CHAPTER 37

PROBABILITY CHARACTERISTICS OF ZERO-CROSSING WAVE HEIGHT

by T. Ohta[1] and A. Kimura[2]

Abstract

This study deals with the probability distribution of zero-crossing wave height applying the definition of the zero-up(down)-cross method faithfully. In this study, gap between wave crest or trough and the envelope at the same location, which has been neglected in the ordinary studies is taken into account in the wave height definition. Its probability distribution is approximated with the Weibull distribution. The probability distribution of the zero-crossing wave height is, then, introduced theoretically together with the theory by modified Tayfun Method and the statistical properties of the gaps. The numerically simulated irregular wave height distributions agree well with the theoretical distribution.

Introduction

The Rayleigh distribution has been used as a probability distribution of zero-crossing wave height. Although this distribution agrees very well for the almost irregular wave height distributions, it was derived theoretically as a probability distribution of "wave amplitude" in the case of narrow band spectrum by Longuet-Higgins(1952). Therefore, the Rayleigh distribution is not the theoretical probability distribution for "wave height", even when the wave spectrum is narrow. Tayfun(1981,1983) tried to derive the probability distribution of the zero-crossing wave height on the basis of its definition faithfully. However, his distribution is considerably different from the Rayleigh distribution when the wave spectrum is wide. We know the Rayleigh distribution

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can be applied to the probability distribution of zero-crossing "wave height" sufficiently. The agreements of the theory and data are mainly around its mean. Very few studies examined the wave height distribution in the very large part, larger than twice the mean wave height, for example. In the design of structural durability and reliability, however, reliable probability in a range over 2.5 times of the mean wave height may become necessary. This study aims at deriving the probability distribution of zero-crossing "wave height", applying the basic definition for the zero-crossing wave height and considering small errors which is inevitably introduced in the ordinary theory.

Definition of Zero-Crossing Wave Height

In a spectrum theory, the envelope for irregular wave profile has been used instead of the amplitude at crest and trough of zero-crossing wave. Fig.1 shows irregular wave profile η, its envelope R and their enlargements around $t=t_2$. $\eta(t_1)$, $\eta(t_2)$ and $\eta(t_3)$ show maxima or minima of η. R_{m_1}, R_{m_2} and R_{m_3} are the simultaneous envelope amplitudes respectively. The zero-crossing wave height is defined, in principle, as a sum of consecutive maximum and minimum of η between two zero-up or down-crossing points. For example, the wave height of the first zero-down-crossing wave in fig.1 is described as $H_1 = \eta(t_1) + \eta(t_2)$. In the ordinary theory, maxima and minima of zero-crossing waves are approximated by the simultaneous envelope amplitudes, then H_1 is given by eq.1.

$$H_1 = R_{m_1} + R_{m_2} \tag{1}$$

When the wave spectrum is very narrow, the envelope changes gradually. Longuet-Higgins(1952) assumed that wave amplitudes are equal to the envelope amplitudes, and we have been applying the Rayleigh distribution as the probability distribution of zero-crossing "wave height", putting $H_1=2R_{m1}$. However, if the wave spectrum is wide, this assumption brings considerable errors. On the basis of eq.1, Tayfun(1981) derived probability distributions of zero-crossing wave height. Since the gaps between $\eta(t_j)$ and R_{m_j}, which are shown by δ_j in fig.1, are order of ν^2 (Tayfun, 1989, $\nu^2 = m_0 m_2 / m_1^2 - 1$, m_n : n-th order spectral moment), he neglected them. However the zero-crossing wave height should be defined, in principle as

$$H_1 = \left(R_{m_1} + R_{m_2}\right) - \left(\delta_1 + \delta_2\right) \tag{2}$$

To derive the probability distribution of zero-crossing wave height which basis on eq.2, it is necessary to make clear the additional probability distribution for R_{m_j} and δ_j. The envelope

