

CHAPTER 320

The equations for integral and mean flow properties in the swash zone

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Abstract

The swash zone is the area of a beach where the waves move the instantaneous shoreline back and forth. This zone is modelled using the nonlinear shallow-water equations (NLSWE) as equations of motion, as is appropriate for gently sloping beaches. Integrated swash zone boundary conditions are derived for wave resolving models: their use gives considerable numerical advantages. Wave-averaged and swash-zone integrated boundary conditions have also been derived. They include numerous terms for which closures are to be found.

Equations and an analytic swash zone solution

In this section we introduce the basic equations used to describe the flow dynamics near the shore. We choose the still water level to be $z = 0$, and define

$$d(x, t) = h(x) + \eta(x, t) \quad (1)$$

where d is the total water depth, $z = -h(x)$ is the seabed, $z = \eta$ is the position of the free surface (see figure 1).

For a plane beach with $h(x) = \alpha x$ the inviscid equations are

$$\begin{aligned} d_{t^*}^* + (d^* u^*)_{x^*} + (d^* v^*)_{y^*} &= 0 \\ u_{t^*}^* + u^* u_{x^*}^* + v^* u_{y^*}^* + g d_{x^*}^* &= -g\alpha \\ v_{t^*}^* + u^* v_{x^*}^* + v^* v_{y^*}^* + g d_{y^*}^* &= 0. \end{aligned} \quad (2)$$

They can be put in a simple dimensionless form with no explicit dependence on the beach slope α (Meyer & Taylor 1972).

The dimensionless variables which eliminate the beach slope α from the equations are:

$$d = \frac{d^*}{\alpha l_0}, \quad u = \frac{u^*}{u_0}, \quad v = \frac{v^*}{u_0}, \quad u_0 = (g l_0 \alpha)^{1/2} \quad (3)$$

$$x = \frac{x^*}{l_0}, \quad y = \frac{y^*}{l_0}, \quad t = \frac{t^*}{t_0}, \quad t_0 = \left(\frac{l_0}{g\alpha} \right)^{1/2} \quad (4)$$

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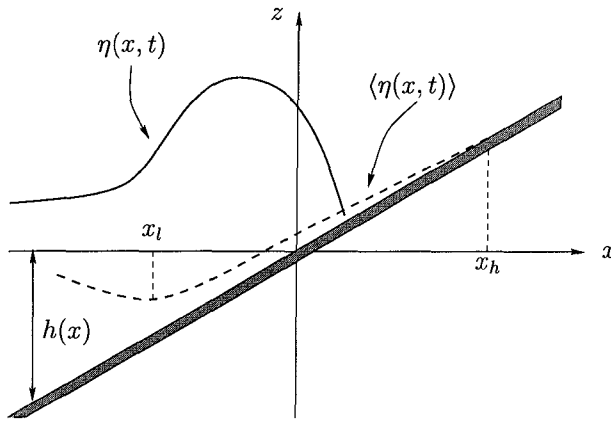


Figure 1: Flow definition in the vicinity of the swash zone.

where l_0 is a reference length which can be specified according to the particular problem under investigation. The equations become:

$$\begin{aligned} d_t + (du)_x + (dv)_y &= 0 \\ u_t + uu_x + vv_y + d_x &= -1 \\ v_t + uv_x + vv_y + d_y &= 0. \end{aligned} \quad (5)$$

For simplicity, friction terms are not included in this analysis and in most of the numerical modelling. Even at laboratory scale they are of secondary importance if beach slope is not too small. Watson et al. (1994) give a quantitative estimate of their importance. On the other hand, the energy dissipation of bores is included in the modelling as discontinuities in the solution.

Equations (5) can be further simplified by approximating for regular waves incident at small angle θ to the beach normal (Ryrie, 1983). We introduce a pseudotime t' and a small parameter ϵ such that:

$$t' = t - \epsilon y \quad (6)$$

gives the only y dependence. It implies that the wave pattern is moving along the shore with phase velocity of $1/\epsilon$. For example, this can occur for a regular train of waves incident towards the shore at a small angle θ to the shore normal, with an alongshore velocity $c/\sin \theta$.

For a weakly three-dimensional flow the following scaling:

$$y' = \epsilon y, \quad v' = v/\epsilon, \quad (7)$$

on substitution into the NLSWE (5), neglecting $\mathcal{O}(\epsilon^2)$ terms and dropping primes, gives the equations:

$$\begin{aligned} d_t + (du)_x &= 0 \\ u_t + uu_x + d_x &= -1 \\ v_t + uv_x - d_t &= 0. \end{aligned} \quad (8)$$

The last of the above set of hyperbolic equations is now decoupled from the first two. The v component of the velocity does not appear in the first two equations of (8) ('onshore problem') and a solution for the third equation ('longshore problem') is found once d and u are known. Characteristic directions in the (x, t) plane for the system (8) are given for the onshore and longshore problem respectively by:

$$\frac{dx}{dt} = u \pm c \quad \text{and} \quad \frac{dx}{dt} = u. \quad (9)$$

Equations (8) can be expressed in characteristic form:

$$\alpha_t + (u + c)\alpha_x = 0, \quad \beta_t + (u - c)\beta_x = 0, \quad \gamma_t + u\gamma_x = 0 \quad (10)$$

where the Riemann invariants are defined as follows:

$$\alpha = 2c + u + t, \quad \beta = 2c - u - t, \quad \gamma = v - \frac{1}{2}u^2 - d - x. \quad (11)$$

Carrier & Greenspan (1958) used a hodograph transformation (velocities are used as coordinates) in solving the onshore problem:

$$\lambda = \alpha - \beta = 2(u + t), \quad \sigma = \alpha + \beta = 4c, \quad u(\sigma, \lambda) = \phi_\sigma / \sigma. \quad (12)$$

