Reflection of P and SV waves at the free surface of a monoclinic elastic half-space

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The propagation of plane waves in an anisotropic elastic medium possessing monoclinic symmetry is discussed. The expressions for the phase velocity of qP and qSV waves propagating in the plane of elastic symmetry are obtained in terms of the direction cosines of the propagation vector. It is shown that, in general, qP waves are not longitudinal and qSV waves are not transverse. Pure longitudinal and pure transverse waves can propagate only in certain specific directions. Closed-form expressions for the reflection coefficients of qP and qSV waves incident at the free surface of a homogeneous monoclinic elastic half-space are obtained. These expressions are used for studying numerically the variation of the reflection coefficients with the angle of incidence. The present analysis corrects some fundamental errors appearing in recent papers on the subject.

1. Introduction

In an anisotropic elastic solid medium, three types of body waves with mutually orthogonal particle motion can be propagated. In general, the particle motion is neither purely longitudinal nor purely transverse. Because of this, the three types of body waves in an anisotropic medium are referred to as qP, qSV and qSH, rather than as P, SV and SH, the symbols used for propagation in an isotropic medium (see, e.g., Keith and Crampin 1977).

A monoclinic medium possesses one plane of elastic symmetry. For wave propagation in the plane of symmetry, SH motion is decoupled from the $P-SV$ motion. While the particle motion of SH waves is purely transverse, it is neither purely longitudinal nor purely transverse in the case of $P-SV$ waves. In a recent paper, Chattopadhyay and Choudhury (1995) discussed the reflection of qP waves at the plane free boundary of a monoclinic half-space. In a subsequent paper, Chattopadhyay *et al* (1996) studied the reflection of qSV waves. Since, in both of these studies, the authors assume that qP waves are purely longitudinal and qSV waves purely transverse, most of the results of these two papers, including the expressions for the reflection coefficients, are erroneous (Singh 1999). The aim of the present study is to derive closed-form algebraic expressions for the reflection coefficients when plane waves of qP or qSV type are incident at the plane free boundary of a monoclinic elastic half-space. Numerical results presented indicate that the anisotropy might affect the reflection coefficients significantly.

2. Basic equations

Consider a homogeneous anisotropic elastic medium of monoclinic type. It has one plane of elastic symmetry and its elastic properties are defined by thirteen elastic modulii. Taking the plane of symmetry as the x_2x_3 - plane, the generalized Hooke's law can be expressed in the form

$$\tau_{11} = c_{11}e_{11} + c_{12}e_{22} + c_{13}e_{33} + 2c_{14}e_{23}, \qquad (1a)$$

$$\tau_{22} = c_{12}e_{11} + c_{22}e_{22} + c_{23}e_{33} + 2c_{24}e_{23}, \qquad (1b)$$

$$\tau_{33} = c_{13}e_{11} + c_{23}e_{22} + c_{33}e_{33} + 2c_{34}e_{23}, \qquad (1c)$$

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$$\tau_{23} = c_{14}e_{11} + c_{24}e_{22} + c_{34}e_{33} + 2c_{44}e_{23}, \tag{1d}$$

$$\tau_{13} = 2(c_{55}e_{13} + c_{56}e_{12}), \tag{1e}$$

$$\tau_{12} = 2(c_{56}e_{13} + c_{66}e_{12}), \tag{1f}$$

where τ_{ij} is the stress tensor and e_{ij} the strain tensor. Further,

$$2e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}, \tag{2}$$

u_i being the displacement vector.

In the case of an orthotropic medium with the planes of symmetry coinciding with the coordinate planes

$$c_{14} = c_{24} = c_{34} = c_{56} = 0. \tag{3a}$$

For a transversely isotropic medium with the axis of symmetry coinciding with the x_1-axis

$$c_{12} = c_{13}, c_{22} = c_{33}, c_{55} = c_{66}, c_{23} = c_{22} - 2c_{44},$$
$$c_{14} = c_{24} = c_{34} = c_{56} = 0. \tag{3b}$$

For an isotropic medium

$$c_{11} = c_{22} = c_{33} = \lambda + 2\mu,$$
$$c_{12} = c_{13} = c_{23} = \lambda,$$
$$c_{44} = c_{55} = c_{66} = \mu,$$
$$c_{14} = c_{24} = c_{34} = c_{56} = 0, \tag{3c}$$

λ and μ being the Lamé parameters.

For plane waves propagating in the x_2x_3-plane

$$\mathrm{u}_i = \mathrm{u}_i(x_2, x_3, t), \partial/\partial x_1 \equiv 0.$$

Equations (1) and (2) now yield

$$\tau_{11} = c_{12}\frac{\partial u_2}{\partial x_2} + c_{13}\frac{\partial u_3}{\partial x_3} + c_{14}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right), \tag{4a}$$

$$\tau_{22} = c_{22}\frac{\partial u_2}{\partial x_2} + c_{23}\frac{\partial u_3}{\partial x_3} + c_{24}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right), \tag{4b}$$

$$\tau_{33} = c_{23}\frac{\partial u_2}{\partial x_2} + c_{33}\frac{\partial u_3}{\partial x_3} + c_{34}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right), \tag{4c}$$

$$\tau_{23} = c_{24}\frac{\partial u_2}{\partial x_2} + c_{34}\frac{\partial u_3}{\partial x_3} + c_{44}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right), \tag{4d}$$

$$\tau_{13} = c_{55}\frac{\partial u_1}{\partial x_3} + c_{56}\frac{\partial u_1}{\partial x_2}, \tag{4e}$$

$$\tau_{12} = c_{56}\frac{\partial u_1}{\partial x_3} + c_{66}\frac{\partial u_1}{\partial x_2}. \tag{4f}$$

The equations of motion without body forces are

$$\frac{\partial}{\partial x_j}\tau_{ij} = \rho\frac{\partial^2 u_i}{\partial t^2}(i = 1, 2, 3), \tag{5}$$

using the summation convention. From equations (4) and (5), we obtain the equations of motion in terms of the displacements in the form

$$c_{66}\frac{\partial^2 u_1}{\partial x_2^2} + 2c_{56}\frac{\partial^2 u_1}{\partial x_2\partial x_3} + c_{55}\frac{\partial^2 u_1}{\partial x_3^2} = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{6a}$$

$$c_{22}\frac{\partial^2 u_2}{\partial x_2^2} + c_{44}\frac{\partial^2 u_2}{\partial x_3^2} + c_{24}\frac{\partial^2 u_3}{\partial x_2^2} + c_{34}\frac{\partial^2 u_3}{\partial x_3^2} + 2c_{24}\frac{\partial^2 u_2}{\partial x_2\partial x_3} + (c_{23} + c_{44})\frac{\partial^2 u_3}{\partial x_2\partial x_3} = \rho\frac{\partial^2 u_2}{\partial t^2}, \tag{6b}$$

$$c_{24}\frac{\partial^2 u_2}{\partial x_2^2} + c_{34}\frac{\partial^2 u_2}{\partial x_3^2} + c_{44}\frac{\partial^2 u_3}{\partial x_2^2} + c_{33}\frac{\partial^2 u_3}{\partial x_3^2} + 2c_{34}\frac{\partial^2 u_3}{\partial x_2\partial x_3} + (c_{23} + c_{44})\frac{\partial^2 u_2}{\partial x_2\partial x_3} = \rho\frac{\partial^2 u_3}{\partial t^2}. \tag{6c}$$

From equations (6a, b, c), it is obvious that the u_1 motion representing SH waves is decoupled from the (u_2, u_3) motion representing qP and qSV waves.

