tilde{n}]_{\overrightarrow{l_0}}(t-t_0) + [D^2 l]_{\overrightarrow{l_0}}\frac{(t-t_0)^2}{2}. \end{aligned} \tag{33}$$

6.2 *The series for g*

The double action of the Lie operator, D on argument of periapsis g can be computed as:

$$D^2 g = \left(\frac{dl}{dt}\frac{\partial}{\partial l} + \frac{dg}{dt}\frac{\partial}{\partial g} + \frac{dL}{dt}\frac{\partial}{\partial L} + \frac{dG}{dt}\frac{\partial}{\partial G} + \frac{\partial}{\partial t}\right)[\dot{g}] \tag{34}$$

setting

$$\tilde{\zeta}_{s,GL} = \frac{(\Phi_s + \Pi_{2(s-1)})G}{e^3 L^2}$$

equation (34) can be written as:

$$\begin{aligned} D^2 g = \sum_{n=0}^{5}\sum_{m=0}^{6}\Big[\alpha\left(\frac{a}{r}\right)^m \mathcal{W}^g_{nm}\cos nE + \alpha^2\Big\{\Big[\left(\frac{a}{r}\right)^m \mathcal{Y}^g_{nm} + \mathcal{S}^g_n\Big]\sin nE \\ + \ \tau\left(\frac{a}{r}\right)^m \mathcal{Z}^g_{nm}\cos nE\Big\}\Big] \end{aligned} \tag{35}$$

where the non-vanishing coefficients are:

$$\mathcal{W}^g_{11} = \frac{m_0^2}{L^3}\tilde{\zeta}_{1,G}$$

$$\mathcal{W}^g_{13} = \frac{m_0^2 G}{4L^5}\tilde{\zeta}_1$$

$$\mathcal{W}^g_{22} = -\frac{G m_0^2}{eL^5}\tilde{\zeta}_1$$

$$\mathcal{W}^g_{33} = -\frac{m_0^2 G}{4L^5}\tilde{\zeta}_1$$

$$\mathcal{Y}^g_{12} = \frac{G}{2eL^2}\left(\tilde{\zeta}_{1,L}\tilde{\zeta}_1 + \frac{G}{2L}\tilde{\zeta}_1\tilde{\zeta}_{1,G}\right)$$

$$\mathcal{Y}^g_{13} = \frac{G^3}{4e^2L^5}\tilde{\zeta}_1\tilde{\zeta}_{1,G}$$

$$\mathcal{Y}^g_{14} = \frac{G^3}{16eL^5}\tilde{\zeta}_1^2$$

$$\mathcal{Y}^g_{15} = \frac{G^3}{8e^3L^6}\left(\frac{1}{e^2L}(e^2+1)\tilde{\zeta}_1^2 - \tilde{\zeta}_1\tilde{\zeta}_{1,L}\right)$$

$$\mathcal{Y}^g_{16} = \frac{\mu G^4}{4e^2L^9}\tilde{\zeta}_1^2$$

$$\mathcal{S}^g_1 = \frac{\tilde{\zeta}_{2,G}}{2}$$

$$\mathcal{Y}^g_{21} = \frac{1}{2}\left(\tilde{\zeta}_{1,L}\tilde{\zeta}_{1,G} - \tilde{\zeta}_{1,GL}\tilde{\zeta}_1 - \frac{G}{2eL^2}\tilde{\zeta}_2\right)$$

$$\mathcal{Y}^g_{23} = \frac{G}{16L^2}\tilde{\zeta}_{1,L}\tilde{\zeta}_1$$

$$\mathcal{Y}^g_{32} = -\frac{G}{4eL^2}\left(\tilde{\zeta}_1\tilde{\zeta}_{1,L} + \frac{1}{4e^2L}(e^2+1)\tilde{\zeta}_1^2\right)$$

$$\mathcal{Y}^g_{34} = \frac{G^3}{16eL^5}\tilde{\zeta}_1^2$$

$$\mathcal{Y}^g_{35} = -\frac{\mu G^3}{4eL^7}\tilde{\zeta}_1^2$$

$$\mathcal{Y}^g_{43} = -\frac{G}{2L^2}\left(\frac{1}{4}\tilde{\zeta}_1\tilde{\zeta}_{1,L} - \frac{G^2}{e^2L^3}\tilde{\zeta}_1^2\right)$$