The characteristic coordinates (σ, λ) are particularly effective for the moving shoreline since the $\sigma = 0$ contour maps the moving shoreline: $d = c = 0$. We note that σ is a space-like coordinate and λ is a time-like coordinate (e.g. see figure 2(d)). Combining the equations for the onshore problem gives a linear equation in ϕ :

$$(\sigma\phi_\sigma)_\sigma - \sigma\phi_{\lambda\lambda} = 0. \quad (13)$$

The simplest periodic solution is that of Carrier and Greenspan (1958):

$$\eta(\sigma, \lambda) = \frac{1}{4}\phi_\lambda - \frac{1}{2}u^2, \quad x(\sigma, \lambda) = \frac{1}{4}\phi_\lambda - \frac{1}{16}\sigma^2 - \frac{1}{2}u^2, \quad t(\sigma, \lambda) = \frac{1}{2}\lambda - u \quad (14)$$

where

$$\phi(\sigma, \lambda) = AJ_0(\sigma)\sin(\lambda), \quad (15)$$

J_0 is the Bessel function of the first kind. Not all the solutions can be transferred back to the (x, t) plane. This occurs for $A > 1$, i.e. when the Jacobian of the transformation vanishes.

Since the equation for the longshore problem is decoupled we can formally integrate this equation (10c):

$$\gamma_t + u\gamma_x = v_\lambda x_\sigma - v_\sigma x_\lambda + u(v_\sigma t_\lambda - v_\lambda t_\sigma) - (\eta_\lambda x_\sigma - \eta_\sigma x_\lambda) = 0. \quad (16)$$

After some algebra the equation can be rearranged to give a solution valid for any ϕ :

$$v = \frac{1}{4}\phi_\lambda + \text{const.} \quad (17)$$

and setting $\text{const.} = 0$ we get

$$v = \frac{1}{4}AJ_0(\sigma)\cos\lambda. \quad (18)$$

Details on some complicated multi-modal solutions are given in Brocchini & Peregrine (1996).

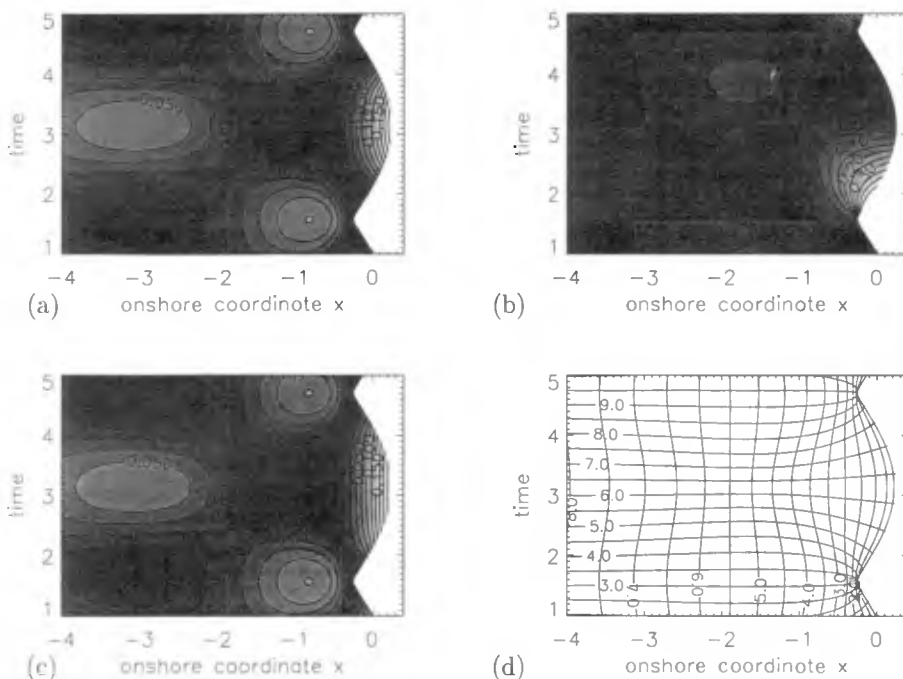


Figure 2: Contour lines for the dimensionless Carrier & Greenspan (1958) solution: (a) the free surface elevation η , (b) the onshore velocity u and (c) the longshore velocity, v , of the weakly three dimensional solution, for the case $A = 1$. Note that the dimensional longshore velocity is a factor ϵ smaller than the dimensionless quantity. (d) shows the (σ, λ) coordinates in the (x, t) plane. Lines of constant λ run from left to right whilst lines of constant σ run from top to bottom.

Contour plots of the free surface elevation and velocity components help visualise the behaviour of the solution, see figure 2. For all the flow properties shown in figure 2 there is a structured pattern of cells. Cells are confined by isolines of zero surface displacement. They also have different sizes and shapes. For all variables the largest values are reached within the most onshore cell. The particular characteristic of the contour structure for u is that it has an antisymmetric cellular pattern within the period. We can therefore state *a priori* that its mean value over a wave period is zero at each $x = \text{const.}$ position. This is not true for both the free surface elevation and the longshore velocity. They have similar cell patterns and for both of them, time averaging over a wave period results in a non-zero contribution.