Let $\mathbf{p}(0, p_2, p_3)$ denote the unit propagation vector, c the phase velocity and k the wave number of plane waves propagating in the x_2x_3-plane. A solution of the equation of motion (6a) representing plane waves is of the form

$$u_1 = A\, \exp[ik(ct - x_2p_2 - x_3p_3)]. \tag{7}$$

From equations (6a) and (7), we find

$$c_{66}\ p_2^2 + 2c_{56}\ p_2p_3 + c_{55}\ p_3^2 = \rho c^2. \tag{8}$$

This equation gives the phase velocity of SH waves propagating in an arbitrary direction in the plane of elastic symmetry of a monoclinic medium.

We seek plane wave solutions of the equations of motion (6b) and (6c) of the form

$$\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = A\begin{pmatrix} d_2 \\ d_3 \end{pmatrix}\exp[ik(ct - x_2p_2 - x_3p_3)], \tag{9}$$

where $\mathbf{d}(0, d_2, d_3)$ is the unit displacement vector, also known as the polarization vector. Inserting the expressions for u_2 and u_3 in the equations of motion (6b) and (6c), we obtain

$$(U-\rho c^2)d_2 + Vd_3 = 0, \tag{10a}$$

$$Vd_2 + (Z-\rho c^2)d_3 = 0, \tag{10b}$$

where

$$\begin{aligned} U(p_2,p_3) &= c_{22}p_2^2 + c_{44}p_3^2 + 2c_{24}p_2p_3, \\ V(p_2,p_3) &= c_{24}p_2^2 + c_{34}p_3^2 + (c_{23}+c_{44})p_2p_3, \\ Z(p_2,p_3) &= c_{44}p_2^2 + c_{33}p_3^2 + 2c_{34}p_2p_3. \end{aligned} \tag{11}$$

Equations (10a) and (10b) yield

$$d_2/d_3 = V/(\rho c^2 - U) = (\rho c^2 - Z)/V. \tag{12}$$

Therefore, ρc^2 satisfies the quadratic equation

$$\rho^2 c^4 - (U+Z)\rho c^2 + (UZ - V^2) = 0, \tag{13a}$$

with solutions

$$2\rho c^2(p_2,p_3) = (U+Z) \pm [(U-Z)^2 + 4V^2]^{1/2}. \tag{13b}$$

The upper sign in equation (13b) is for qP waves and the lower sign is for qSV waves.

Eliminating ρc^2 from the two equations in (12), we find

$$(d_2^2 - d_3^2)V = d_2d_3(U-Z). \tag{14a}$$

Inserting the expressions for U, V and Z from equation (11), we obtain

$$\begin{aligned} &[c_{24}(d_3^2-d_2^2) + (c_{22}-c_{44})d_2d_3]p_2^2 + [c_{34}(d_3^2-d_2^2) \\ &\quad + (c_{44}-c_{33})d_2d_3]p_3^2 + [(c_{23}+c_{44})(d_3^2-d_2^2) \\ &\quad + 2(c_{24}-c_{34})d_2d_3]p_2p_3 = 0. \end{aligned} \tag{14b}$$

We may write equation (14a) in the form

$$\frac{d_2d_3}{d_3^2-d_2^2} = V/(Z-U). \tag{14c}$$

Noting that $U = U(p_2,p_3)$ etc., equation (14c) can be used to find the direction of the displacement vector $\mathbf{d}$ for a given direction of propagation $\mathbf{p}$. Putting $\tan e = p_2/p_3, \tan\phi = d_2/d_3$, we find

$$\phi = \frac{1}{2}\tan^{-1}(\Omega), \frac{\pi}{2} + \frac{1}{2}\tan^{-1}(\Omega), \tag{14d}$$

where

$$\Omega = 2\frac{c_{24}\tan^2 e + (c_{23}+c_{44})\tan e + c_{34}}{[(c_{44}-c_{22})\tan^2 e + 2(c_{34}-c_{24}) \times \tan e + c_{33} - c_{44}]}. \tag{14e}$$

For an isotropic medium [see equation (3c)], we find $\Omega = \tan 2e$ so that $\phi = e, \pi/2 + e$ corresponding to the longitudinal P waves and transverse SV waves. The same is true for a transversely isotropic medium with the axis of symmetry perpendicular to the plane of propagation.

Equation (9) will represent a longitudinal wave if the displacement vector $\mathbf{d}$ is parallel to the propagation vector $\mathbf{p}$, i.e., if $d_2 = p_2, d_3 = p_3$. In that case, equation (14b) yields

$$\begin{aligned} &c_{24}p_2^4 + (c_{23}-c_{22}+2c_{44})p_2^3p_3 - 3(c_{24}-c_{34})p_2^2p_3^2 \\ &\quad - (c_{23}-c_{33}+2c_{44})p_2p_3^3 - c_{34}p_3^4 = 0. \end{aligned} \tag{15}$$

Equation (15) gives the directions of propagation for which P waves are purely longitudinal. From equation (10), the corresponding phase velocity is given by

$$\begin{aligned} \rho c_1^2 &= U + (p_3/p_2)V \\ &= (p_2/p_3)V + Z. \end{aligned} \tag{16}$$

Similarly, equation (9) will represent a transverse wave if the displacement vector $\mathbf{d}$ is perpendicular to the propagation vector $\mathbf{p}$, i.e., if $\mathbf{d.p} = d_2p_2 + d_3p_3 = 0$. In this case also equation (14b) leads to equation (15), as expected. The phase velocity of transverse qSV waves is given by

$$\begin{aligned} \rho c_2^2 &= U - (p_2/p_3)V \\ &= Z - (p_3/p_2)V. \end{aligned} \tag{17}$$

For an orthotropic medium [see equation (3a)], equations (11) and (15) to (17) yield:

$$\text{(i) } p_2 = 0, c_1 = (c_{33}/\rho)^{1/2}, c_2 = (c_{44}/\rho)^{1/2}; \tag{18a}$$

$$\text{(ii) } p_3 = 0, c_1 = (c_{22}/\rho)^{1/2}, c_2 = (c_{44}/\rho)^{1/2}; \tag{18b}$$

$$\text{(iii) } \left(\frac{p_2}{p_3}\right)^2 = \frac{c_{23}-c_{33}+2c_{44}}{c_{23}-c_{22}+2c_{44}}. \tag{18c}$$

These are the directions along which qP waves are purely longitudinal and qSV waves purely transverse in an orthotropic medium. Of course, similar

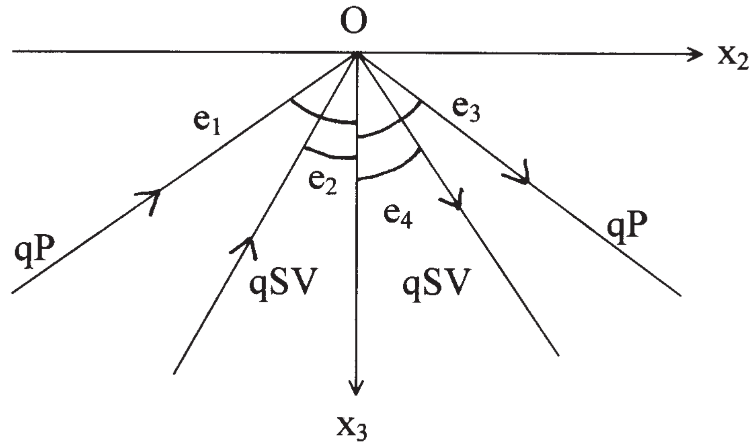

Figure 1. Reflection of qP and qSV waves at the plane free boundary ($x_3 = 0$) of a monoclinic half-space ($x_3 \geq 0$).