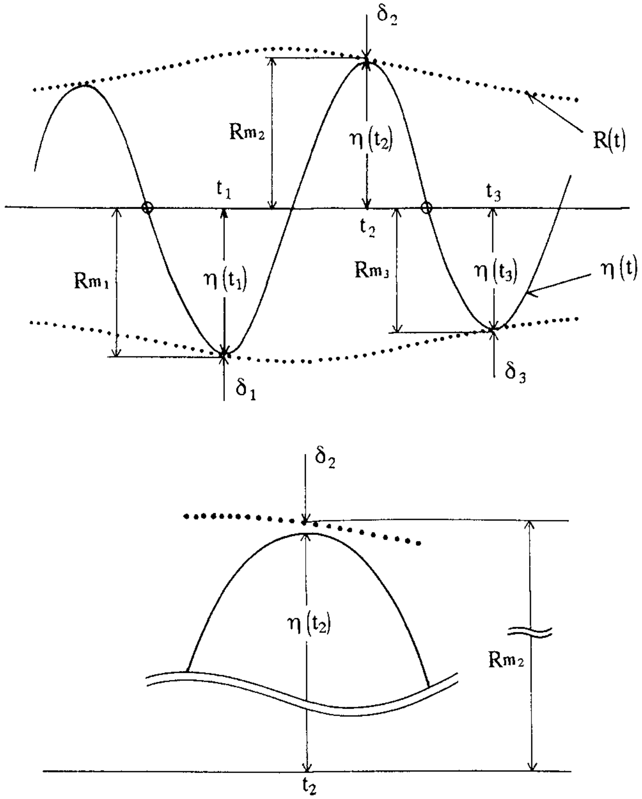


Fig. 1 Components of wave height

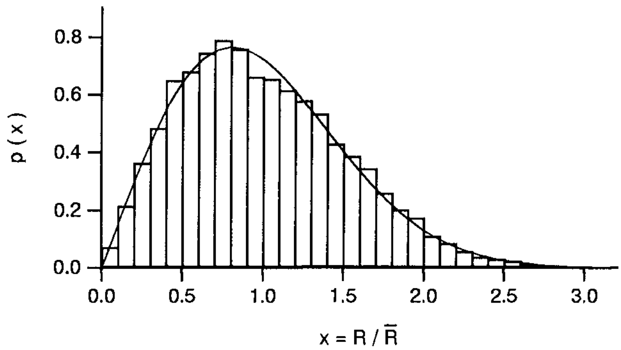


Fig. 2 Probability distribution of R (r = 5)

amplitudes R(t) follow the Rayleigh distribution , however, R_{m_j} is not statistically uniform samples from the population of envelope amplitude. In other words, because of the uneven interval between consecutive R_{m_j}, its distribution may be depart from the Rayleigh distribution. And there is no theoretical probability distribution for δ_j. In this study, the probability distribution for R_{m_j} and δ_j are investigated experimentally through numerical simulations.

Numerical Simulations

The irregular wave profiles η were simulated by FFT method (8192 points, Δt=0.05s). Next wave spectrum S(f) with variable shape factor r (r=4,5,6,7,8,9,10,15,20) is used in the simulations.

$$S(f)=\left(\frac{f}{f_p}\right)^{-r}\exp\left[\frac{r}{4}\left\{1-\left(\frac{f}{f_p}\right)^{-4}\right\}\right] \tag{3}$$

where, f_p is a peak frequency of S(f). The envelope R(t) was calculated by the following equation.

$$R(t)=\sqrt{\eta^2(t)+\hat{\eta}^2(t)} \tag{4}$$

where, $\hat{\eta}$ is the Hilbelt transformation of η. The histgram in fig.2 shows the frequency distribution of R(t). The theoretical probability distribution for R(t) is the Rayleigh distribution, however, we attempted to apply the Weibull distribution as a more general distribution (eq.5).

$$p(x)=\frac{\alpha}{2\gamma}x^{\alpha-1}\exp\left(-\frac{x^\alpha}{2\gamma}\right) \tag{5}$$

where, $x=R/\bar{R}$

$$\gamma=\frac{1}{2}\left[\Gamma\left\{(1+\alpha)/\alpha\right\}\right]^{-\alpha}$$

α is the shape parameter, γ is the scale parameter and Γ is the Gamma function. In fig.2, the solid line shows the Weibull distribution whose shape parameter α=2.027 and scale parameter γ=0.639. In the case of α=2.0 and γ=0.637, it is the Rayleigh distribution. Therefore the probability distribution for R(t) agrees well with the Rayleigh distribution. The histgram in fig.3 illustrates the frequency distribution of R_{m_j}. In this case, the spectral shape parameter r is 5. Applying the Weibull distribution to this frequency distribution, we obtained a result that the shape parameter α_2=1.864 and the scale parameter γ_2=0.697. The solid line in fig.3 shows that Weibull distribution. In fig.4, the histgram