Time averaging of flow properties is rather involved for the Carrier & Greenspan solution because of difficulties in relating (x, t) and (σ, λ) . Let

$$\langle G(x, t) \rangle = \frac{1}{T} \int_T G(x, t) dt = \frac{1}{T} \lim_{t \rightarrow T} H(x, t) \quad (19)$$

be the time average of $G(x, t) = g(\sigma, \lambda)$ where

$$H(x, t) = \int_0^t G(x, s) ds. \quad (20)$$

To evaluate this type of average which is based on time integration on curves of $x = \text{const}$ we solve the following partial differential equation along the curves $x = \text{const}$ in the (σ, λ) plane:

$$\frac{\partial H}{\partial t} = G(x, t) = g(\sigma, \lambda). \quad (21)$$

But for constant x we have

$$\frac{dH}{d\lambda} = \frac{\partial H}{\partial t} \left(\frac{\partial t}{\partial \sigma} \frac{d\sigma}{d\lambda} + \frac{\partial t}{\partial \lambda} \right). \quad (22)$$

Finally a set of two o.d.e.s for $\langle G(x, t) \rangle$ and σ is obtained:

$$\begin{aligned} \frac{d\langle G(x, t) \rangle}{d\lambda} &= \frac{1}{T} \lim_{t \rightarrow T} g(\sigma, \lambda) \left[\frac{\partial t}{\partial \lambda} \left(-\frac{\partial x}{\partial \lambda} / \frac{\partial x}{\partial \sigma} \right) + \frac{\partial t}{\partial \sigma} \right] \\ \frac{d\sigma}{d\lambda} &= -\frac{\partial x}{\partial \lambda} / \frac{\partial x}{\partial \sigma}. \end{aligned} \quad (23)$$

The first equation is the total derivative of $\langle G(x, t) \rangle$ in terms of $G(x, t)$ while the second states that we are integrating along curves where $x = \text{const}$. This method is used to compute the time average of velocities and mass fluxes.

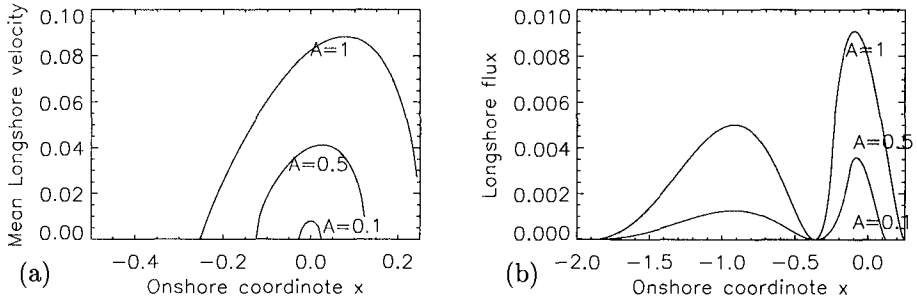


Figure 3: (a) Mean longshore velocity in the swash zone and (b) mean longshore mass flux inside and near to the swash zone for $A = 1$, $A = 0.5$ and $A = 0.1$.

A mean drift is associated with the swash zone width while the average is identically zero outside the swash zone (see figure 3(a)). This closely resembles what happens for the Stokes' drift for 2-D water waves where a mass flux is associated with the motion of the free surface, with zero mass flux below the trough level, and hence is expected. However a more interesting result relates to the mean longshore mass flux (see figure 3(b)). Correlation between solution for the total water depth and the longshore velocity is such that the mean longshore mass flux is non-zero for a wide range of the solution domain.

A more detailed analysis on swash zone mean flow properties is given in Brocchini & Peregrine (1995; 1996).

The model equations for integrated flow properties

In order to average flow properties in the swash zone and to deal with the moving free surface we investigate a model with a boundary chosen at the lower limit of the swash zone.

The swash zone limits are x_l , x_h , the seaward (lower) and shoreward (higher) limits of the swash zone respectively. The seaward boundary may for example be, the lowest limit x_l of the moving shoreline in a group of waves. Integral properties of the water shoreward of that point are considered.

We integrate the basic flow equations across the swash zone. In the following we adopt Einstein's summation convention and use Greek suffices for the two dimensional horizontal flow properties. The general set of equations (5) and the related energy equation are put in conservation form:

$$\begin{aligned}
 d_t + (ud)_x + (vd)_y &= 0 \\
 (ud)_t + \left(u^2 d + \frac{1}{2} d^2\right)_x + (uvd)_y + d + \tau_1 &= 0 \\
 (vd)_t + (uvd)_x + \left(v^2 d + \frac{1}{2} d^2\right)_y + \tau_2 &= 0 \\
 \frac{1}{2} \left[d(u^2 + v^2) + d^2\right]_t + \left(\frac{1}{2} u^3 d + \frac{1}{2} v^2 u d + u d^2\right)_x \\
 + \left(\frac{1}{2} v^3 d + \frac{1}{2} u^2 v d + v d^2\right)_y + u(d + \tau_1) + v \tau_2 &= 0.
 \end{aligned} \tag{24}$$

where a dimensionless bed friction $\tau = (\tau_1, \tau_2)$ is included.

We also assume the wave motion as occurring at a 'fast time scale' t while the swash zone boundaries x_h (run-up) and x_l (run-down) are supposed to vary (if varying in time) on a 'slow time scale' T such as $t = \epsilon T$. We do not use this notation since the separation of the two scales is generally quite clear. Equations (24) are integrated over the swash zone width, with constant x_l , to give:

$$\begin{aligned}
 \frac{\partial V}{\partial t} &= \int_{x_l}^{x_h} d_t \, dx = Q_1(t)|_{x_l} - \frac{\partial P_2}{\partial y} \\
 \frac{\partial P_1}{\partial t} &= \int_{x_l}^{x_h} (ud)_t \, dx = S_{11}(t)|_{x_l} - V(t) - \frac{\partial M_{12}}{\partial y} - \Upsilon_1(t) \\
 \frac{\partial P_2}{\partial t} &= \int_{x_l}^{x_h} (vd)_t \, dx = S_{12}(t)|_{x_l} - \frac{\partial M_{22}}{\partial y} - \Upsilon_2(t) \\
 \frac{\partial E}{\partial t} &= \int_{x_l}^{x_h} \left[\frac{1}{2}(u^2 + v^2)d + d^2\right]_t \, dx \\
 &= F(t)Q_1(t)|_{x_l} - P_1(t) - \frac{\partial}{\partial y} \int_{x_l}^{x_h} F Q_2 \, dx - \Gamma(t).
 \end{aligned} \tag{25}$$

This introduces a number of new flow properties and flow properties integrated over the swash zone. These are listed in table 1 where $\mathbf{u} = (u_1, u_2)$ is the horizontal velocity vector.