directions exist for wave propagation in the x_1x_2- and x_1x_3-planes, which are also planes of elastic symmetry. For a transversely isotropic medium [equation (3b)], equation (15) reveals that qP waves are longitudinal and qSV waves transverse for all directions of propagation. Therefore, for wave propagation in a plane perpendicular to the axis of elastic symmetry of a transversely isotropic medium, qP waves are longitudinal and qSV waves transverse. From equations (16) and (17), the phase velocities of qP and qSV waves are given by

$$c_1 = (c_{22}/\rho)^{1/2}, c_2 = (c_{44}/\rho)^{1/2}. \tag{19}$$

3. Reflection of qP and qSV waves

Consider a homogeneous, monoclinic, elastic half-space occupying the region $x_3 \geq 0$ (figure 1). The plane of elastic symmetry is taken as the x_2x_3-plane. Plane qP or qSV waves are incident at the traction-free boundary $x_3 = 0$. We consider plane strain problem for which

$$u_1 = 0, u_2(x_2, x_3, t), u_3 = u_3(x_2, x_3, t). \tag{20}$$

Incident qP or qSV waves will generate reflected qP and qSV waves. The total displacement field is given by

$$u_2 = \sum_{j=1}^{4} A_j e^{iP_j},$$
$$u_3 = \sum_{j=1}^{4} B_j e^{iP_j}, \tag{21}$$

where

$$P_1 = \omega[t - (x_2 \sin e_1 - x_3 \cos e_1)/c_1],$$
$$P_2 = \omega[t - (x_2 \sin e_2 - x_3 \cos e_2)/c_2],$$
$$P_3 = \omega[t - (x_2 \sin e_3 + x_3 \cos e_3)/c_3],$$
$$P_4 = \omega[t - (x_2 \sin e_4 + x_3 \cos e_4)/c_4], \tag{22}$$

ω being the angular frequency. We distinguish quantities corresponding to various waves by using the subscript (1) for incident qP waves, (2) for incident qSV waves, (3) for reflected qP waves and (4) for reflected qSV waves. Thus, for example, for the incident qP waves, c_1 denotes the phase velocity, e_1 the angle of incidence, $P_1(x_2, x_3, t)$ the phase factor, A_1 the amplitude factor of the u_2 component of the displacement and B_1 that of the u_3 component.

Since each of the incident qP, incident qSV, reflected qP and reflected qSV waves must satisfy the equations of motion, we have, as in equations (12) and (13b),

$$A_i = F_i B_i (i = 1, 2, 3, 4; \text{no summation}), \tag{23}$$

where

$$F_i = V_i/(\rho c_i^2 - U_i) = (\rho c_i^2 - Z_i)/V_i, \quad (i = 1, 2, 3, 4) \tag{24}$$

$$2\rho c_i^2 = (U_i + Z_i) + [(U_i - Z_i)^2 + 4V_i^2]^{1/2}, \quad (i = 1, 3) \tag{25}$$

$$2\rho c_i^2 = (U_i + Z_i) - [(U_i - Z_i)^2 + 4V_i^2]^{1/2}. \quad (i = 2, 4) \tag{26}$$

The expressions for U_i, V_i and Z_i are obtained from the expressions for U, V and Z given in equation (11) on substituting suitable values for (p_2, p_3). For incident qP waves, $p_2 = \sin e_1, p_3 = -\cos e_1$; for incident qSV waves, $p_2 = \sin e_2, p_3 = -\cos e_2$; for reflected qP waves, $p_2 = \sin e_3, p_3 = \cos e_3$; and, for reflected qSV waves, $p_2 = \sin e_4, p_3 = \cos e_4$ (see figure 1). We thus obtain

$$U_1 = c_{22} \sin^2 e_1 + c_{44} \cos^2 e_1$$
$$- 2c_{24} \sin e_1 \cos e_1,$$

$$V_1 = c_{24}\sin^2 e_1 + c_{34}\cos^2 e_1 - (c_{23}+c_{44})\sin e_1 \cos e_1,$$

$$Z_1 = c_{44}\sin^2 e_1 + c_{33}\cos^2 e_1 - 2c_{34}\sin e_1\cos e_1; \tag{27}$$

$$U_3 = c_{22}\sin^2 e_3 + c_{44}\cos^2 e_3 + 2c_{24}\sin e_3\cos e_3,$$

$$V_3 = c_{24}\sin^2 e_3 + c_{34}\cos^2 e_3 + (c_{23}+c_{44})\sin e_3\cos e_3,$$

$$Z_3 = c_{44}\sin^2 e_3 + c_{33}\cos^2 e_3 + 2c_{34}\sin e_3\cos e_3. \tag{28}$$

(U_2, V_2, Z_2) are obtained from (U_1, V_1, Z_1) on replacing e_1 by e_2 and (U_4, V_4, Z_4) are obtained from (U_3, V_3, Z_3) on replacing e_3 by e_4.

The total displacement field given by equation (21) must satisfy the traction-free boundary conditions, viz.,

$$\tau_{23} = \tau_{33} = 0 \text{ at } x_3 = 0. \tag{29}$$

Equations (4c), (4d), (21) and (29) yield

$$\begin{aligned}
&\left[(c_{24}A_1 + c_{44}B_1)\frac{\sin e_1}{c_1} - (c_{44}A_1 + c_{34}B_1)\frac{\cos e_1}{c_1}\right] \times e^{iP_1(x_2,0)}\\
&+ \left[(c_{24}A_2 + c_{44}B_2)\frac{\sin e_2}{c_2} - (c_{44}A_2 + c_{34}B_2)\frac{\cos e_2}{c_2}\right] \times e^{iP_2(x_2,0)}\\
&+ \left[(c_{24}A_3 + c_{44}B_3)\frac{\sin e_3}{c_3} + (c_{44}A_3 + c_{34}B_3)\frac{\cos e_3}{c_3}\right] \times e^{iP_3(x_2,0)}\\
&+ \left[(c_{24}A_4 + c_{44}B_4)\frac{\sin e_4}{c_4} + (c_{44}A_4 + c_{34}B_4)\frac{\cos e_4}{c_4}\right] \times e^{iP_4(x_2,0)} = 0,
\end{aligned} \tag{30}$$

$$\begin{aligned}
&\left[(c_{23}A_1 + c_{34}B_1)\frac{\sin e_1}{c_1} - (c_{34}A_1 + c_{33}B_1)\frac{\cos e_1}{c_1}\right] \times e^{iP_1(x_2,0)}\\
&+ \left[(c_{23}A_2 + c_{34}B_2)\frac{\sin e_2}{c_2} - (c_{34}A_2 + c_{33}B_2)\frac{\cos e_2}{c_2}\right] \times e^{iP_2(x_2,0)}\\
&+ \left[(c_{23}A_3 + c_{34}B_3)\frac{\sin e_3}{c_3} + (c_{34}A_3 + c_{33}B_3)\frac{\cos e_3}{c_3}\right] \times e^{iP_3(x_2,0)}\\
&+ \left[(c_{23}A_4 + c_{34}B_4)\frac{\sin e_4}{c_4} + (c_{34}A_4 + c_{33}B_4)\frac{\cos e_4}{c_4}\right] \times e^{iP_4(x_2,0)} = 0.
\end{aligned} \tag{31}$$

Since equations (30) and (31) are to be satisfied for all values of x_2, we must have

$$P_1(x_2,0) = P_2(x_2,0) = P_3(x_2,0) = P_4(x_2,0). \tag{32}$$

Equations (22) and (32) imply

$$\frac{\sin e_1}{c_1(e_1)} = \frac{\sin e_2}{c_2(e_2)} = \frac{\sin e_3}{c_3(e_3)} = \frac{\sin e_4}{c_4(e_4)} = 1/c_a, \tag{33}$$

where c_a is the apparent phase velocity. This is the form of Snell's law for a monoclinic medium.