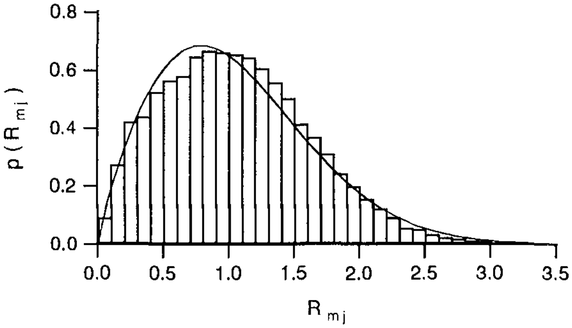


Fig. 3 Probability distribution of R_{mj} ($r=5$)

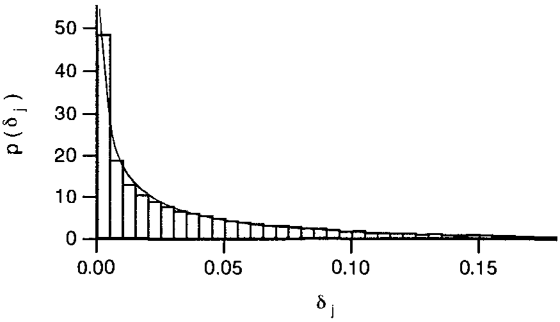


Fig. 4 Probability distribution of δ_j ($r=5$)

Tab. 1 Parameters for Weibull distribution

r	v	α_1	γ_1	α_2	γ_2	$\overline{\delta_j}$	$\overline{R_{m_j}}$
4	0.5571	0.6325	0.08649	1.829	0.8065	0.08469	1.154
5	0.4041	0.6394	0.06427	1.864	0.6975	0.05435	1.062
6	0.3247	0.6418	0.05081	1.898	0.6223	0.03802	0.9959
7	0.2768	0.6430	0.04274	1.939	0.5666	0.02891	0.9459
8	0.2444	0.6394	0.03767	1.975	0.5193	0.02356	0.9037
9	0.2205	0.6467	0.03262	1.921	0.4792	0.01955	0.8677
10	0.2018	0.6546	0.02900	1.941	0.4495	0.01701	0.8395
15	0.1514	0.6690	0.01988	1.941	0.3548	0.01043	0.7432
20	0.1256	0.6942	0.01462	1.988	0.2999	0.00775	0.6854

shows the frequency distribution of δ_j when the spectral shape parameter r is 5. Solid line shows the Weibull distribution with shape parameter α_1=0.639 and scale parameter γ_1=0.0643. The results for other spectral shape parameters, whose range

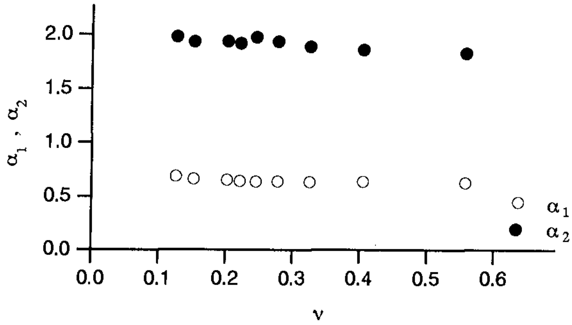


Fig. 5 Shape parameters of Weibull distribution

is from 4 to 20, are given in tab.1. Fig.5 shows shape parameters of the Weibull distribution. The open circles show the shape parameters of the Weibull distribution for δ_j and the closed circles show those for R_{m_j}. Horizontal axis shows the spectral band width parameter ν. As ν increases, the shape parameter of the Weibull distribution has tendency to decrease. The distributions obtained become a little flatter shape compared with the Rayleigh distribution. We concluded that uneven sampling from the population of R(t) causes this deviation from the Rayleigh distribution.

Probability Distribution of Wave Height

It is confirmed so far that the probability distributions for R_{m_j} and δ_j are well approximated by the Weibull distribution. To derive the probability distribution of wave height with the definition as eq.2, it is necessary to investigate the following points.