The set of equations (25) for the fully three-dimensional motion can be reduced by Ryrie's approximation to a simpler set of equations valid for weakly three-dimensional motion. Second order terms are neglected after the formal substitution (7) is made. Then, at the leading order the set of equations (25)

Name	Explicit expression	Flow property
Q_μ	$u_\mu d$	Local mass flow
$S_{\mu\nu}$	$u_\mu u_\nu d + \delta_{\mu\nu} \frac{1}{2} d^2$	Local momentum flux tensor
F	$\frac{1}{2} \mathbf{u}^2 + d$	Local energy density
V	$\int_{x_l}^{x_h} d \, dx$	Volume of water in swash zone
P_μ	$\int_{x_l}^{x_h} Q_\mu \, dx$	Momentum of water in swash zone
$M_{\mu\nu}$	$\int_{x_l}^{x_h} S_{\mu\nu} \, dx$	Integrated momentum flux tensor
E	$\int_{x_l}^{x_h} (\frac{1}{2} \mathbf{u}^2 d + d^2) \, dx$	Energy of water in swash zone
Υ_μ	$\int_{x_l}^{x_h} \tau_\mu \, dx$	Friction force in swash zone
Γ	$\int_{x_l}^{x_h} u_\mu \tau_\mu \, dx$	Work done by friction in swash zone

Table 1: Definition of the flow properties adopted in equations (25).

becomes:

$$\begin{aligned}
 \frac{\partial V}{\partial t} &= \int_{x_l}^{x_h} d_t \, dx = Q_1(t)|_{x_l} \\
 \frac{\partial P_1}{\partial t} &= \int_{x_l}^{x_h} (ud)_t \, dx = S_{11}(t)|_{x_l} - V(t) - \Upsilon_1(t) \\
 \frac{\partial P_2}{\partial t} &= \int_{x_l}^{x_h} (vd)_t \, dx = S_{12}(t)|_{x_l} + \frac{\partial M_{22}}{\partial t} - \Upsilon_2(t) \\
 \frac{\partial \bar{E}}{\partial t} &= \int_{x_l}^{x_h} (\frac{1}{2} u^2 d + d^2)_t \, dx = \bar{F}(t)Q_1(t)|_{x_l} - P_1(t) - \bar{\Gamma}(t).
 \end{aligned} \tag{26}$$

Table 2 lists definitions which, because of the approximation, differ from those of table 1. We distinguish newly defined flow properties by using overbars in equations (26) and table 2.

Name	Explicit expression	Flow property
$\bar{F}$	$\frac{1}{2} u_1^2 + d$	Local energy density
$\bar{E}$	$\int_{x_l}^{x_h} \frac{1}{2} (u_1^2 d + d^2) \, dx$	Energy of water in swash zone
$\bar{\Gamma}$	$\int_{x_l}^{x_h} u_1 \tau_1 \, dx$	Work done by friction in swash zone

Table 2: Definition of some flow properties adopted in equations (26).

In the idealized case of no dissipation ($\Upsilon_1 = \Upsilon_2 = \bar{\Gamma} = 0$) the set of partial differential equations (26) is such that it can be recursively solved for the integral

flow properties in the swash zone (i.e. V , P_1 , P_2 and $\bar{E}$) once the local flow properties Q_1 , S_{11} , S_{12} and $\bar{F}$ are known at the seaward boundary of the swash zone. The result of the equation for the water volume only depends on the known variable Q_1 . Once this equation is solved for V the result can be substituted into the second equation and so on.

We believe that equations (25) can be used as swash zone boundary conditions for wave resolving models. Use of integrated swash zone properties gives considerable numerical advantage, since adequate modelling of the details of the swash zone flows requires small time steps.

The swash zone boundary conditions for wave-averaged models

A natural extension to averaging the swash zone is to seek appropriate boundary conditions at the seaward swash zone limit for wave-averaged models of the incident motions (both long and short period motions). This requires definition of the contributions to the motion of the boundary $x = x_l$ from both long-period motions (e.g. low frequency waves, currents, etc..) and averaged short-period motions. The problem here is that there is no definitive model for the waves approaching the swash zone, especially for the most usual case where waves are already breaking. The following discussion develops swash-zone boundary conditions by assuming that appropriate short-wave properties are known.

To obtain dynamic equations for wave-averaged flow properties we start with the equations of motion and divide the basic flow properties u , v and d into long-period motions and the short-period wave contributions:

$$u = \langle u \rangle + \tilde{u}, \quad v = \langle v \rangle + \tilde{v}, \quad d = \langle d \rangle + \tilde{d}, \quad \langle \tilde{u} \rangle = \langle \tilde{v} \rangle = \langle \tilde{d} \rangle = 0. \quad (27)$$

Contributions to $\langle u \rangle$, $\langle v \rangle$ and $\langle d \rangle$ come from all motions whose typical time scale is significantly longer than the typical short-wave period. These can be for example either bound long waves associated with the set-down occurring under a group of short-period waves or free long waves caused by a time-varying break-point or any sort of current. On the other hand pure short-period contributions appear as correlations of wave-type terms (e.g. $\langle \tilde{u}\tilde{d} \rangle$, $\langle \tilde{u}\tilde{v} \rangle$, etc..).