From equations (25), (27) and (28), we note that even if $e_1 = e_3, c_1 \neq c_3$. Therefore, from equation (33), the angle of reflection of qP waves is not equal to the angle of incidence of qP waves. Similarly, the angle of reflection of qSV waves is not equal to the angle of incidence of qSV waves. Chattopadhyay and Choudhury (1995) and Chattopadhyay *et al.* (1996) assume that the angle of reflection of qP (qSV) waves is equal to the angle of incidence of qP (qSV) waves. Therefore, the reflection coefficients obtained in these studies are incorrect.

In the case of an orthotropic medium, $c_{14} = c_{24} = c_{34} = c_{56} = 0$. Consequently, $c_1 = c_3$ if $e_1 = e_3$. Equation (33) then reveals that the angle of reflection of qP (qSV) waves is equal to the angle of incidence of qP (qSV) waves.

Using the relations (23), (32) and (33) in equations (30) and (31), we obtain

$$a_1B_1 + a_2B_2 + a_3B_3 + a_4B_4 = 0, \tag{34a}$$

$$b_1B_1 + b_2B_2 + b_3B_3 + b_4B_4 = 0, \tag{34b}$$

where

$$\begin{aligned}
a_1 &= c_{24}F_1 + c_{44} - (c_{44}F_1 + c_{34})\cot e_1,\\
a_2 &= c_{24}F_2 + c_{44} - (c_{44}F_2 + c_{34})\cot e_2,\\
a_3 &= c_{24}F_3 + c_{44} + (c_{44}F_3 + c_{34})\cot e_3,\\
a_4 &= c_{24}F_4 + c_{44} + (c_{44}F_4 + c_{34})\cot e_4,\\
b_1 &= c_{23}F_1 + c_{34} - (c_{34}F_1 + c_{33})\cot e_1,\\
b_2 &= c_{23}F_2 + c_{34} - (c_{34}F_2 + c_{33})\cot e_2,\\
b_3 &= c_{23}F_3 + c_{34} + (c_{34}F_3 + c_{33})\cot e_3,\\
b_4 &= c_{23}F_4 + c_{34} + (c_{34}F_4 + c_{33})\cot e_4.
\end{aligned} \tag{35}$$

3.1 *Incident qP waves*

In the case of incident qP waves, $A_2 = B_2 = 0$ and equations (34a, b) become

$$a_1B_1 + a_3B_3 + a_4B_4 = 0, \tag{36a}$$

$$b_1B_1 + b_3B_3 + b_4B_4 = 0. \tag{36b}$$

On solving, we obtained the amplitude ratios in the form

$$\frac{B_3}{B_1} = (a_4b_1 - a_1b_4)/\Delta, \tag{37a}$$

$$\frac{B_4}{B_1} = (a_1b_3 - a_3b_1)/\Delta, \tag{37b}$$

where

$$\Delta = a_3b_4 - a_4b_3. \tag{38}$$

Using equation (23), we find

$$\frac{A_3}{A_1} = \frac{F_3}{F_1}\left(\frac{B_3}{B_1}\right), \frac{A_4}{A_1} = \frac{F_4}{F_1}\left(\frac{B_4}{B_1}\right). \tag{39}$$

3.2 *Incident qSV waves*

For incident qSV waves, $A_1 = B_1 = 0$, so that

$$a_2B_2 + a_3B_3 + a_4B_4 = 0, \tag{40a}$$

$$b_2B_2 + b_3B_3 + b_4B_4 = 0, \tag{40b}$$

$$\frac{B_3}{B_2} = (a_4b_2 - a_2b_4)/\Delta, \tag{41a}$$

$$\frac{B_4}{B_2} = (a_2b_3 - a_3b_2)/\Delta, \tag{41b}$$

$$\frac{A_3}{A_2} = \frac{F_3}{F_2}\left(\frac{B_3}{B_2}\right), \frac{A_4}{A_2} = \frac{F_4}{F_2}\left(\frac{B_4}{B_2}\right). \tag{41c}$$

3.3 *Isotropic half-space*

Using equation (3c), it can be shown that, for an isotropic half-space,

$$\begin{aligned} c_1 &= c_3 = [(\lambda + 2\mu)/\rho]^{1/2} = \alpha, c_2 = c_4 \\ &= (\mu/\rho)^{1/2} = \beta, e_1 = e_3 = e, e_2 = e_4 = f, \end{aligned}$$

$$\frac{\sin e}{\alpha} = \frac{\sin f}{\beta},$$

$$F_1 = -F_3 = -\tan e, F_2 = -F_4 = \cot f,$$

$$a_1 = a_3 = 2\mu, a_2 = a_4 = -\mu\cos 2f/\sin^2 f,$$

$$b_1 = -b_3 = -2\mu(\alpha/\beta)^2\cos 2f/\sin 2e,$$

$$b_2 = -b_4 = -2\mu\cot f. \tag{42}$$

Putting these values in equations (37), (39) and (41), we deduce the amplitude ratios for an isotropic half-space in the form

$$\frac{A_3}{A_1} = -\frac{B_3}{B_1} = \frac{\sin 2e\sin 2f - (\alpha/\beta)^2\cos^2 2f}{\sin 2e\sin 2f + (\alpha/\beta)^2\cos^2 2f},$$

$$\frac{A_4}{A_1} = \frac{\cot f}{\tan e}\left(\frac{B_4}{B_1}\right) = \frac{(\alpha/\beta)^2\cot e\sin 4f}{\sin 2e\sin 2f + (\alpha/\beta)^2\cos^2 2f},$$

$$\frac{A_3}{A_2} = \frac{\tan e}{\cot f}\left(\frac{B_3}{B_2}\right) = \frac{4\sin^2 e\cos 2f}{\sin 2e\sin 2f + (\alpha/\beta)^2\cos^2 2f},$$

$$\frac{A_4}{A_2} = -\frac{B_4}{B_2} = -\frac{A_3}{A_1} = \frac{B_3}{B_1}. \tag{43}$$

The above expressions for the amplitude ratios for an isotropic half-space coincide with the corresponding results of Ben-Menahem and Singh (1981, pp. 93 and 95).