1. The joint probability distribution between consecutive envelope amplitudes R_{m_j} and $R_{m_{j+1}}$; $p(R_{m_j},R_{m_{j+1}})$
2. The joint probability distribution between δ_j and δ_{j+1} ; $p(\delta_j,\delta_{j+1})$
3. The joint probability distribution between $R_m=(R_{m_j}+R_{m_{j+1}})$ and $\delta=(\delta_j+\delta_{j+1})$; $p(R_m,\delta)$

First of all, the probability distribution for R_{m_j} and $R_{m_{j+1}}$ are also approximated by the Weibull distribution. The shape parameter is denoted by α_2. Since the interval between R_{m_j} and $R_{m_{j+1}}$ is about half of the mean period, R_{m_j} and $R_{m_{j+1}}$ must have correlation. With only above limited conditions, however, it is not possible to determine the joint probability distribution theoretically. In this study, we use the 2-dimensional Weibull distribution (Kimura,1981) as the joint probability distribution for the consecutive envelope amplitudes, $p(R_{m_j}, R_{m_{j+1}})$.

$$p\left(R_{m_j}, R_{m_{j+1}}\right) = \frac{\alpha_2^2}{4\left(\gamma_2^2-\rho^2\right)} R_{m_j}^{\alpha_2-1} R_{m_{j+1}}^{\alpha_2-1} \cdot \exp\left\{-\frac{\gamma_2}{2\left(\gamma_2^2-\rho^2\right)}\left(R_{m_j}^{\alpha_2}+R_{m_{j+1}}^{\alpha_2}\right)\right\} \cdot I_0\left(\frac{R_{m_j}^{\alpha_2/2} R_{m_{j+1}}^{\alpha_2/2}}{\gamma_2^2-\rho^2}\rho\right) \tag{6}$$

where, I_0 is the modified Bessel function of the first kind (0-th order), ρ is the correlation parameter between R_{m_j} and $R_{m_{j+1}}$.

$$\rho=\kappa\gamma_2 \tag{7}$$

and κ is given as following (Kimura and Ohta,1992).

$$\kappa=\sqrt{\mu_{13}^2+\mu_{14}^2}/m_0 \tag{8}$$

$$\mu_{13}=\int_{f_d}^{f_u} S(f)\cos\left\{2\pi(f-\bar{f})t_m\right\} df$$

$$\mu_{14}=\int_{f_d}^{f_u} S(f)\sin\left\{2\pi(f-\bar{f})t_m\right\} df$$

$$f_d=\left(-0.186/r+0.735\right)f_p \qquad (4\le r\le 20)$$

$$f_u=\left(1.61/r+1.62\right)f_p \qquad (4\le r\le 20)$$

$$\bar{f}=m_1/m_0 \quad , \quad t_m=\sqrt{m_0/m_2}$$

Because R_{m_j} is not normalized by its mean, the scale parameter γ_2 is given as follows.

$$\gamma_2=\frac{1}{2}\left[\Gamma\left\{(1+\alpha_2)/\alpha_2\right\}\right]^{-\alpha_2}\overline{R_{m_j}}^{\alpha_2} \tag{9}$$

Fig.6 shows that the 2-dimensional Weibull distribution agrees well with the simulated joint frequency distribution between R_{m_j} and $R_{m_{j+1}}$. Three cases of r=5,10 and 20 were shown in fig.6. Similar agreements are obtained in other cases (r=4,6,7,8,9,15).

Second, the probability distribution for δ_j and δ_{j+1} are also approximated by the Weibull distribution. The shape parameter is described by α_1. Open circles in fig.7 show the correlation coefficients between δ_j and δ_{j+1}, and fig.8 illustrates the distribution of δ_j and δ_{j+1} in the case r=5. Although calculated