By substituting in the shallow water equations (24) in this way for each flow variable and by phase averaging we obtain a set of equations for the long-period flow properties $\langle u \rangle$, $\langle v \rangle$ and $\langle d \rangle$:

$$\frac{\partial \langle d \rangle}{\partial t} + \frac{\partial}{\partial y} [\langle u \rangle \langle d \rangle + \langle \tilde{Q}_1 \rangle] + \frac{\partial}{\partial y} [\langle v \rangle \langle d \rangle + \langle \tilde{Q}_2 \rangle] = 0 \quad (28)$$

$$\frac{\partial}{\partial t} [\langle u \rangle \langle d \rangle + \langle \tilde{Q}_1 \rangle] + \frac{\partial}{\partial y} [\langle d \rangle (\langle u \rangle^2 + \langle \tilde{u}^2 \rangle + \frac{1}{2} \langle d \rangle) + 2 \langle u \rangle \langle \tilde{Q}_1 \rangle + \langle \tilde{S}_{11} \rangle] \quad (29)$$

$$+ \frac{\partial}{\partial y} [\langle d \rangle (\langle u \rangle \langle v \rangle + \langle \tilde{u}\tilde{v} \rangle) + \langle u \rangle \langle \tilde{Q}_2 \rangle + \langle v \rangle \langle \tilde{Q}_1 \rangle + \langle \tilde{S}_{12} \rangle] + \langle d \rangle = 0$$

$$\frac{\partial}{\partial t} [\langle v \rangle \langle d \rangle + \langle \tilde{Q}_2 \rangle] + \frac{\partial}{\partial y} [\langle d \rangle (\langle v \rangle^2 + \langle \tilde{v}^2 \rangle + \frac{1}{2} \langle d \rangle) + 2 \langle v \rangle \langle \tilde{Q}_2 \rangle + \langle \tilde{S}_{22} \rangle] \quad (30)$$

$$+ \frac{\partial}{\partial y} [\langle d \rangle (\langle u \rangle \langle v \rangle + \langle \tilde{u}\tilde{v} \rangle) + \langle v \rangle \langle \tilde{Q}_1 \rangle + \langle u \rangle \langle \tilde{Q}_2 \rangle + \langle \tilde{S}_{12} \rangle] = 0$$

where $\langle \tilde{Q}_\mu \rangle = \langle \tilde{u}_\mu \tilde{d} \rangle$ is the mass flow due to wave motion while the contribution of the momentum flux due to wave motion is given by the radiation stress term $\langle \tilde{S}_{\mu\nu} \rangle$.

The above set can be also written in matrix form in order to clarify the forcing action of the short-period contributions. The set of equations can, thus, be rewritten as:

$$AU_t + BU_x + CU_y + DU = \mathcal{E} \quad (31)$$

where

$$U = \begin{bmatrix} \langle d \rangle \\ \langle u \rangle \\ \langle v \rangle \end{bmatrix} \quad \text{and} \quad \mathcal{E} = - \begin{bmatrix} \langle \tilde{Q}_1 \rangle_x + \langle \tilde{Q}_2 \rangle_y \\ \langle \tilde{Q}_1 \rangle_t + \langle \tilde{S}_{11} \rangle_x + \langle \tilde{S}_{12} \rangle_y + \langle d \rangle \\ \langle \tilde{Q}_2 \rangle_t + \langle \tilde{S}_{12} \rangle_x + \langle \tilde{S}_{22} \rangle_y \end{bmatrix} \quad (32)$$

are respectively the column vector of the unknown variables and the purely short-wave forcing.

This set of equations is not 'closed' unless a particular wave theory for the short-period wave field is adopted to compute terms like the mass flow and the radiation stress. For instance, it is common practice to use linear wave theory while dissipation effects induced by wave-breaking are parameterized in the solution.

However linear wave theory is not a good representation for the wave field near the shore. The Carrier & Greenspan (1958) solutions and their extension described above can be more appropriate. To help crystallise ideas we note that one approach to this problem would be to extend the Carrier & Greenspan solution to $A > 1$ by computing the surf and swash zone behaviour for waves which have incoming Riemann invariants as for the Carrier & Greenspan solution with $A > 1$. The reflection, mass flow, radiation stress and the averaged swash zone properties, V , P and E could be tabulated as functions of the short-wave parameters. We assume that if a suitable model is found for the short-wave motion this can be described in terms of only an amplitude A a frequency ω and a wave number $\mathbf{k}$ once the mean depth and mean flow velocity are given.

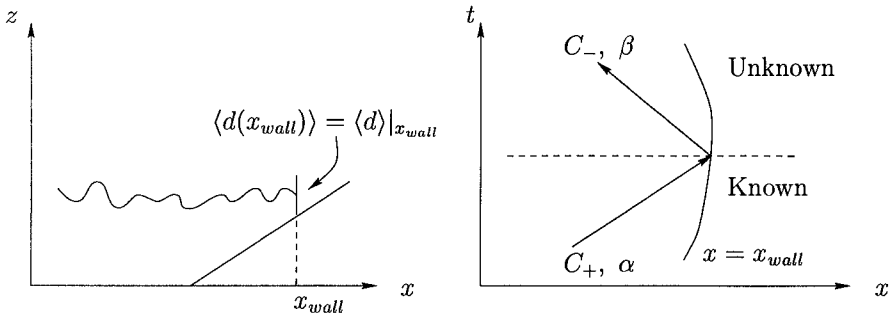


Figure 4: Schematic of: (a) the 'physical' (x, z) plane and (b) the 'characteristic' (x, t) plane for the rigid wall problem.

In order to obtain suitable boundary conditions for wave-averaged models we need to clarify what sort of information is available and what are the unknowns. To achieve this we first use a simple model of a moving boundary that is a rigid wall moving at velocity $U(t)\mathbf{i}$ which varies on a sufficiently slow time scale that its acceleration is unimportant (see figure 4). The flow is one dimensional and the medium is assumed uniform so that A, ω and $\mathbf{k}$ do not depend on (x, t) .