4. Numerical results and conclusions

The reflection coefficients given by Chattopadhyay and Choudhury (1995) and Chattopadhyay *et al.* (1996) for the reflection of qP and qSV waves at the plane free boundary of a monoclinic elastic half-space are incorrect because of two erroneous assumptions made by these authors, namely, qP waves are longitudinal (qSV waves are transverse) and the angle of reflection of qP (qSV) waves is equal to the angle of incidence of qP (qSV) waves. In the present study, we have obtained the correct reflection coefficients by solving the problem *ab initio.*

Equations (37) and (39) give the amplitude ratios when plane qP waves are incident at the plane-free boundary of a monoclinic elastic half-space. In these equations, A_3/A_1 and A_4/A_1 are the amplitude ratios for the horizontal component of the displacement and B_3/B_1 and B_4/B_1 are the amplitude ratios for the vertical component of the displacement. Similarly, equation (41) gives the amplitude ratios for incident qSV waves. From equations (21) and (23), we note that, for example, the total displacement of the incident qP waves is

$$(A_1^2 + B_1^2)^{1/2}e^{iP_1} = (1 + F_1^2)^{1/2}B_1e^{iP_1}.$$

Therefore, the reflection coefficients can be expressed in the form

$$R_{PP} = \left(\frac{1+F_3^2}{1+F_1^2}\right)^{1/2}.\frac{B_3}{B_1}, R_{PS} = \left(\frac{1+F_4^2}{1+F_1^2}\right)^{1/2}.\frac{B_4}{B_1} \tag{44a}$$

for incident qP waves, and

$$R_{SP} = \left(\frac{1+F_3^2}{1+F_2^2}\right)^{1/2} \cdot \frac{B_3}{B_2}, R_{SS} = \left(\frac{1+F_4^2}{1+F_2^2}\right)^{1/2} \cdot \frac{B_4}{B_2} \quad \text{(44b)}$$

for incident qSV waves. The reflection coefficients are in terms of the four angles e_i and the four velocities $c_i\ (e_i), i = 1, 2, 3, 4$. For an incident qP wave, e_1 and, therefore, $c_1(e_1)$ is supposed to be known. One has to compute e_3 and e_4 for given e_1. The velocities $c_3(e_3)$ and $c_4(e_4)$ can then be computed from explicit algebraic formulae. We give below the procedure for computing e_3 and e_4 for given e_1 in the case of incident qP waves and for given e_2 in the case of incident qSV waves.

The Snell's law for a monoclinic medium is given by equation (33) in which the apparent velocity c_a can be written as $c_a = c/p_2$, where $\mathbf{p}(0, p_2, p_3)$ is the propagation vector. We define dimensionless apparent velocity $\overline{c}$ through the relation

$$\overline{c} = c_a/\beta = c/(p_2\beta), \quad (45)$$

where $\beta = (c_{44}/\rho)^{1/2}$. Equation (13a) then becomes

$$\overline{c}^4 - (\overline{U} + \overline{Z})\overline{c}^2 + (\overline{UZ} - \overline{V}^2) = 0 \quad (46)$$

where

$$\begin{aligned}
\overline{U} &= U/(c_{44}p_2^2) = p^2 + 2\ \overline{c}_{24}p + \overline{c}_{22},\\
\overline{V} &= V/(c_{44}p_2^2) = \overline{c}_{34}p^2 + (1 + \overline{c}_{23})p + \overline{c}_{24},\\
\overline{Z} &= Z/(c_{44}p_2^2) = \overline{c}_{33}p^2 + 2\ \overline{c}_{34}p + 1,\\
p &= p_3/p_2, \overline{c}_{ij} = c_{ij}/c_{44}.
\end{aligned} \quad (47)$$

For incident qP waves, $p = -\cot e_1$; for incident qSV waves, $p = -\cot e_2$; for reflected qP waves, $p = \cot e_3$; for reflected qSV waves, $p = \cot e_4$. For a given p, equation (46) can be solved for $\overline{c^2}$, the two roots corresponding to qP and qSV waves. However, for a given $\overline{c}$, equation (46) is a bi-quadratic in p, corresponding to incident qP, incident qSV, reflected qP and reflected qSV. The positive roots corresponding to the reflected waves and the negative roots corresponding to the incident waves. On inserting the expressions for $\overline{U}, \overline{Z}$ and $\overline{V}$ from equation (47) into equation (46), the bi-quadratic in p becomes

$$g_0p^4 + g_1p^3 + g_2p^2 + g_3p + g_4 = 0, \quad (48)$$

where

$$\begin{aligned}
g_0 &= \overline{c}_{33} - \overline{c}_{34}^2,\\
g_1 &= 2(\overline{c}_{24}\overline{c}_{33} - \overline{c}_{23}\overline{c}_{34}),\\
g_2 &= 1 + \overline{c}_{22}\overline{c}_{33} + 2\overline{c}_{24}\overline{c}_{34} - (1 + \overline{c}_{23})^2 - (1 + \overline{c}_{33})\overline{c}^2,\\
g_3 &= 2[\overline{c}_{22}\overline{c}_{34} - \overline{c}_{23}\overline{c}_{24} - (\overline{c}_{24} + \overline{c}_{34})\overline{c}^2],\\
g_4 &= \overline{c}^4 - (1 + \overline{c}_{22})\overline{c}^2 + \overline{c}_{22} - \overline{c}_{24}^2.
\end{aligned} \quad (49)$$

If we define $q = 1/p = p_2/p_3$, the bi-quadratic transforms to

$$g_4q^4 + g_3q^3 + g_2q^2 + g_1q + g_0 = 0. \quad (50)$$

For angles of incidence, for which both reflected qP and reflected qSV waves exist, equation (50) will possess two positive roots, the smaller positive root (say q_4) corresponding to reflected SV and the larger positive root (q_3) corresponding to reflected qP. Further,

$$e_3 = \tan^{-1}(q_3), e_4 = \tan^{-1}(q_4). \quad (51)$$

For an isotropic medium (see equation (3c))

$$\begin{aligned}
g_0 &= \gamma, g_1 = 0,\\
g_2 &= 2\gamma - (1 + \gamma)\overline{c}^2, g_3 = 0,\\
g_4 &= (\overline{c}^2 - 1)(\overline{c}^2 - \gamma),
\end{aligned} \quad (52)$$

where

$$\gamma = (\lambda + 2\mu)/\mu = (\alpha/\beta)^2.$$

Equation (48) reduces to

$$\gamma p^4 + [2\gamma - (1 + \gamma)\overline{c}^2]p^2 + (\overline{c}^2 - 1)(\overline{c}^2 - \gamma) = 0,$$

i.e.,

$$\gamma(p^2 - \overline{c}^2 + 1)(p^2 - \overline{c}^2/\gamma + 1) = 0. \quad (53)$$

In the present case, the Snell's law (23) becomes

$$\frac{\sin e}{\alpha} = \frac{\sin f}{\beta} = 1/c_a.$$

Equation (45) shows that

$$\overline{c} = c_a/\beta = \mathrm{cosec} f = \sqrt{\gamma}\ \mathrm{cosec}\ e. \quad (54)$$