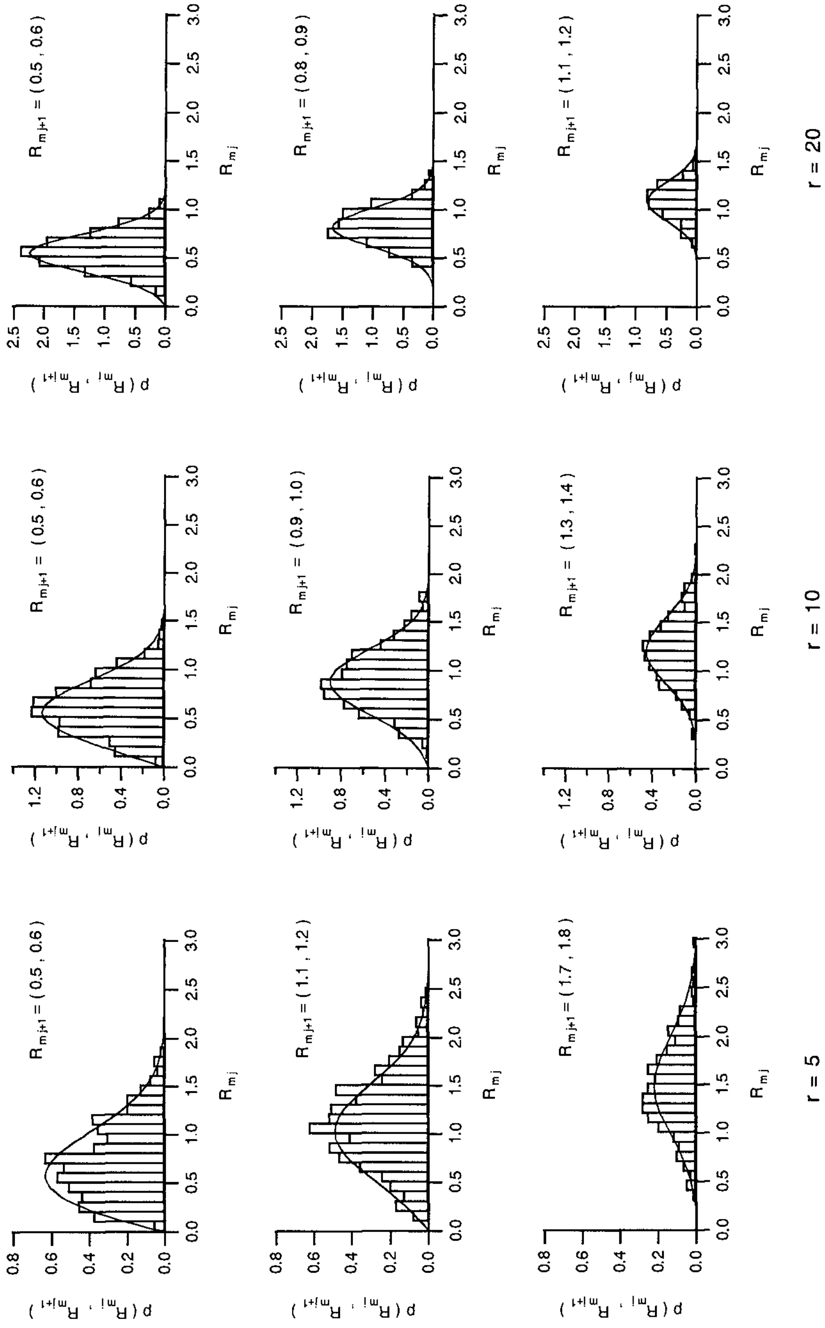


Fig. 6 Joint probability distribution of R_{mj} and R_{mj+1}

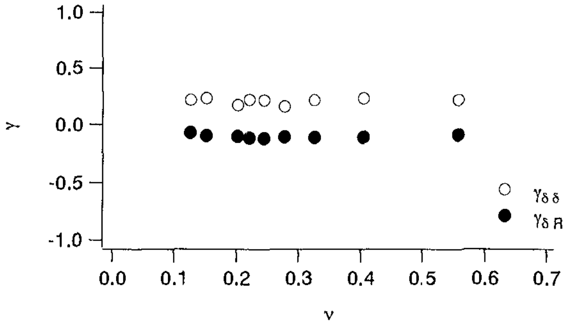


Fig. 7 Correlation coefficients

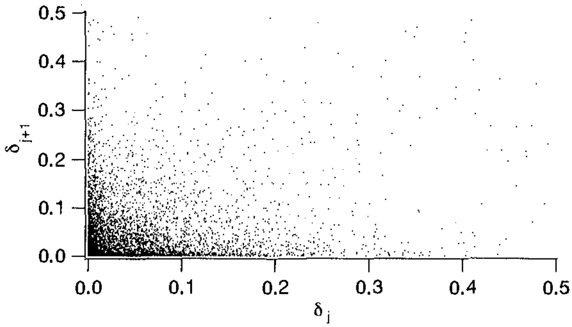


Fig. 8 Distribution of δ_j and δ_{j+1} (r = 5)

correlation coefficient is about 0.2 in this case, we can not see apparent correlation as shown in fig.8. Considering δ_j and δ_{j+1} are independent, then, we tried to apply the product of Weibull distribution as a joint probability distribution between δ_j and δ_{j+1}, $p(\delta_j, \delta_{j+1})$.

