CHAPTER 203

LONGSHORE CURRENT INSTABILITIES: GROWTH TO FINITE AMPLITUDE.

Nicholas Dodd[1] and Edward B. Thornton[2]

Abstract.

The growth of shear instabilities in the longshore current to finite amplitude is studied. It is shown, by considering near-critical conditions and the self-interaction of the fastest growing mode (FGM), that the basic flow is supercritical and that, therefore, the disturbance may be expected to evolve to a finite size, with frequency equal to that predicted by linear stability theory, and final amplitude proportional to the linear growth rate. This is consistent with observation. The mean flow may therefore also be expected to evolve to a new form.

1 Introduction.

The observations of Oltman-Shay *et al.* (1989) (referred to hereinafter as OSHB89), which were made in the presence of a strong longshore current and on a barred beach, show clear evidence of periodic motions at infragravity periods (> 50 s), but with short wavelengths compared to edge waves of similar frequencies. The motions are not attributable to edge or any other form of surface gravity waves. These wave-like disturbances propagate alongshore in the same direction as the longshore current and with a speed proportional to the strength of that current (OSHB89), and, to a first order of approximation, they are non-dispersive (unlike edge waves). Bowen and Holman (1989) (referred to hereinafter as BH89) suggest that the observations are of a shear instability in the longshore current. The observed frequency (for a particular wavenumber k) is then considered to be that associated with the fastest growing unstable mode (if more than one such mode is present). A detailed analysis (Dodd *et al.*, 1992—referred to hereinafter

[1]Institute for Marine and Atmospheric Research, Utrecht University, Princetonplein 5, 3584 CC, Utrecht, The Netherlands.
[2]Oceanography Department, Naval Postgraduate School, Monterey, CA 93943, U.S.A.

as DOT92) has confirmed that these wave-like motions are indeed shear instabilities. DOT92 also found reasonably good agreement between the shape of the observed variance spectra (especially that for the cross-shore velocity component) and that calculated from the predicted growth rate. They also demonstrated that bottom friction may provide a damping mechanism, which might explain why such instabilties are not seen more often.

So far, only the linear, temporal instability problem has been tackled. However, according to linear theory, unstable modes grow exponentially in time and so that theory may be expected to break down sooner or later, depending on the growth rates of the disturbances, which in turn depend primarily on the longshore current offshore shear and the damping (DOT92 estimate that the fastest growing shear instabilities will grow by a factor e in 300 to 400 seconds). However, the observations of OSHB89 and DOT92 were over lengths of time far exceeding these comparatively short times: typically one to four hours. This implies that the observed oscillations (instabilities) are fully developed, in some sense, and that finite-amplitude effects are of importance. Nevertheless, using just the linear stability theory, and assuming that only the fastest growing unstable mode is important (for any k), DOT92 do notice good agreement between theory and observation (in frequency-wavenumber space). The conclusion seems to be that linearly unstable modes do grow and equilibrate (i.e., evolve to a finite amplitude and a steady form), with their final amplitudes being proportional to their linear growth rate, at least to a "first approximation".

In this paper we consider a weakly nonlinear theory. Unfortunately, this restricts us to so-called near-critical conditions (i.e., the longshore current shear is only just large enough to overcome the bottom friction and allow unstable modes to develop). Therefore, we shall also be restricted to only a small band of wavenumbers, centered on one *critical* value, k_c, whereas the linear analysis (DOT92) predicts a wider band of unstable wavenumbers whose shape and width correspond to that of the observed spectra. Notwithstanding this, it seems reasonable to expect that the analysis (based on the pioneering work of Stuart (1960) and others) will have physical relevance because it is centred on the wavenumber that will have the largest growth rate, and so and will at least reveal the qualitative long-time behaviour of the disturbances. The aim of this analysis is to confirm that the linear instabilities will evolve to a steady final form and to predict the amplitude of these forms.

In §2 the existing theory of the longshore current and of the shear instabilities of the longshore current are briefly reviewed, and the relation of each to the other is shown. In §3 finite-amplitude effects are introduced, and an evolution equation governing the long-time growth of the instabilities is derived. Results are shown in §4, and some conclusions are presented in §5.

2 Linear Theory.

We adopt a right-handed coordinate system in which y is the alongshore, x the cross-shore, and z the vertical coordinate. The water depth, $z = -h$, is assumed uniform alongshore: i.e., $h = h(x)$. The total horizontal velocity field is denoted by $\underline{u} = (u, v)$, and we write the depth and time averaged horizontal velocity field as $\underline{U}(x, y, t)$, and decompose $\underline{u}$ into

$$\underline{u} = \tilde{\underline{u}}(x, y, z, t) + \underline{U}(x, y, t), \tag{1}$$

where time averaging is performed over one period (T_g) of the incident waves. $\tilde{\underline{u}}$ represents deviations from the averaged velocity field; in particular, it represents the contribution to $\underline{u}$ from the incoming wind waves, and the definition (1) $\Rightarrow$ $\langle \int_{-h}^{\eta} \tilde{u}_i \, dz \rangle = 0$, where the triangular brackets denote the aforementioned time average. $\underline{U}$ will only depend on the horizontal coordinates and a long timescale, and will therefore contain both the mean flow (i.e. the longshore current) and any long-time perturbations in that flow (i.e. shear instabilities in the longshore current). Similarly, $\eta(x, y, t)$ is decomposed into $\eta = \tilde{\eta} + \langle \eta \rangle = \tilde{\eta}(x, y, t) + \zeta(x, y, t)$. Shear instabilities in the longshore current possess typical periods of $O(100$–$1000\,\mathrm{s})$, and so their period is an order of magnitude greater than T_g. These timescales are clearly consistent with the above decomposition.