A simple exercise of bookkeeping shows what are known and unknown parameters for both long and short wave motions (see table 3 and figure 4).

Natural boundary conditions for this flow are:

$$\langle d \rangle|_{x_{wall}} = \text{const} , \quad \frac{dx_{wall}}{dt} = \langle u \rangle = U(t) \quad (33)$$

Consequently it is possible to explicitly compute the long-period motion variable β (i.e. the outgoing Riemann invariant) from the known incoming variable α

$$\alpha = 2\langle c \rangle + \langle u \rangle , \quad \beta = 2\langle c \rangle - \langle u \rangle , \quad \implies \quad \beta = \alpha - 2U. \quad (34)$$

The incoming waves of amplitude A_{in} , wavenumber $\mathbf{k}_{in}$ frequency ω_{in} may be perfectly reflected at the wall, but to determine the outgoing values of A , $\mathbf{k}$ and ω the local behaviour at the moving wall must be considered.

In a frame of reference moving with the wall, the waves have an unchanging intrinsic frequency σ which is readily found from the Doppler shift:

$$\omega_{in} = \sigma + U\mathbf{i} \cdot \mathbf{k}_{in}. \quad (35)$$

If the waves are perfectly reflected then the outward propagating wave has $A_{out} = A_{in}$ and $\mathbf{k}_{out} = -(\mathbf{k}_{in} \cdot \mathbf{i})\mathbf{i} + (\mathbf{k}_{in} \cdot \mathbf{j})\mathbf{j}$ giving:

$$\omega_{out} = \sigma + U\mathbf{i} \cdot \mathbf{k}_{out} = \sigma - U\mathbf{i} \cdot \mathbf{k}_{in} = \omega_{in} - 2U\mathbf{i} \cdot \mathbf{k}_{in}. \quad (36)$$

	Long-period	Short-period
Known	$\alpha = \alpha(\langle u \rangle, \langle d \rangle) , \quad \langle d \rangle _{x_l}$	$A_{in} , \quad \omega_{in} , \quad \mathbf{k}_{in}$
Unknown	$\beta = \beta(\langle u \rangle, \langle d \rangle) , \quad x_l(t)$	$A_{out} , \quad \omega_{out} , \quad \mathbf{k}_{out}$

Table 3: Known and unknown flow variables for the rigid wall problem.

In comparing the moving swash zone boundary with the above simple boundary, we note that it is necessary to model the swash zone boundary as a 'leaky wall' such that fluxes of mass and momentum are allowed across $x = x_{wall}$ which for the 'leaky' case may be identified with $x = x_l$. We also note that:

1. sufficient information must be available to determine the motion of the swash zone boundary, $x_l(t)$, as well as to determine β ;
2. as above, a local wave model is needed to determine the properties of the outgoing wave, if any.

We concentrate on the first of these, and look at integration of swash-zone equations similar to (25), or (26), to provide the necessary information. It is clear that in addition to the kinematic type of boundary conditions that occur on the rigid wall (equations 33) a third condition is necessary for the longshore flow in the swash zone.

There is now an important choice to be made: how much of the swash motion, if any, should be assigned to the long time scales? Consideration of the moving

rigid wall example leads us to consider all motion relative to the point $x = x_l(y, t)$ to be short wave motion and $x_l(y, t)$ to be 'driven' by the long-period motions. As a result, we consider all the quantities defined in tables 1 and 2 to be defined with velocities relative to x_l and also to be considered as known once incident short-wave parameters A_{in} , k_{in} and σ are known.

Inside the swash zone most the short-wave quantities averaged over the short wave motion, such as $\langle V \rangle$, $\langle P \rangle$ etc., are to be determined from a short-wave model of the swash. There are exceptions for the longshore current quantities P_2 , $M_{\mu 2}$ which are not entirely dependent on the local waves. When a bore, or flow which was near a bore, enters the swash zone it makes a large contribution to the longshore velocity, which often may only respond to bed friction on the longer time scale, e.g. see Ryrie (1983). This means that a decomposition of long-wave and short-wave contributions is also necessary within the swash zone.

We achieve this decomposition by assuming that swash motion is almost entirely assigned to short-wave contributions and that the only long-wave contribution, other than the motion of x_l , comes from parameterising the longshore drift due to wave-breaking by a longshore current velocity $W = W(y, t)$. In similar fashion to equation (27), which applies outside the swash zone, we separate short-wave from long-wave contribution for the flow properties inside the swash zone:

$$d = \hat{d}, \quad u = \frac{\partial x_l}{\partial t} + \hat{u}, \quad v = W + \hat{v}, \quad (37)$$

where a short-wave property inside the swash zone is defined as $\hat{G}$ rather than $\tilde{G}$ which pertains to short-wave contributions outside the swash zone. This decomposition permits the integrated terms depending on the longshore velocity to be divided between long and short waves:

$$\langle P_2 \rangle = \langle W \hat{V} \rangle + \langle \hat{P}_2 \rangle, \quad (38)$$

$$\langle M_{12} \rangle = \langle W \hat{P}_1 \rangle + \langle \hat{M}_{12} \rangle + \frac{\partial x_l}{\partial t} (\langle W \hat{V} \rangle + \langle \hat{P}_2 \rangle), \quad (39)$$

$$\langle M_{22} \rangle = \langle W^2 \hat{V} \rangle + 2 \langle W \hat{P}_2 \rangle + \langle \hat{M}_{22} \rangle. \quad (40)$$

All the integral properties which are to be considered as short-wave terms have a hat symbol. In principle it is possible to choose a different short-wave solution for the two regions separated by $x = x_l$ also on many sandy beaches there is a strong difference in the character of the bed in the swash zone compared with the bed just outside the swash zone.

In order to avoid greater complexity, the longshore variation of x_l has been assumed to be negligible in the above derivation. This is essentially a geometrical matter, and the above boundary conditions can be interpreted as being applicable when the x direction is normal to the mean lowest boundary of the swash.