$$\mathcal{Y}^g_{45} = -\frac{\mu G^3}{4e^2L^7}\tilde{\zeta}_1^2$$

$$\mathcal{Y}^g_{54} = -\frac{G^3}{16eL^5}\tilde{\zeta}_1^2$$

$$\mathcal{Z}^g_{11} = \left(\tilde{\eta}_{1,L}\tilde{\zeta}_{1,G} + \frac{m_0^2}{2L^3}\tilde{\zeta}_{2,G}\right)$$

$$\mathcal{Z}^g_{13} = -\left(\frac{G}{4L^2}\tilde{\zeta}_1\tilde{\eta}_{1,L} + \frac{m_0^2 G}{8L^5L^3}\tilde{\zeta}_2\right)$$

$$\mathcal{Z}^g_{22} = -\frac{G}{eL^2}\left[\frac{m_0^2}{2L^3}\tilde{\zeta}_2 + \tilde{\zeta}_1\tilde{\eta}_{1,L}\right]$$

$$\mathcal{Z}_{33}^{g} = -\frac{G}{4L^2}\left[\frac{m_0^2}{2L^3}\tilde{\zeta}_2 + \tilde{\zeta}_1\tilde{\eta}_{1,L}\right]$$

which can be rewritten as:

$$D^2 g = \sum_{n=0}^{5}\sum_{m=0}^{6}\left[-\frac{1}{\mu_0}(\alpha_1 m_{10}^{n(1)} + \alpha_2 m_{20}^{n(2)})\mathbb{W}_{nm}^{g}\cos nE + \frac{1}{\mu_0^2}(\alpha_1 m_{10}^{n(1)} + \alpha_2 m_{20}^{n(2)})^2 \right.$$
$$\left. \times\,(\mathbb{Y}_{nm}^{g}\sin nE + \tau\mathbb{Z}_{nm}^{g}\cos nE)\right] \tag{36}$$

where the non-vanishing coefficients are:

$$\mathbb{W}_{11}^{g} = -\frac{m_0^2 G}{eL^4}$$

$$\mathbb{W}_{13}^{g} = \frac{m_0^2 G e}{4L^4}$$

$$\mathbb{W}_{22}^{g} = -\frac{G m_0^2}{L^4}$$

$$\mathbb{W}_{33}^{g} = -\frac{m_0^2 G e}{4L^4}$$

$$\mathbb{Y}_{12}^{g} = \frac{G}{2eL^2}\left(L - \frac{G^2}{2L}\right)$$

$$\mathbb{Y}_{13}^{g} = -\frac{G^4}{4e^2L^5}$$

$$\mathbb{Y}_{14}^{g} = \frac{G^3 e}{16L^3}$$

$$\mathbb{Y}_{15}^{g} = \frac{G^3}{8eL^5}$$

$$\mathbb{Y}_{16}^{g} = \frac{\mu G^4}{4L^7}$$

$$\mathbb{S}_{1}^{g} = \frac{G}{eL}$$

$$\mathbb{Y}_{21}^{g} = \frac{1}{2}G\left(\frac{1}{L} - \frac{1}{e^2L} - 1\right)$$

$$\mathbb{Y}_{23}^{g} = \frac{G}{16L}$$

$$\mathbb{Y}_{32}^{g} = -\frac{G}{4eL}\left(1 + \frac{1}{4}(e^2+1)\right)$$

$$\mathbb{Y}_{34}^{g} = \frac{G^3 e}{16L^3}$$

$$\mathbb{Y}^g_{35} = -\frac{\mu G^3 e}{4L^5}$$

$$\mathbb{Y}^g_{43} = -\frac{G}{2L^2}\left(\frac{1}{4}L - \frac{G^2}{eL}\right)$$

$$\mathbb{Y}^g_{45} = -\frac{\mu G^3}{4L^5}$$

$$\mathbb{Y}^g_{54} = -\frac{G^3 e}{16L^3}$$

$$\mathbb{Z}^g_{11} = \left(-\frac{2\mu_0^2 G}{eL^4} + \frac{m_0^2 G}{eL^4}\right)$$