The shallow water momentum equations may be written as

$$\rho(\zeta + h)[\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j}] = -\rho g(\zeta + h)\frac{\partial \zeta}{\partial x_i} - \frac{\partial S_{ij}}{\partial x_j} - \langle \tau_i \rangle, \tag{2}$$

for $i = 1, 2$ (see Mei, 1989), where $\langle \tau_i \rangle$ is the ith component of bottom friction, and S_{ij} are components of the radiation stress tensor. If, for the moment, we assume that all averaged quantities are time independent—and are therefore dependent on a timescale infinitely, rather than finitely longer than T_g—and make use of the assumed alongshore uniformity, then continuity $\Rightarrow$ $\underline{U} = (0, V(x))$ (see Mei, 1989). Therefore, the left side of (2) is identically zero, and the cross-shore momentum equation can be written as

$$0 = -g\frac{\partial \zeta_s}{\partial x} - \frac{1}{\rho(\zeta_s + h)} \frac{\partial S_{11}^{(0)}}{\partial x} \tag{3}$$

and the corresponding alongshore equation as

$$0 = -\frac{\partial S_{12}^{(0)}}{\partial x} - \langle \tau_2^{(0)} \rangle, \tag{4}$$

where $\zeta = \zeta_s(x)$ is the mean, wave-induced change in the water level (i.e., the set-up/set-down). The superscript '(0)' is used to show that these terms are only included at their leading order (i.e., only the time-independent current $V(x)$, and $\zeta_s(x)$ are included). $\zeta_s(x)$, can be found from (3); $V(x)$, is found from (4).

In the model of Thornton and Guza (1986)—referred to hereinafter as TG86—which we shall examine in detail, an analytical form is derived, by assuming that the incoming waves are narrow-banded in frequency but Rayleigh distributed in wave height, that the beach is plane with a slope s, and that the longshore current, $V \ll \langle|\underline{\tilde{u}}|\rangle$ (the weak current assumption; see Mei, 1989). The alongshore momentum equation (4) then becomes

$$0 = -\frac{\partial}{\partial x} S_{12}^{(1)} - \rho c_f \langle|\underline{\tilde{u}}|\rangle V, \tag{5}$$

and we can write

$$V(x) = -\frac{1}{\rho c_f \langle|\underline{\tilde{u}}|\rangle} \frac{\partial}{\partial x} S_{12}^{(1)} = \mathcal{K}_1 h^{9/10} \{1 + \mathcal{K}_2 h^{23/4}\}^{-6/5}, \tag{6}$$

where c_f is the bottom friction coefficient, and the constants $\mathcal{K}_1$ and $\mathcal{K}_2$ can be found in TG86. ζ_s is small compared with the depth for almost all x. Only very close to the idealised shoreline $x = 0$ is $\zeta_s > h$. This makes virtually no difference to the ensuing analysis, so we shall ignore ζ_s in calculations from now on, though we retain it in the derivations for the remainder of this section.

Following BH89, we may examine the linear stability of the mean, laminar, inviscid flow $\underline{U} = (0, V(x))$ by superimposing a disturbance on that flow. Thus, $\underline{U}$ is decomposed as

$$\underline{U} = (u'(x,y,t), V(x) + v'(x,y,t)). \tag{7}$$

Here, u' and v' represent perturbations in the components of the mean flow $(0, V(x))$; ζ is treated similarly: $\zeta = \zeta_s(x) + \zeta'(x,y,t)$. Thus, ζ now consists of ζ_s and an additional contribution due to the perturbations in the mean flow. Substituting (7) back into the momentum equations, we get

$$u'_t + Vu'_y + u'u'_x + v'u'_y = -g[\zeta_s + \zeta']_x - \frac{\{S_{11}^{(1)}\}_x + \langle\tau_1^{(1)}\rangle}{\rho(\zeta_s + \zeta' + h)}, \tag{8}$$

$$v'_t + Vv'_y + u'V_x + u'v'_x + v'v'_y = -g\zeta'_y - \frac{\{S_{12}^{(1)}\}_x + \langle\tau_2^{(1)}\rangle}{\rho(\zeta_s + \zeta' + h)}, \tag{9}$$

where the superscript '(1)' denotes the inclusion of the perturbations in these quantities. If we now subtract (3) and (4) from (8) and (9) respectively, we get