Therefore, the roots of equation (53) are given by

$$p^2 = \overline{c}^2 - 1 = \cot^2 f, \quad (55)$$

corresponding to SV waves, and

$$p^2 = \overline{c}^2/\gamma - 1 = \cot^2 e, \quad (56)$$

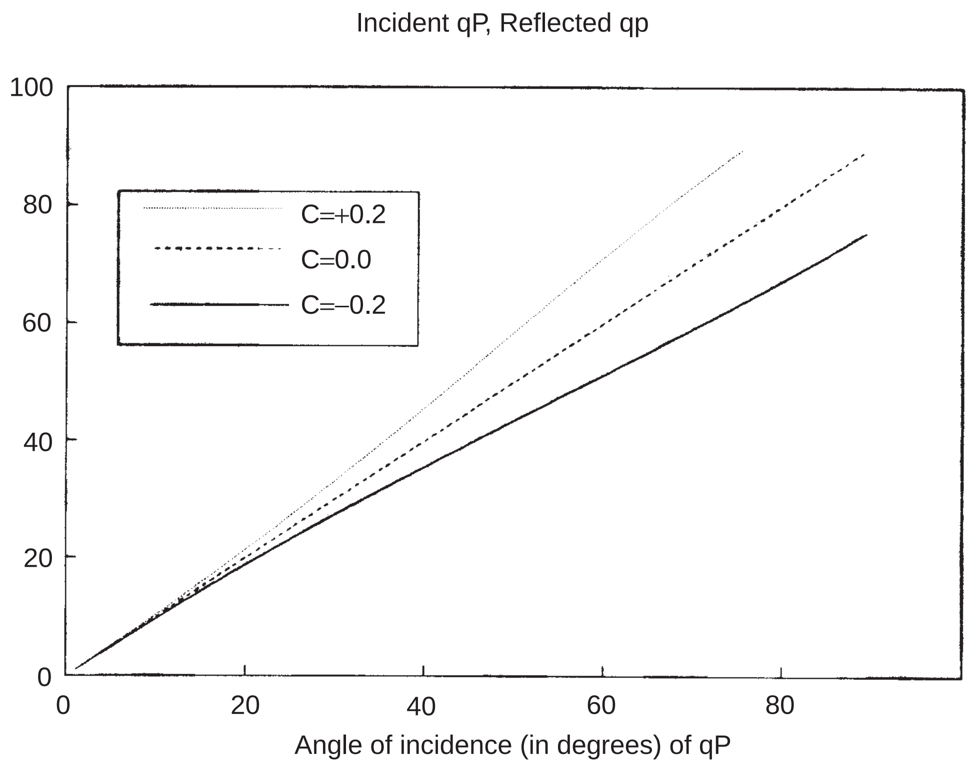

Figure 2. Variation of the angle of reflection (e_3) of qP waves with the angle of incidence (e_1) of qP waves for three values of the anisotropy parameter C.

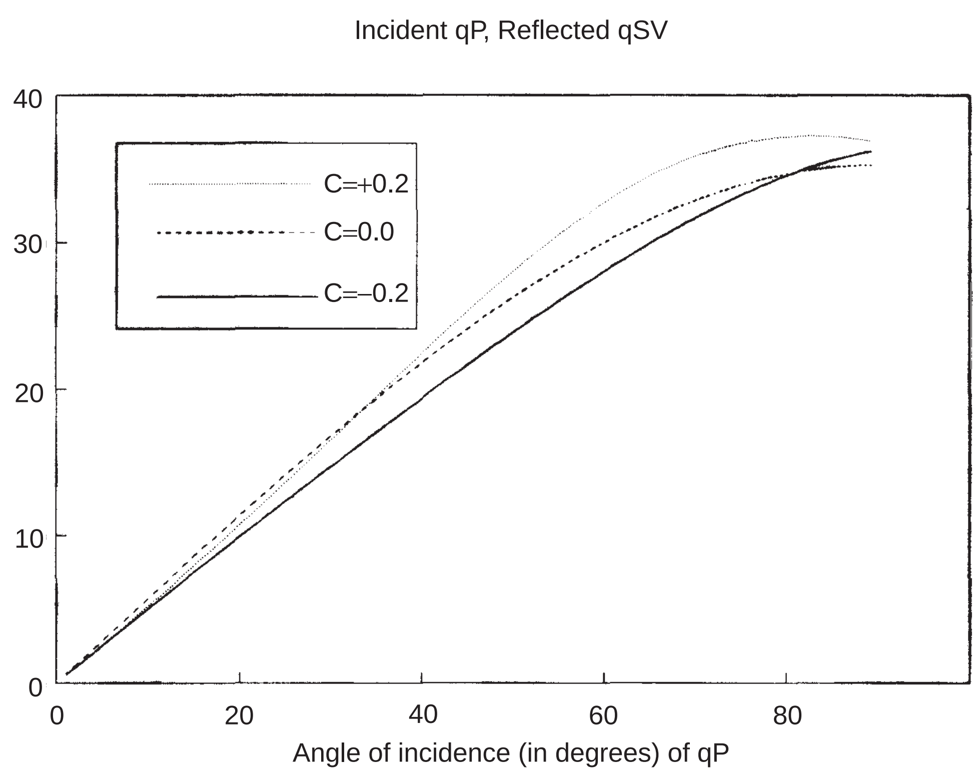

Figure 3. Variation of the angle of reflection (e_4) of qSV waves with the angle of incidence (e_1) of qP waves.

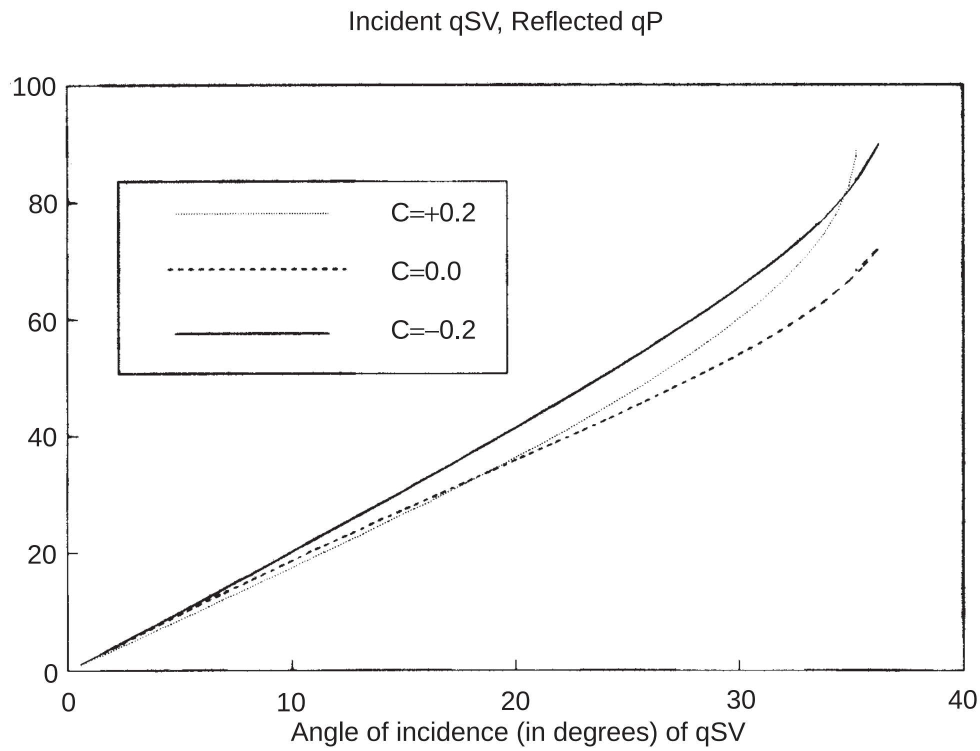

Figure 4. Variation of the angle of reflection (e_3) of qP waves with the angle of incidence (e_2) of qSV waves.