$$\mathbb{Z}^g_{13} = -\frac{eG}{2L^4}\left(\mu_0^2 - \frac{m_0^2}{2}\right)$$

$$\mathbb{Z}^g_{22} = -\frac{G}{eL^2}\left[-\frac{m_0^2}{L^2} + \frac{2\mu_0^2 e}{L^2}\right]$$

$$\mathbb{Z}^g_{33} = -\frac{Ge}{4L^4}[-m_0^2 + 2\mu_0^2].$$

Then the solution

$$\begin{aligned} g(t) &= [g]_{\overrightarrow{g_0}} + [Dg]_{\overrightarrow{g_0}}(t - t_0) + [D^2 g]_{\overrightarrow{g_0}}\frac{(t - t_0)^2}{2} \\ &= [g]_{\overrightarrow{g_0}} + [\dot{g}]_{\overrightarrow{l_0}}(t - t_0)_{\overrightarrow{l_0}}\frac{(t - t_0)^2}{2} + [D^2 g]_{\overrightarrow{g_0}}\frac{(t - t_0)^2}{2}. \end{aligned} \tag{37}$$

6.3 *The series for L*

The double action of the Lie operator, D on momenta L can be computed as:

$$D^2 L = \left(\frac{dl}{dt}\frac{\partial}{\partial l} + \frac{dg}{dt}\frac{\partial}{\partial g} + \frac{dL}{dt}\frac{\partial}{\partial L} + \frac{dG}{dt}\frac{\partial}{\partial G} + \frac{\partial}{\partial t}\right)[\dot{L}] \tag{38}$$

which can be written as:

$$D^2 L = \sum_{n=0}^{5}\sum_{m=0}^{5}\left(\frac{a}{r}\right)^m [\alpha \mathcal{W}^L_{nm}\sin nE + \alpha^2\{\mathcal{Y}^L_{nm}\cos nE + \tau \mathcal{Z}^L_{nm}\sin nE\}] \tag{39}$$

where the non-vanishing coefficients are:

$$\mathcal{W}^L_{12} = \frac{m_0^2}{L^3}\tilde{\zeta}_1$$

$$\mathcal{W}^L_{23} = \frac{m_0^2 e}{2L^3}\tilde{\zeta}_1$$

$$\mathcal{Y}^L_{02} = \tilde{\zeta}_1\tilde{\zeta}_{1,L}$$

$$\mathcal{Y}^L_{04} = \frac{G^2}{8L^3}\tilde{\zeta}_1^2 - \left(\frac{a}{r}\right)^5 \frac{\tilde{\zeta}_1^2}{2}\frac{G^2}{L^3}\frac{\mu}{L^2}$$

$$\mathcal{Y}^L_{05} = -\frac{\mu G^2}{2L^5}\tilde{\zeta}_1^2$$

$$\mathcal{Y}^L_{11} = -\frac{\tilde{\zeta}_2}{2}$$

$$\mathcal{Y}^L_{13} = \frac{e}{4}\tilde{\zeta}_1\tilde{\zeta}_{1,L} - \frac{G^2}{4eL^3}\left(1 + \frac{1}{L^2}\right)\tilde{\zeta}_1^2$$

$$\mathcal{Y}^L_{15} = \frac{3\mu G^2}{4eL^5}\tilde{\zeta}_1^2$$

$$\mathcal{Y}^L_{22} = \tilde{\zeta}_1\tilde{\zeta}_{1,L}$$

$$\mathcal{Y}^L_{25} = -\frac{\mu G^2}{L^5}\tilde{\zeta}_1^2$$

$$\mathcal{Y}^L_{33} = \left[\frac{e}{4}\tilde{\zeta}_1\tilde{\zeta}_{1,L} + \frac{G^2}{4eL^5}(L^2 - 1)\tilde{\zeta}_1^2\right]$$

$$\mathcal{Y}^L_{35} = \frac{\mu G^2}{4eL^5}\tilde{\zeta}_1^2$$

$$\mathcal{Y}^L_{44} = -\frac{G^2}{8L^3}\tilde{\zeta}_1^2$$

$$\mathcal{Z}^L_{12} = \frac{m_0^2}{2L^3}\tilde{\zeta}_2 + \tilde{\zeta}_1\tilde{\eta}_{1,L}$$