$$u'_t + Vu'_y + u'u'_x + v'u'_y = -g\zeta'_x - \frac{[\{S_{11}^{(1)}\}_x + \langle\tau_1^{(1)}\rangle]}{\rho(\zeta_s + \zeta' + h)} + \frac{\{S_{11}^{(0)}\}_x}{\rho(\zeta_s + h)} \tag{10}$$

$$v'_t + Vv'_y + u'V_x + u'v'_x + v'v'_y = -g\zeta'_y - \frac{[\{S_{12}^{(1)}\}_x + \langle\tau_2^{(1)}\rangle]}{\rho(\zeta_s + \zeta' + h)} + \frac{\{S_{12}^{(0)}\}_x + \langle\tau_2^{(0)}\rangle}{\rho(\zeta_s + h)} \tag{11}$$

These equations differ from (8) and (9) in that the mean motion has been subtracted out. Thus, (10) and (11) may be regarded as perturbation momentum equations.

Equations (10) and (11) are linearised by assuming that u' and v' are much smaller than the longshore current, and that, therefore, products of perturbed quantities are negligible. The quantities are further simplified by assuming that $S_{ij}^{(1)} \approx S_{ij}^{(0)}$. ζ_s may also be neglected when compared with h. This gives us

$$u'_t + Vu'_y = -g\zeta'_x - \frac{[\langle\tau_1^{(1)}\rangle - \langle\tau_1^{(0)}\rangle]}{\rho h} = -g\zeta'_x - \frac{\tilde{\tau}_1}{h}, \quad (12)$$

$$v'_t + Vv'_y + u'V_x = -g\zeta'_y - \frac{[\langle\tau_2^{(1)}\rangle - \langle\tau_2^{(0)}\rangle]}{\rho h} = -g\zeta'_y - \frac{\tilde{\tau}_2}{h}. \quad (13)$$

Finally, a rigid-lid approximation is imposed, which enables us to introduce a stream function Ψ, where $u' = -\Psi_y/h$ and $v' = \Psi_x/h$, so (12) and (13) can be combined into a single vorticity equation in Ψ. Assuming a harmonic dependence in t and y, $\Psi(x,y,t) = \mathrm{Re}\{\phi(x)e^{i(ky-\omega t)}\}$, then a stability equation analogous to the Rayleigh equation (see Drazin and Reid, 1981) is derived. The form of the stability equation depends on the form of the bottom friction terms on the right of (12) and (13). Here, we use the weak current assumption (for consistency with the TG86 model). In this case,

$$\frac{\tilde{\tau}_1}{h} = \frac{2\mu}{h}u' \text{ and } \frac{\tilde{\tau}_2}{h} = \frac{\mu}{h}v' \text{ where } \mu = c_d\langle|\underline{\tilde{u}}|\rangle = c_d\frac{2}{\pi}u_0(x), \quad (14)$$

and c_d is also a bottom friction coefficient, theoretically the same as c_f and therefore determined empirically, but for our purposes treated as a parameter; $\frac{2}{\pi}u_0$ is given by equation (16) of TG86. We write the resulting stability equation as

$$(V - i\mu/kh - c)\mathcal{L}\phi = h(V_x/h)_x\phi + (i/k)(\mu/h)_x\phi_x - (i\mu/kh)k^2\phi, \quad (15)$$

where $\mathcal{L}\phi = \phi_{xx} - (h_x/h)\phi_x - k^2\phi$, but we shall ignore the last two terms that appear on the right of (15) hereafter. The exclusion of these terms makes only a small difference in results (Dodd, 1992), but the simplified equation is desirable from the point of view of the ensuing weakly nonlinear analysis. Equation (15) is solved subject to the no-normal-flow boundary conditions $\phi(0) = \phi(\infty) = 0$. If it is assumed that k is real, then ω and $c = \omega/k$ are in general complex. If ω possesses a positive imaginary part, then the mode grows exponentially (as $t \to \infty$), and is therefore deemed unstable. Denoting $\Re(\omega) = \omega_r$ and $\Im(\omega) = \omega_i$, then ω_r defines the frequency, $c_r = \omega_r/k$ the phase speed, and ω_i the growth rate. If we ignore the bottom friction terms (i.e., put $c_d = 0$), then the stability equation of BH89 is arrived at.

Results of a linear stability analysis of (15) for the V profile of TG86 are shown in Fig. 1. For $c_d = 0$, the fastest growing point on the growth rate curve is situated at $k \approx .055\ \mathrm{m}^{-1}$ (wavelength $\lambda = 2\pi/k \approx 114$ m), and has an associated period (Fig. 1c) of about 360 s. We refer to this disturbance as the fastest growing mode (FGM) (this terminology can be slightly misleading since for the TG86