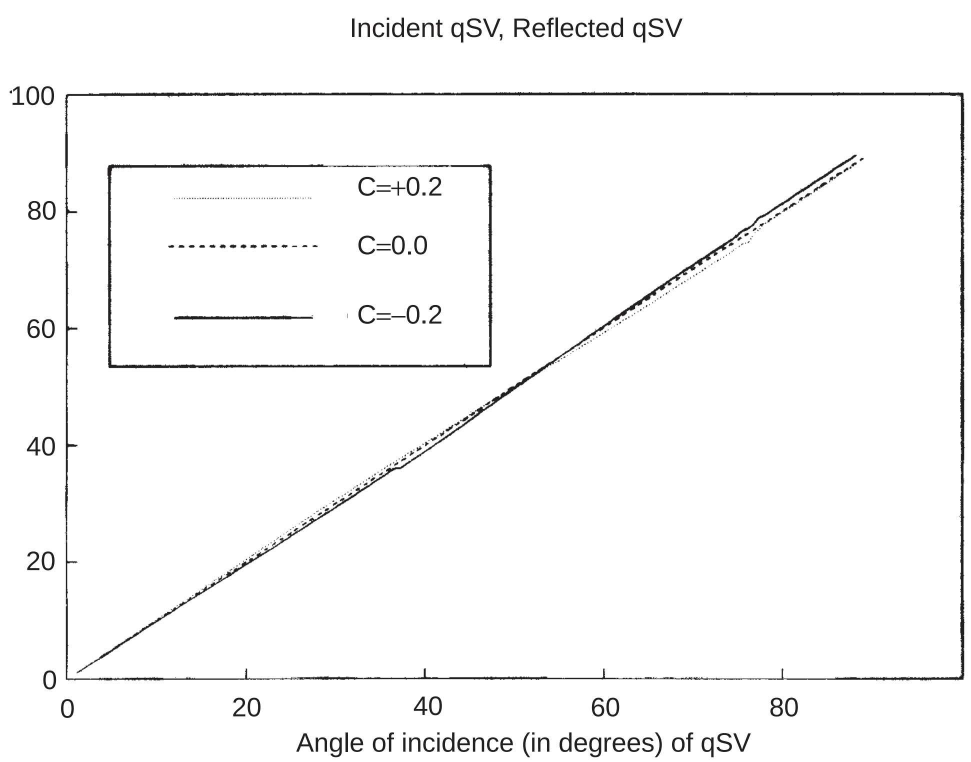

Figure 5. Variation of the angle of reflection (e_4) of qSV waves with the angle of incidence (e_2) of qSV waves.

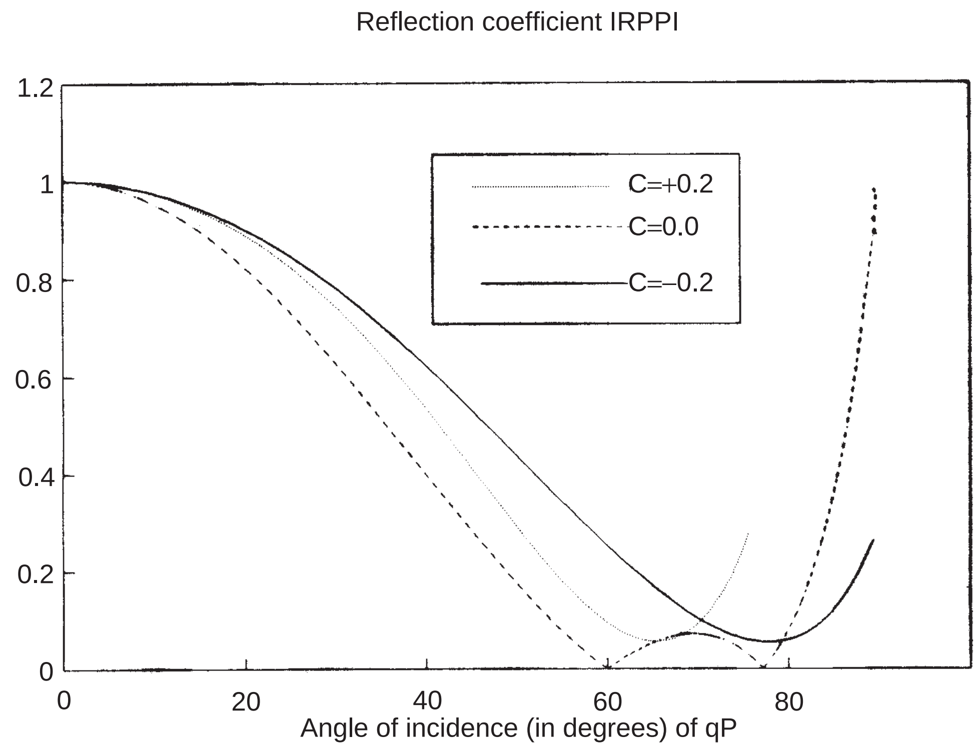

Figure 6. Variation of the reflection coefficient $|R_{PP}|$ with the angle of incidence (e_1) of qP waves.

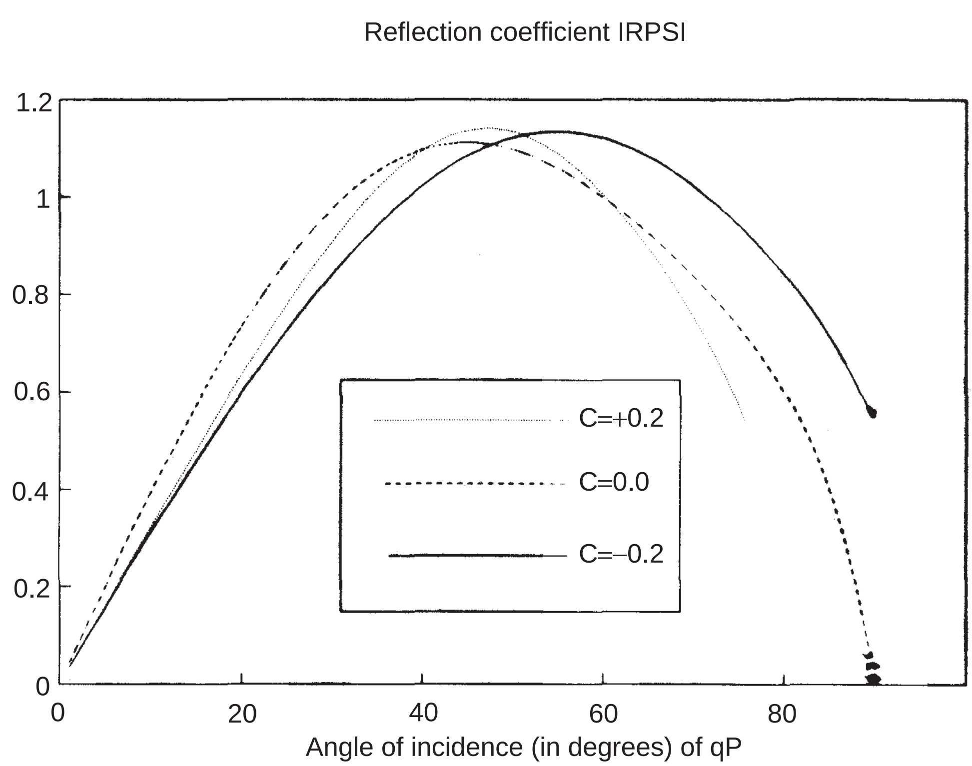

Figure 7. Variation of the reflection coefficient $|R_{PS}|$ with the angle of incidence (e_1) of qP waves.