$$\mathcal{Z}^L_{23} = \frac{e}{2}\left[\frac{m_0^2}{2L^3}\tilde{\zeta}_2 + \tilde{\zeta}_1\tilde{\eta}_{1,L}\right]$$

which can be rewritten as:

$$D^2L = \sum_{n=0}^{5}\sum_{m=0}^{5}\left[-\frac{1}{\mu_0}\left(\alpha_1 m_{10}^{n(1)} + \alpha_2 m_{20}^{n(2)}\right)\mathbb{W}^g_{nm}\sin nE + \frac{1}{\mu_0^2}(\alpha_1 m_{10}^{n(1)} + \alpha_2 m_{20}^{n(2)})^2 \right.$$
$$\left. \times\, (\mathbb{Y}^g_{nm}\cos nE + \tau\mathbb{Z}^g_{nm}\sin nE)\right] \tag{40}$$

where the non-vanishing coefficients are:

$$\mathbb{W}^L_{12} = \frac{m_0^2 e}{L^2}$$

$$\mathbb{W}^L_{23} = \frac{m_0^2 e^2}{2L^2}$$

$$\mathbb{Y}^L_{02} = L$$

$$\mathbb{Y}^L_{04} = \frac{G^2 e^2}{8L}$$

$$\mathbb{Y}_{05}^{L} = -\frac{\mu G^2 e^2}{2L^3}$$

$$\mathbb{Y}_{11}^{L} = eL$$

$$\mathbb{Y}_{13}^{L} = \frac{eL}{4} - \frac{G^2 e}{4L}\left(1 + \frac{1}{L^2}\right)$$

$$\mathbb{Y}_{15}^{L} = \frac{3\mu G^2 e}{4L^3}$$

$$\mathbb{Y}_{22}^{L} = L$$

$$\mathbb{Y}_{25}^{L} = -\frac{\mu G^2 e^2}{L^3}$$

$$\mathbb{Y}_{33}^{L} = \frac{eL}{4} + \frac{G^2 e}{4L^3}(L^2 - 1)$$

$$\mathbb{Y}_{35}^{L} = \frac{\mu_0 G^2 e}{4L^3}$$

$$\mathbb{Y}_{44}^{L} = -\frac{G^2 e}{8L}$$

$$\mathbb{Z}_{12}^{L} = \frac{e}{L^2}(2\mu_0^2 - m_0^2)$$

$$\mathbb{Z}_{23}^{L} = \frac{m_0^2 e^2}{2L^2}(2\mu_0^2 - m_0^2).$$

Then the solution yields:

$$\begin{aligned} L(t) &= [L]_{\overrightarrow{L_0}} + [DL]_{\overrightarrow{L_0}}(t - t_0) + [D^2 L]_{\overrightarrow{L_0}}\frac{(t - t_0)^2}{2} \\ &= [L]_{\overrightarrow{L_0}} + [\dot{L}]_{\overrightarrow{L_0}}(t - t_0) + [D^2 L]_{\overrightarrow{L_0}}\frac{(t - t_0)^2}{2}. \end{aligned} \tag{41}$$

6.4 *The series for G*

The double action of the Lie operator, D on momenta G equal to zero

$$G(t) = [G]_{\overrightarrow{G_0}}. \tag{42}$$

7. Conclusion

The following concluding remarks and notes can be outlined:

- The ejection of mass from any body depends on many parameters, amongst the most important of which are the central condensation (which means, more or less, the degree of rigidity of the body) and the velocity of rotation which provides the external layers with the angular momentum that activates the process of ejection of mass. But to simplify the model, we assumed that the rate of mass ejection depends explicitly, solely on the mass of the body.

- Since the different models of variable mass assume that the mass loss takes place isotropically, i.e., there is no preformed direction in the space, it is expected to find the Hamiltonian free from dependence on the inclination. This reflects the absence of H in the Hamiltonian of the problem. Therefore $\dot{h} = 0 \Rightarrow h = \text{const.}$
- Also, as the stars are assumed point masses, the Hamiltonian is free from orientation angles (g, h). This means that G and H are kept constants.
- The effect of one body ejects mass on the other body is declared through the appearance of the non-linear term factored by $\alpha_1\alpha_2$.

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