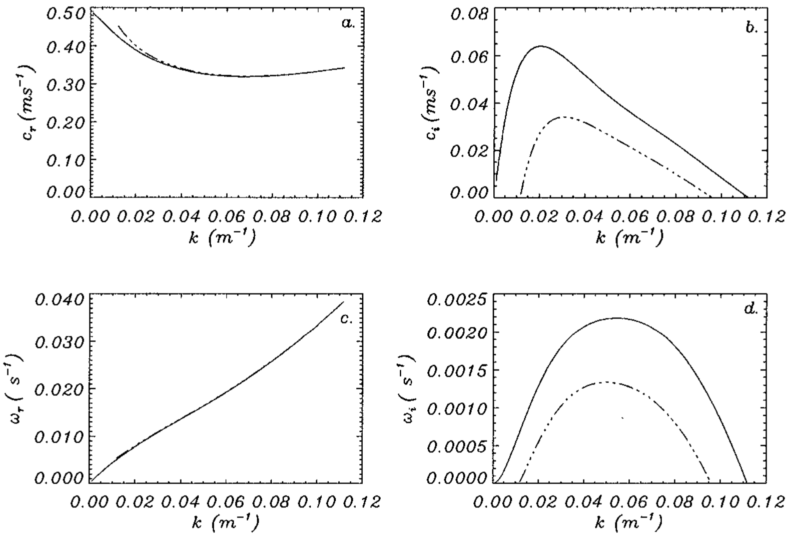

Figure 1: Linear stability for the TG86 model. (a) Real phase velocity, (b) imaginary part of phase velocity, (c) radial frequency, and (d) growth rate. Solid line: $c_d = 0$. Chained line: $c_d = .003$. ($c_f = .009$).

profile there is only one unstable mode for any k; the mode—eigenfunction—is continuously dependent upon k).

The inclusion of dissipation ($c_d = .003$) reduces growth rates, though not quite uniformly. The position of the FGM is now at $k \approx .045\ \mathrm{m}^{-1}$. Frequencies (or phase velocities) are barely affected by the inclusion of friction. Clearly, at the FGM $\partial\omega/\partial k = \partial\omega_r/\partial k$, and in its vicinity, $\partial\omega/\partial k \approx \partial\omega_r/\partial k$, and so the group velocity will be the true velocity of displacement. It can also be seen that, to a first approximation, the motions are non-dispersive. Of course, as c_d is increased the growth rates are reduced still further, and the width of the band of unstable wavenumbers diminishes. The dimensionless parameter c_d^{-1} plays a similar rôle in this problem to that of the Reynolds number in the viscous stability problem. If c_d^{-1} is decreased beyond a critical value, ${c_d}_c^{-1}$ (and therefore c_d increased), then all (small) disturbances will be completely damped and no instability will develop. At $c_d^{-1} = {c_d}_c^{-1}$ all wavelengths are damped except for $k = k_c$, which is a neutral disturbance. Therefore, $k = k_c$ and $c_d^{-1} = {c_d}_c^{-1}$ are referred to as *critical conditions*. In Fig.2, the neutral stability curve for (15) for the TG86 model is shown. If c_d^{-1} is increased slightly above its critical value, then we have so-called *near-critical conditions*

$$k = k_c \quad \text{and} \quad c_d^{-1} = {c_d}_c^{-1} + \Delta c_d^{-1}. \tag{16}$$

Under these conditions, a small set of wavelengths, centred on the critical wavenum-

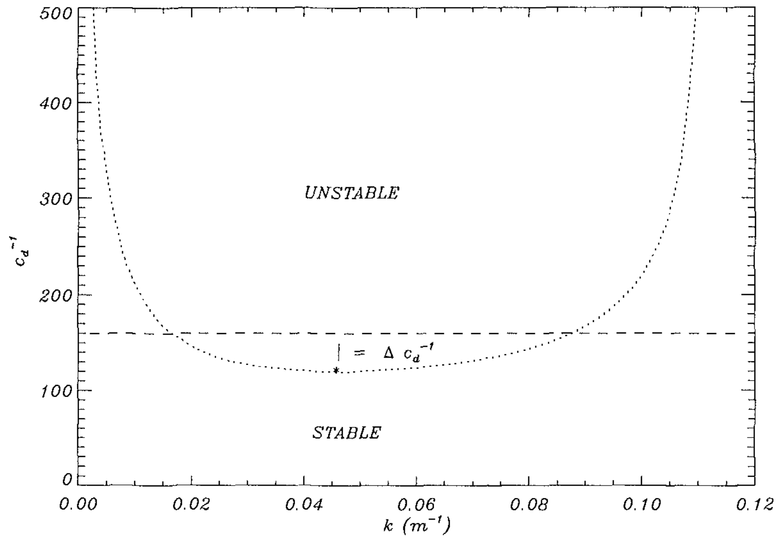

Figure 2: Neutral stability curve for the TG86 model. The asterisk shows the position of critical conditions $(k_c, c_{d_c}^{-1})$

ber $k = k_c$, becomes unstable, and propagates in the direction of the current at group velocity $c_g \approx c_{g_r} \approx c_r(k = k_c)$. It is at these near-critical conditions that the weakly nonlinear analysis of the next section applies.