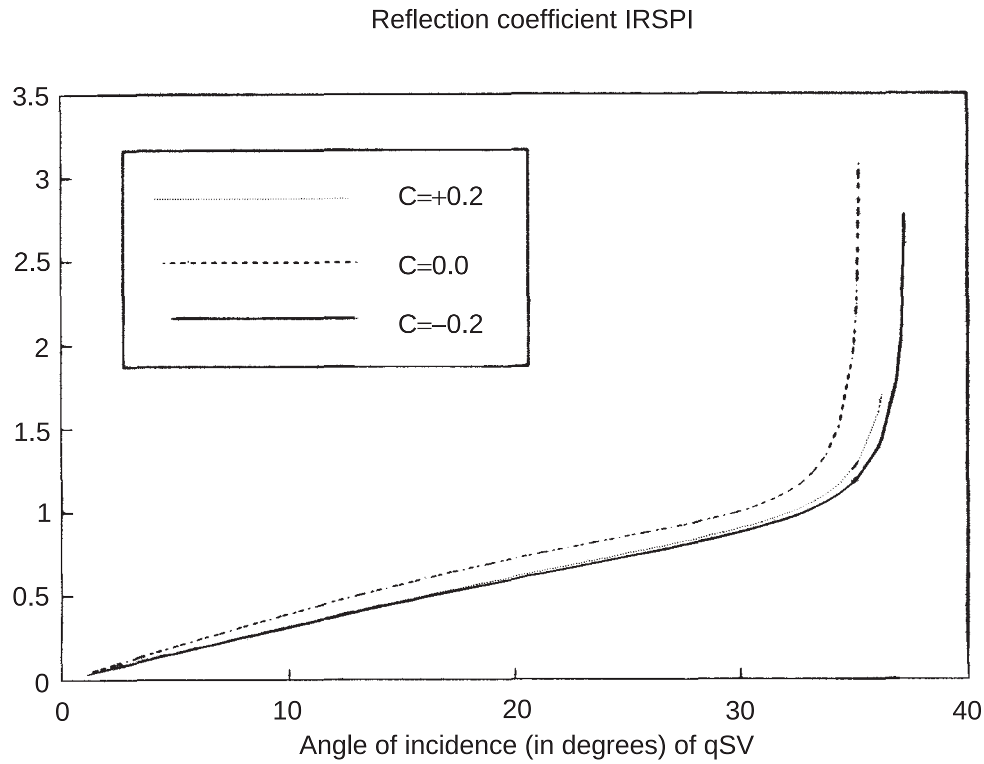

Figure 8. Variation of the reflection coefficient $|R_{SP}|$ with the angle of incidence (e_2) of qSV waves.

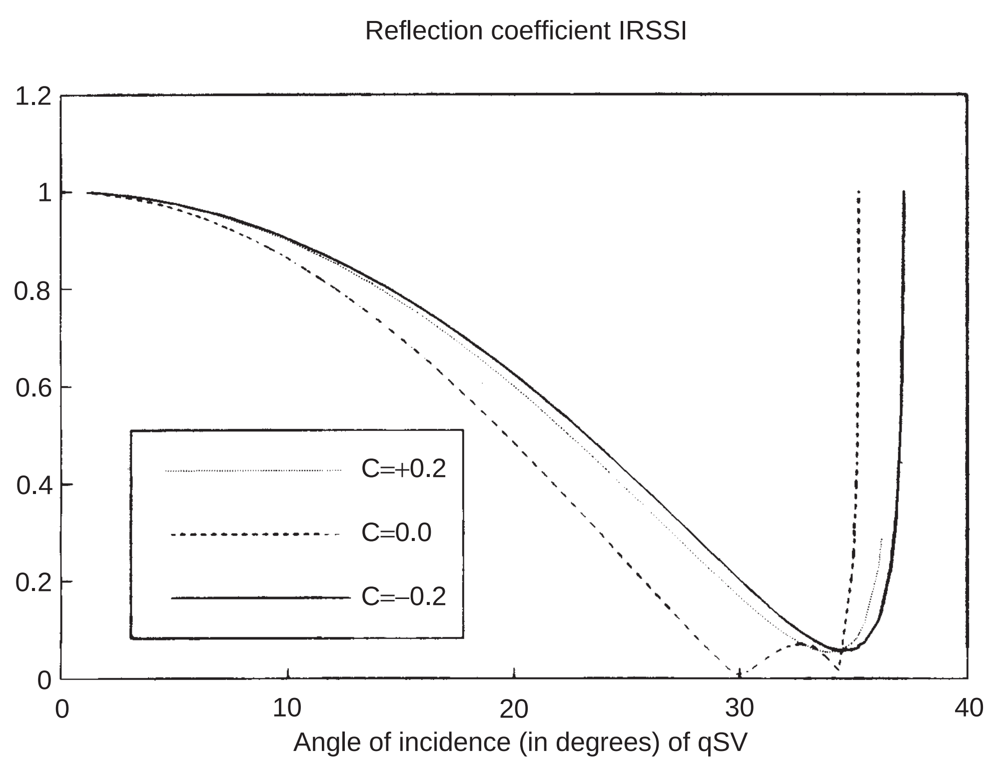

Figure 9. Variation of the reflection coefficient $|R_{SS}|$ with the angle of incidence (e_2) of qSV waves.

corresponding to P waves. Thus, we may choose $(q = 1/p)$

$$q_1 = -\tan e, q_2 = -\tan f,$$
$$q_3 = \tan e, q_4 = \tan f, \tag{57}$$

as the four roots of the bi-quadratic equation (50) for an isotropic medium. This choice acts as a guiding factor in computing the angles of reflection of qP and qSV waves in a monoclinic medium. For an orthotropic medium (see equation (3a)), it can be shown that $g_1 = g_3 = 0$. Therefore, equation (50) reduces to a quadratic equation in q^2. Thus, we may choose

$$q_1 = -q_3, q_2 = -q_4$$

so that the angle of reflection of qP (qSV) waves is equal to the angle of incidence of qP (qSV) waves. This is not true for a monoclinic material.

As observed earlier, for monoclinic media the angle of reflection of qP (qSV) waves is not equal to the angle of incidence of qP (qSV) waves. For numerical computation of results, we have assumed that

$$c_{22}/c_{44} = 19.8/6.67, c_{33}/c_{44} = 24.9/6.67,$$
$$c_{23}/c_{44} = 7.8/6.67, c_{24}/c_{44} = c_{34}/c_{44} = C.$$

Figure 2 gives the angle of reflection of qP waves for various values of the angle of incidence of qP waves for three values of C. In figure 2, for $C > 0$, the angle of reflection is greater than the angle of incidence. In contrast, for $C < 0$, the angle of reflection is less than the angle of incidence. Figure 3 gives the angle of reflection of qSV waves for various values of the angle of incidence of qP waves. Figures 4 and 5 are for incident qSV waves.

The variation of the reflection coefficient R_{PP} for incident qP-reflected qP waves as defined in equation (44a) with the angle of incidence of qP waves is shown in figure 6. The variation of the reflection coefficients R_{PS}, R_{SP} and R_{SS} is shown in figures 7–9. From figures 6–9 we observe that the anisotropy has a significant effect on the reflection coefficients.

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