$$p\left(\delta_j , \delta_{j+1}\right) = \frac{\alpha_1^2}{4\gamma_1^2}\,\delta_j^{\alpha_1 - 1}\,\delta_{j+1}^{\alpha_1 - 1}\exp\left\{-\frac{1}{2\gamma_1}\left(\delta_j^{\alpha_1} + \delta_{j+1}^{\alpha_1}\right)\right\} \tag{10}$$

Since δ_j is not normalized by its mean, the scale parameter γ_1 is given as

$$\gamma_1 = \frac{1}{2}\left[\Gamma\left\{(1+\alpha_1)/\alpha_1\right\}\right]^{-\alpha_1}\bar{\delta}_j^{\,\alpha_1} \tag{11}$$

Fig.9 illustrates a comparison between the product of the Weibull distribution and the simulated joint frequency distribution of δ_j and δ_{j+1} in the cases of r=5,10 and 20.

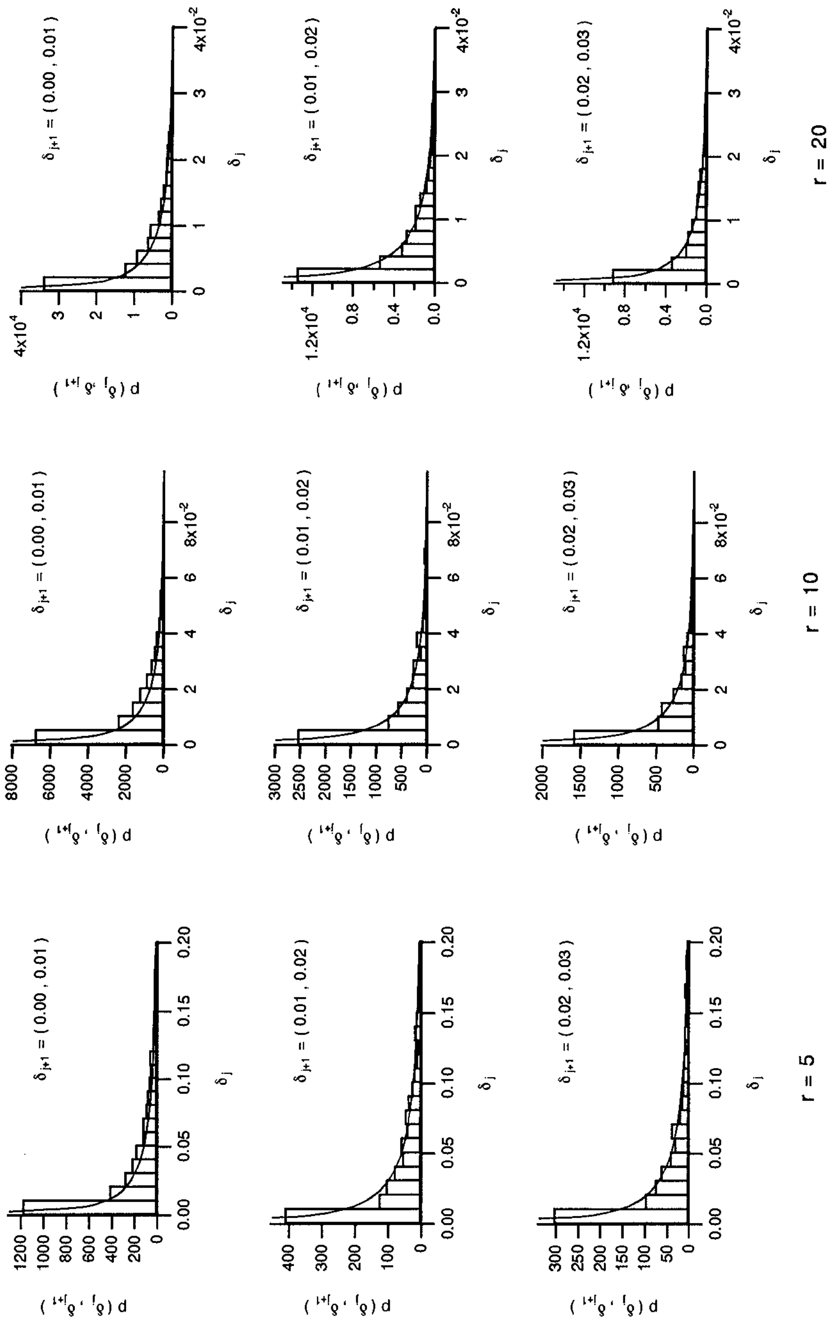


Fig. 9 Joint probability distribution of δ_j and δ_{j+1}

Third, the joint probability distribution between $R_m=(R_{m_j}+R_{m_{j+1}})$ and $\delta=(\delta_j+\delta_{j+1})$ is determined by using above results. Closed circles in fig.7 show the correlation coefficients between R_m and δ. Although the calculated values are about -0.1, we consider R_m and δ to be independent. Therefore, the joint probability distribution between R_m and δ is given by the product of the probability distribution for R_m and it for δ.