To obtain explicit expression of the first boundary condition we consider the flow of mass into the swash zone relative to $x = x_l$. Average of equation (25) is:

$$\frac{\partial \langle V \rangle}{\partial t} + \frac{\partial \langle P_2 \rangle}{\partial y} = \langle Q_1 \rangle|_{x_l} \quad (41)$$

where now the right hand side contains the relative flow velocity $(u - \partial x_l / \partial t)$. The left hand side is also rewritten in terms of the local variables inside the

swash zone and then after using (38) for $\langle P_2 \rangle$, after integration across the swash zone width and averaging over the short waves we obtain

$$\frac{\partial \langle \hat{V} \rangle}{\partial t} + \frac{\partial \langle W \hat{V} \rangle}{\partial y} + \frac{\partial \langle \hat{P}_2 \rangle}{\partial y} = \left\langle \left(u - \frac{\partial x_l}{\partial t} \right) d \right\rangle = \langle u \rangle \langle d \rangle + \langle \tilde{u} \tilde{d} \rangle - \frac{\partial x_l}{\partial t} \langle d \rangle. \quad (42)$$

The first term on the left hand side of this equation is the rate of change of the total volume of water in the swash zone; the second and third terms are the change in volume due to the lateral variation of longshore currents respectively associated with the longshore velocity inside the swash zone W and with the wave contribution $\langle \hat{P}_2 \rangle$. The right hand side of (42) is the increase of water in the swash zone through its lower boundary. The right hand side of (42) is evaluated at $x = x_l$ using outer variables.

A similar derivation for the average balance of onshore momentum in the swash zone gives

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\langle \hat{P}_1 \rangle + \frac{\partial x_l}{\partial t} \langle \hat{V} \rangle \right] + \frac{\partial \langle W \hat{P}_1 \rangle}{\partial y} + \frac{\partial \langle \hat{M}_{12} \rangle}{\partial y} \\ & + \frac{\partial x_l}{\partial t} \left[\frac{\partial \langle W \hat{V} \rangle}{\partial y} + \frac{\partial \langle \hat{P}_2 \rangle}{\partial y} \right] + g\alpha \langle \hat{V} \rangle + \langle \Upsilon_1 \rangle = \left\langle \left(u - \frac{\partial x_l}{\partial t} \right)^2 d + \frac{1}{2} g d^2 \right\rangle = \quad (43) \\ & \left(\langle u \rangle - \frac{\partial x_l}{\partial t} \right)^2 \langle d \rangle + \frac{1}{2} g \langle d \rangle^2 + 2 \langle \tilde{u} \tilde{d} \rangle \left(\langle u \rangle - \frac{\partial x_l}{\partial t} \right) + \langle \tilde{u}^2 \rangle \langle d \rangle + \langle \tilde{u} \tilde{d} \rangle + \frac{1}{2} g \langle \tilde{d}^2 \rangle. \end{aligned}$$

In this equation dimensional expressions have been inserted to clarify the origin of terms. The first two terms on the left hand side are the rate of change of the mean momentum in the swash zone. The group of terms with y derivatives are the contribution from longshore velocity gradients which include both long period contributions (terms with W) and short period contributions (terms with $\langle \hat{P}_2 \rangle$ and $\langle \hat{M}_{12} \rangle$). The following terms are the action of gravity and friction on the water in the swash zone. The right hand side is the momentum transfer into the swash zone at $x = x_l$ where the mean flow velocity relative to the swash zone limit appears in both long and short period terms. Finally pure short period contributions appear. Conservation of longshore momentum similarly yields

$$\begin{aligned} & \frac{\partial \langle W \hat{V} \rangle}{\partial t} + \frac{\partial \langle \hat{P}_2 \rangle}{\partial t} + \frac{\partial \langle W^2 \hat{V} \rangle}{\partial y} + 2 \frac{\partial \langle W \hat{P}_2 \rangle}{\partial y} + \frac{\partial \langle \hat{M}_{22} \rangle}{\partial y} + \langle \Upsilon_2 \rangle = \left\langle \left(u - \frac{\partial x_l}{\partial t} \right) v d \right\rangle = \quad (44) \\ & \left(\langle u \rangle - \frac{\partial x_l}{\partial t} \right) \langle v \rangle \langle d \rangle + \langle \tilde{u} \tilde{v} \rangle \langle d \rangle + \langle \tilde{u} \tilde{d} \rangle \langle v \rangle + \langle \tilde{u} \tilde{v} \rangle \langle d \rangle + \langle \tilde{u} \tilde{d} \rangle \langle v \rangle + \langle \tilde{v} \tilde{d} \rangle \left(\langle u \rangle - \frac{\partial x_l}{\partial t} \right) + \langle \tilde{u} \tilde{v} \tilde{d} \rangle. \end{aligned}$$

The above three equations are boundary conditions for the long wave motion and x_l if the short wave properties on each side of $x = x_l$ have 'known' models. Whereas in the rigid wall model only the two boundary conditions are required, now we also need to evaluate the changes in W . In addition, the short-wave models may be chosen to permit the reflection of short waves. Note, the short-wave swash model must be used relative to $x = x_l$, hence the motion of x_l is seen as the long period contribution to the swash zone. A delicate matter concerning the different wave models needed on each side of x_l also needs attention.

The short-wave contribution

In this section we briefly analyse the relevance of the short-wave contributions with particular emphasis on those appearing in the equations (42), (43) and (44) which are used to define the boundary conditions at the lower limit of the swash zone. Computation of the contributions has been performed by integrating the analytical solution for $A \leq 1$ and by integrating the fully numerical solution when wave-breaking occurs, i.e. for $A > 1$.