Finally, it is important to note that c_f and c_d are actually one and the same (= .009, NSTS, Feb. 4th, 1980; see TG86). The value of $c_d = .003$ was chosen in Fig. 1 so that the effects of bottom friction could be seen without the instabilities being damped entirely (as they are for $c_d = .009$). Thus, it is also important to realise that decreasing (or increasing) the bottom friction coefficient not only reduces (increases) damping of the instabilities, but also increases (reduces) the longshore current shear (cf. (6))—if we were to be consistent and put $c_d = c_f = .003$ for the stability analysis in Fig. 1, we should expect all quantities to be three times as large. Of course, the case $c_d = c_f = 0$, which would predict an infinitely large V, would be physically meaningless.

From now on, we take c_f and c_d to be the same, (called c_d) but continue to treat the bottom friction coefficient as a parameter.

3 Weakly Nonlinear Theory.

If we dispense with the assumption of small amplitude, then we can define mean momentum equations appropriate for the finite-amplitude disturbance by averaging (8) and (9) over one alongshore wavelength λ:

$$\overline{u'u'_x} + \overline{v'u'_y} = -g\frac{\partial \zeta_s}{\partial x} - \overline{\frac{\{S_{11}\}_x + \langle \tau_1^{(1)} \rangle}{\rho(\zeta' + \zeta_s + h')}}, \tag{17}$$

$$\overline{u'v'_x} = -\overline{\frac{\{S_{12}\}_x + \langle \tau_2^{(1)} \rangle}{\rho(\zeta' + \zeta_s + h)}} \tag{18}$$

where the overbar denotes such an average. Amended perturbation equations may also be derived, by subtracting (17) and (18) from the full equations (8) and (9). If we again use the weak current assumption, we get

$$u'_t + Vu'_y + u'u'_x + v'u'_y - \overline{u'u'_x} - \overline{v'u'_y} = -g\zeta'_x - 2\frac{\mu}{h}u', \tag{19}$$

$$v'_t + Vv'_y + u'V_x + u'v'_x + v'v'_y - \overline{u'v'_x} = -g\zeta'_y - \frac{\mu}{h}v'. \tag{20}$$

The difference between these equations and those of the preceeding section is a different definition of what constititutes a mean.

By cross-differentiating (19) and (20) and then subtracting the second equation from the first, we get

$$\left\{\frac{\partial}{\partial t} + u'\frac{\partial}{\partial x} + (v' + V)\frac{\partial}{\partial y}\right\}\left(\frac{\Pi + V_x}{h}\right) - \frac{X_x}{h} = \frac{\mu}{h}\left\{\frac{\partial}{\partial y}\left(\frac{u'}{h}\right) - \frac{\partial}{\partial x}\left(\frac{v'}{h}\right)\right\}, \tag{21}$$

where $\Pi = v'_x - u'_y$ and $X = \overline{u'v'_x}$. Note that all the mean motion has been subtracted out of (21). This motion may straightforwardly be retained in an equation such as (21), but we prefer to keep the equations for mean and fluctuating motions separate. Recall that, for a prescribed longshore current profile ($V = V(x)$), linear theory predicts a stream function Ψ of the form $\Psi(x, y, t) = \mathrm{Re}\{\phi(x)e^{i(ky - \omega_r t)}e^{\omega_i t}\}$. It has been shown (DOT92) that the frequency so predicted (ω_r) is in good agreement with the observed frequency for a given wavenumber, k. Furthermore, linear theory consistently predicts that $\omega_r \gg \omega_i$, even in the absence of bottom friction, so that it may safely be assumed that Ψ grows appreciably only over times in excess of the linear period, $2\pi/\omega_r$. It is, therefore, natural to regard $e^{\omega_i t}$ as an amplitude modulation function. Clearly, however, Ψ will not continue growing without bound, and so $e^{\omega_r t}$ must be considered to be the "short-time" expression of a more general (complex) amplitude modulation function $a(t)$. Thus, for the nonlinear problem we have

$$\Psi(x, y, t) = \mathrm{Re}\{a(t)\phi(x)e^{i(ky - \omega_r t)}\}. \tag{22}$$

The initial time $t = t_0$, present through the constant $a_0 = a(t = t_0)$, provides a measure of how long the linear solution will be valid and the amplitude remain small, and therefore of how long linear theory remains applicable. By hypothesis, $a(t) \sim e^{\omega_i t}$ as $t \to -\infty$, and $a_0 \to 0$. The long-time asymptote of $a(t)$, $a_\infty = \lim_{t \to \infty} a(t)$ (if such a limit exists) is the amplitude to which the initially small disturbances will evolve.

The weakly nonlinear problem can now be formulated by introducing a small parameter ϵ, such that at near-critical conditions we have $c_d^{-1} = c_{dc}^{-1} + \epsilon^2\kappa$ (cf. equation (16)). We expect ω_i and Δc_d^{-1} to be closely related, and expanding ω_i in a Taylor series about c_{dc}^{-1} we find that $\omega_i(k = k_c, c_d^{-1} = c_{dc}^{-1} + \Delta c_d^{-1}) = \partial\omega_i/\partial c_d^{-1}(k = k_c, c_d^{-1} = c_{dc}^{-1})\epsilon^2\kappa + O(\epsilon^4)$. The amplitude a is dependent on a "slow" time coordinate $T = \epsilon^2 t$. We ignore the small but finite sidebands around $k = k_c$, and restrict ourselves to just the one, critical wavenumber; thus the disturbance will grow only by self-interaction.