Using above results, we derive the probability distribution of zero-crossing wave height. First, the probability distribution for R_m, which is denoted by $p(R_m)$, is given as eq.(12).

$$p\left(R_m\right) = \int_0^{R_m} \frac{\alpha_2^2}{4\left(\gamma_2^2-\rho^2\right)} R_{m_j}^{\alpha_2-1}\left(R_m - R_{m_j}\right)^{\alpha_2-1}$$

$$\cdot \exp\left[-\frac{\gamma_2}{2\left(\gamma_2^2-\rho^2\right)}\left\{R_{m_j}^{\alpha_2}+\left(R_m-R_{m_j}\right)^{\alpha_2}\right\}\right]$$

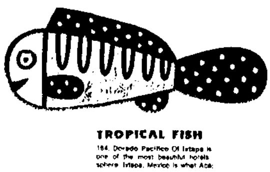
$$\cdot I_0\left[\frac{R_{m_j}^{\alpha_2/2}\left(R_m-R_{m_j}\right)^{\alpha_2/2}}{\gamma_2^2-\rho^2}\rho\right] dR_{m_j} \tag{12}$$

Second, the probability distribution for δ, which is described by $p(\delta)$, is given as following.

$$p(\delta) = \int_0^{\delta} \frac{\alpha_1^2}{4\gamma_1^2}\delta_j^{\alpha_1-1}\left(\delta-\delta_j\right)^{\alpha_1-1}$$

$$\cdot \exp\left[-\frac{1}{2\gamma_1}\left\{\delta_j^{\alpha_1}+\left(\delta-\delta_j\right)^{\alpha_1}\right\}\right] d\delta_j \tag{13}$$

The zero-crossing wave height is defined as $H=R_m-\delta$, and the probability distribution of H can be obtained by



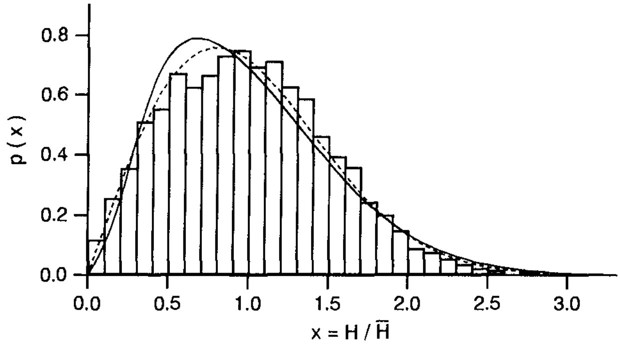


Fig.10 (a) Wave height distribution (r = 5)