For waves that break momentum and energy are transferred from the short-waves to the longshore currents hence terms depending on the longshore drift velocity W are to be taken into account as major contributions to the swash zone motions in equations (42), (43) and (44) (see figure 5). Note that the curves have been obtained by joining discrete points which represent the results of single computations. This explains the lack of smoothness in some of the curves.

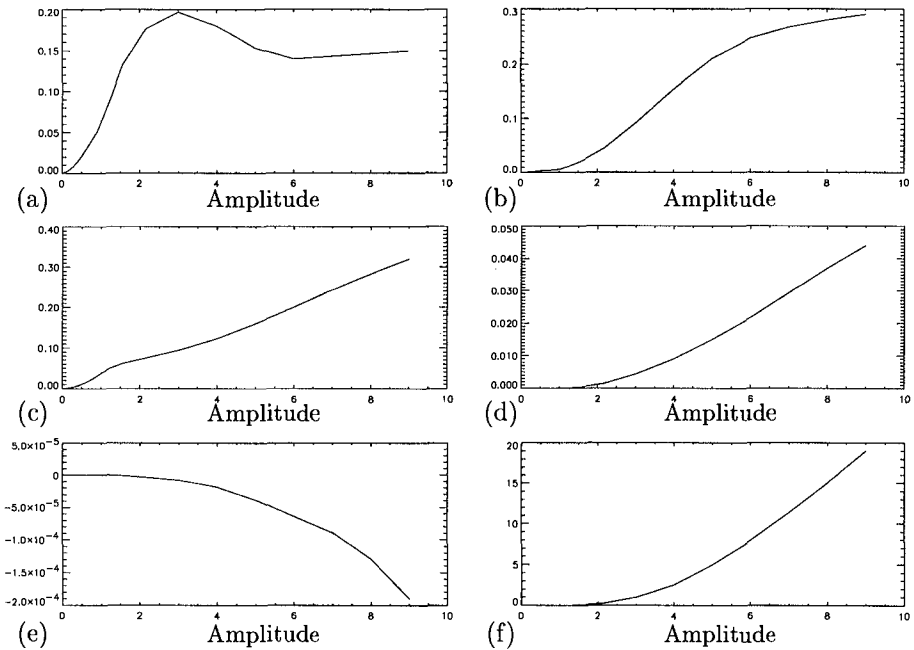


Figure 5: Short-wave contribution at the boundary x_l and inside the swash zone. From top left to bottom right they are the short-wave correlations (a) $\langle \hat{u}^2 \rangle$, (b) $\langle \hat{d}^2 \rangle$ the integral properties (c) $\langle \hat{V} \rangle$, and the rate of change of the three global integral properties (d) $\langle P_2 \rangle$, (e) $\langle M_{12} \rangle$ and (f) $\langle M_{22} \rangle$. Note that for $A \leq 1$ numerical integration of the analytical solution is shown while for $A > 1$ fully numerical solution of the NLSWE is represented.

The input of longshore momentum from the bores is balanced by bed friction but this is generally on the longer time scale. Although bores die out when they meet the shoreline and cause run-up, the water that forms the swash has acquired longshore momentum from the bore before it enters the swash zone. A

fully detailed consideration of this case also awaits further study; here we present the rate of change in time of the quantities $\langle \hat{V} \rangle$, $\langle \hat{P}_2 \rangle$ and $\langle \hat{M}_{22} \rangle$.

The figure presented here indicates that not all the terms that are introduced into the boundary conditions (42, 43 and 44) are necessary. Inspection of figure (5) reveals that the largest term to be retained in case of wave breaking is the rate of change of $\langle \hat{M}_{22} \rangle$. This is not surprising as $\langle \hat{M}_{22} \rangle$ is the only integral term proportional to W^2 (see equations 38) and hence it is more heavily influenced by the longshore drift induced by wave breaking. We can further comment that the non-diagonal contribution $\partial \langle \hat{M}_{12} \rangle / \partial t$ seems to be negligible in this particular case in which we have used $x_l = \text{const}$. However, in view of equation (39), here reported for ease of inspection,

$$\langle M_{12} \rangle = \langle W \hat{P}_1 \rangle + \langle \hat{M}_{12} \rangle + \frac{\partial x_l}{\partial t} (\langle W \hat{V} \rangle + \langle \hat{P}_2 \rangle) \quad (45)$$

we believe that this term may give larger contribution for a time-varying x_l as the rate of change of the boundary position explicitly contributes to this term.

Further investigation should help to clarify both which are the most significant terms when considered in the context of the equations and what are the effects of friction terms.

Acknowledgements

Support from E.U., D.G. XII, contracts : ERBCHBICT930678 and MAS2-CT92-0047 is gratefully acknowledged. We wish to thank Dr Gary Watson for many useful conversations.

References

- BROCCHINI, M. & PEREGRINE, D.H. (1995). Flow properties of the swash zone. *Proc. Coastal Dynamics '95*, ASCE, 221-232.
- BROCCHINI, M. & PEREGRINE, D.H. (1996). Integral flow properties of the swash zone and averaging. *J. of Fluid Mech.* **317**, 241-273.
- CARRIER, G.F. & GREENSPAN, H.P. (1958). Water waves of finite amplitude on a sloping beach. *J. of Fluid Mech.* **4**, 97-109.
- MEYER, R. E. & TAYLOR, A. D. (1972) Run-up on beaches. In *Waves on beaches*. (ed. R. E. Meyer). Academic Press, New York.
- RYRIE, S.C. (1983). Longshore motion generated on beaches by obliquely incident bores. *J. Fluid Mech.* **129**, 193-212.
- WATSON, G., BARNES, T.C.D. & PEREGRINE, D.H. (1994). The generation of low frequency waves by a single wave group incident on a beach. *Proc. 24th Internat. Conf. Coast. Engng.*, ASCE **1**, 776-790.