Physically, we can think of the above conditions as being brought about by waves reaching the coast and building up the longshore current to the extent that the offshore shear in the current just overcomes the critical damping (c_{dc}^{-1}), so that instabilities with wavelength $2\pi/k_c$ develop. It is also implicitly assumed that the mean longshore current from which the instabilities begin to develop is steady.

The ansatz (22) is expanded to allow for self-interaction and for near-critical conditions:

$$\Psi = \sum_{n=1}^{\infty} \frac{1}{2}\{\psi_n(x,t)e^{ni(k_c y - \omega_r t)} + \hat{\psi}_n(x,t)e^{-ni(k_c y - \omega_r t)}\}\,, \tag{23}$$

where a circumflex denotes a complex conjugate. Note that (23) still retains the phase velocity c_r, associated with the linear theory at critical conditions. The functions ψ_n are expanded as $\psi_1 = \epsilon a\phi_1(x) + \epsilon^3 a^2\hat{a}\phi_{11}(x) + \ldots$, and $\psi_2 = \epsilon^2 a^2\phi_2(x) + \ldots$, *etc.*, where $\phi_1(0) = \phi_1(\infty) = \phi_{11}(0) = \phi_{11}(\infty) = \phi_2(0) = \phi_2(\infty) = 0$, *etc.*. Thus, the $O(\epsilon^0)$ mean flow ($V(x)$) becomes unstable, giving rise to $O(\epsilon)$ fundamental disturbances ($\psi_1 e^{i(k_c y - \omega_r t)}$ and c.c.). These, in turn, interact with each other, producing either 1st harmonic or mean components at $O(\epsilon^2)$. These components interact with components at $O(\epsilon)$ to produce additional fundamental components and components at 2nd harmonic. This, it turns out, is as far as we need to go, and all terms of $O(\epsilon^4)$ are ignored; for more details see Stuart (1960). Note that the mean terms are not included in (23), as they have already been subtracted from the momentum equations (see (19) and (20)); all these terms (up to $O(\epsilon^2)$) are present in the mean equations (17) and (18). For the TG86 model, (18) becomes

$$\rho h\overline{u'v'_x} = -\frac{\partial}{\partial x}\overline{S_{12}} - \rho(c_{dc} + \Delta c_d^{-1})\overline{\langle|\underline{\tilde{u}}|\rangle}V. \tag{24}$$

More generally, we can write

$$V_2(x,t) = V_1(x)\left(1 + \frac{\epsilon^2\kappa}{c_{dc}^{-1}}\right) + \epsilon^2|a(t)|^2 f(x) \tag{25}$$

where V_1 is the solution at $O(\epsilon^0)$ (i.e., the TG86 model, (6), at critical conditions). $\Psi(x,y,t)$ and (23) are substituted into (21), and the various harmonics are collected (see Dodd (1992) for the resulting equation). The mean equation (up to $O(\epsilon^2)$), is already given by (25). For the fundamental, the equation is (at $O(\epsilon)$) the linear problem of the previous section, (15) (but at *critical* conditions so that $\mu = \mu_c = c_{dc}\langle|\tilde{\underline{u}}|\rangle$ and $k = k_c$). At 1st harmonic ($O(\epsilon^2)$) there is an inhomogeneous equation—inhomogeneous because the fundamentals force the first harmonic by self-interaction. Finally, the fundamental also has a contribution at $O(\epsilon^3)$. It is this equation that requires a secularity (or compatibility) condition for its unique solution, and this condition yields the amplitude evolution equation. From this equation, and its c.c., it is easy to derive

$$\frac{d|a|^2}{dt} = 2\sigma|a|^2 - \ell|a|^4, \tag{26}$$

where ℓ is a real constant called the Landau constant, and it can be shown that $\sigma \approx \omega_i$. All these equations and expressions are given in Dodd (1992).

The equation (26) is readily integrated to give

$$|a|^2 = \frac{Ce^{2\omega_i t}}{\left(1 + \frac{\ell}{2\omega_i}Ce^{2\omega_i t}\right)} \quad \text{where} \quad C = \frac{2\omega_i|a_0|^2 e^{-2\omega_i t_0}}{2\omega_i - \ell|a_0|^2}, \tag{27}$$

and where σ has been replaced by ω_i. We are only concerned with cases where the longshore current is linearly unstable, i.e., $\omega_i > 0$. In this case, as $t \to \infty$, $|a|^2 \to 2\omega_i/\ell$, for any a_0, as long as $\ell > 0$. The flow may then be expected to evolve to a new, steady form (i.e., to equilibrate). When $\ell > 0$ the flow is said to be supercritical. Conversely, if $\ell < 0$, then any infinitesimal disturbance will become unbounded after a finite time. In this case, the flow is deemed subcritical, because this breakdown can occur *below* critical conditions, for a *finite* disturbance (see Drazin and Reid, 1981). Physically, we expect the flow to settle down to a new steady form after some time, and that this form will have a period $2\pi/\omega_r$ (i.e., $\ell > 0$). For a sufficiently small value of t the disturbance $a(t)$ amplifies like $\exp(\omega_i t)$, corresponding to linear instability.