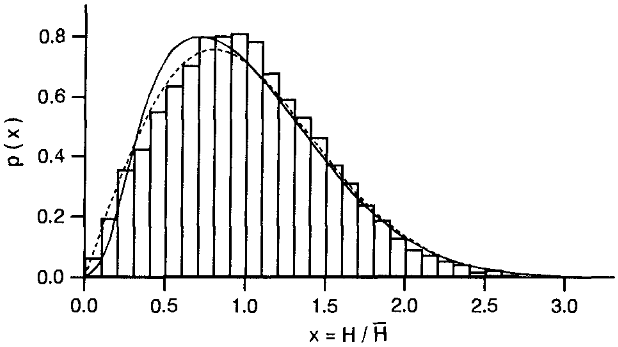


Fig.10 (b) Wave height distribution (r = 10)

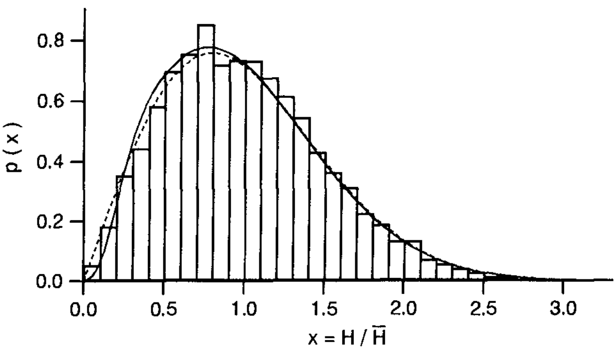


Fig.10 (c) Wave height distribution (r = 20)

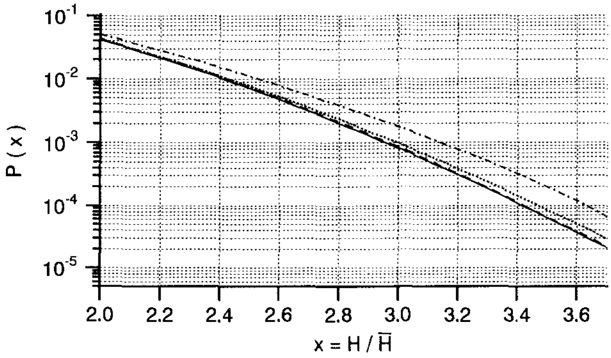


Fig.11 Exceedance probability
(chain line : r=5, dotted line : r=10, broken line : r=20, solid line : Rayleigh)

$$
\begin{aligned}
p(H) = & \int_H^\infty \int_0^{R_m} \frac{\alpha_2^2}{4\left(\gamma_2^2-\rho^2\right)} R_{m_j}^{\alpha_2-1}\left(R_m-R_{m_j}\right)^{\alpha_2-1} \\
& \cdot \exp\left[-\frac{\gamma_2}{2\left(\gamma_2^2-\rho^2\right)}\left\{R_{m_j}^{\alpha_2}+\left(R_m-R_{m_j}\right)^{\alpha_2}\right\}\right] \\
& \cdot I_0\left[\frac{R_{m_j}^{\alpha_2/2}\left(R_m-R_{m_j}\right)^{\alpha_2/2}}{\gamma_2^2-\rho^2}\rho\right] dR_{m_j} \\
& \cdot \int_0^{R_m-H} \frac{\alpha_1^2}{4\gamma_1^2}\delta_j^{\alpha_1-1}\left(R_m-H-\delta_j\right)^{\alpha_1-1} \\
& \cdot \exp\left[-\frac{1}{2\gamma_1}\left\{\delta_j^{\alpha_1}+\left(R_m-H-\delta_j\right)^{\alpha_1}\right\}\right] d\delta_j \quad dR_m
\end{aligned} \tag{14}
$$

As results of the numerical calculation of eq.(14), the probability distributions of zero-crossing wave height are obtained as shown in fig.10(a)-(c). (a) is the case when the spectral shape parameter r=5, (b) r=10 and (c) r=20. The solid line shows the present theory and the broken line shows the Rayleigh distribution. Fig.11 shows the exceedance probability of wave height. The chain line, dotted line and broken line show the distribution when r=5,10 and 20, and solid line shows the Rayleigh distribution respectively. We can see considerably larger probability than the Rayleigh distribution when r=5.

Conclusion

The probability distributions of zero-crossing wave height, when the gap between maximum(minimum) of wave profile and the simultaneous envelope amplitude is taken into account, were derived. As the result, larger probability of exceedance than the Rayleigh distribution was obtained in the range of larger wave height, when the spectrum is wide.

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