4 Results.

For the TG86 model, $k_c \approx .0453$ m^{-1}, and $c_{d_c}^{-1} \approx .0083527$. Finding the value of ℓ requires a lot of computation: the linear (fundamental) problem must first be solved; then the problem for the first harmonic; and finally the the integrals needed to find ℓ must be calculated. At the above critical conditions, it was found that $\ell \approx 12300\ s^{-1}$. For $\Delta c_d^{-1} = 40$ (corresponding to a decrease in the bottom friction coefficient of about .0028 from critical conditions), $a_\infty = \sqrt{2\sigma/\ell} \approx .00052$. This figure is only meaningful when it multiplies the appropriate quantities. In Fig. 3 the resulting amplitudes of the velocities of with the fundamental and 1st harmonic disturbances are shown. The longshore current maximum is about .53 ms^{-1}, so the fundamental disturbances reach about .066 of this value. The velocities associated with the first harmonic are at $O(\epsilon^2)$, and are therefore smaller

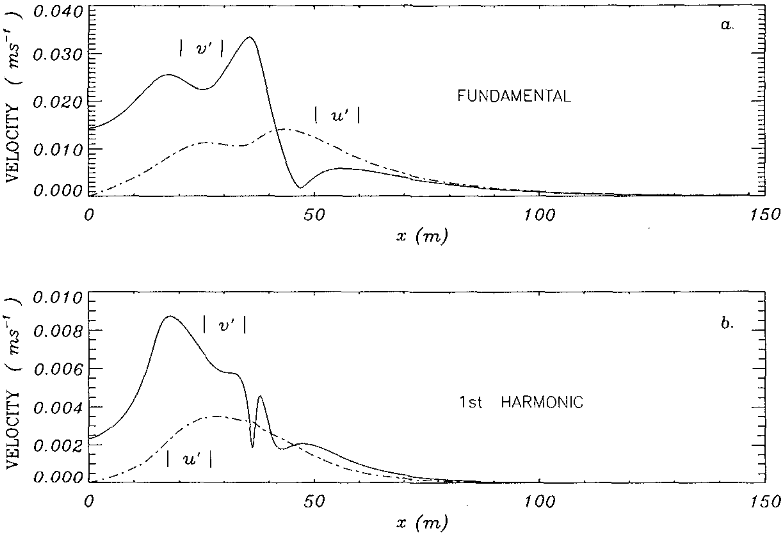

Figure 3: Moduli of complex perturbation velocities: $\Delta c_d^{-1} = 40$.

still. Also at this order is the term representing the deformation of the longshore current; it and $V(x)$ are shown in Fig. 4. The effect of the mean deformation on the TG86 profile can just be discerned in Fig. 4a. For a larger value of Δc_d^{-1} these amplitudes would all be larger ($|a| \propto \sqrt{\Delta c_d^{-1}}$), and vice versa. However, even with $\Delta c_d^{-1} = 40$, a large band of unstable wavenumbers is admitted, whose effects on each other are ignored in this analysis. Furthermore, the larger Δc_d^{-1} becomes, the further away from critical conditions we get, and the ordering assumptions behind the analysis breaks down. Nevertheless, for $\Delta c_d^{-1} = 200$, the same velocities are shown in Fig. 5. The perturbation velocities reach about 0.075 ms^{-1} (about one sixth of the current maximum), which is more typical of the observations of OSHB89, and DOT92.

The analysis of the previous section actually applies for any near-critical conditions; i.e., we can find a Landau constant for any point on the neutral curve shown in Fig. 2. We calculated its value for $k_c = k_1 = .0209\ \mathrm{m}^{-1}$ and $k_c = k_2 = .08085\ \mathrm{m}^{-1}$ (both of which share the same critical value of $c_d = c_{dc} = .00689$, $c_{dc}^{-1} \approx 145$). For k_1, $\ell_1 \approx 6900\ \mathrm{s}^{-1}$, and for k_2, $\ell_2 \approx 200,000\ \mathrm{s}^{-1}$. Although these values are not absolute indicators of the resulting perturbation amplitudes (because they depend on the normalisation of ϕ_1, which was not uniform for the numerical scheme used herein), they do tell us that (1) for resulting amplitudes for k_1 are bigger than for k_2, and (2) for all values of k tried so far, $\ell > 0$. Of course, these and all other such points on the neutral stability curve will only

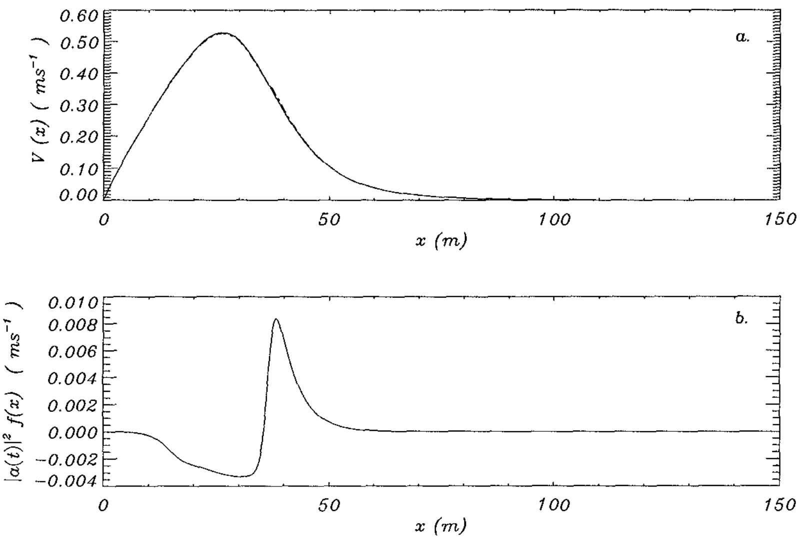

Figure 4: (a) $V(x)$ for TG86 (solid line), and modified mean current due to instability, (b) mean component due to instability: $\Delta c_d^{-1} = 40$.

exhibit instability *after* the overall critical condition (i.e., that at $k \approx .0453\ \mathrm{m}^{-1}$) has been surpassed, and will therefore be of lesser importance.

5 Conclusions.

It has been shown that, at least for the TG86 V profile and for the stability equation (15), the longshore current is supercritical and disturbances may be expected to evolve to a finite, steady amplitude ($\ell > 0$). This is in agreement with the observations of OSHB89 and DOT92. Only three positions on the neutral stability curve have been examined, but it seems very likely that $\ell > 0$ for all k between k_1 and k_2 at least. Most importantly, this is true for the overall critical condition at $k = .0453\ \mathrm{m}^{-1}$ (see Fig. 2).

At true near-critical conditions (represented here by $\Delta c_d^{-1} = 40$), the fundamental disturbances evolve to only about 1/15th of the V-maximum, whereas the observed disturbances were more typically one third of this value. This may be indicative of a number of things: (a) The V and h profiles measured at Duck, North Carolina (where the observations of OSHB89 were made) were significantly different from the TG86 model, and the barred beach profile and stronger current shear there may give rise to larger amplitudes (though without doing the weakly nonlinear analysis for these data it is hard to say). If this were so, it might provide

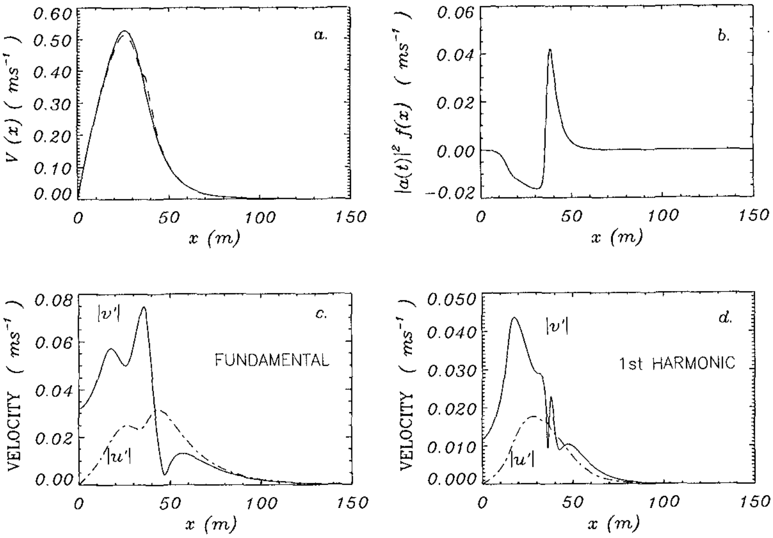

Figure 5: Linear stability for the TG86 model. (a) Original (solid line) and modified mean current, (b) mean component due to instability, (c) and (d) moduli of complex velocities due to instabilities: $\Delta c_d^{-1} = 200$.

a further, accompanying (and highly intriguing) mechanism for explaining why longshore currents on a plane beach do not 'apparently' become unstable: even if an instability were to develop, it will be restricted by nonlinearity to a small amplitude. Of course, linear growth rates for Duck are much larger than those for the TG86 model (Santa Barbara, California; DOT92), but the weakly nonlinear analysis only applies for near-critical conditions. The problem is then one of whether or not V continues to increase above critical conditions faster than do the instabilities that would presumably appear as soon as these conditions are first exceeded. In the time series shown by OSHB89 (Fig. 8 of that paper) it appears that the instabilities develop rather faster than V, as they seem to be established well before the end of the record. Whether this is true in general is not known. If it is not, then the weakly nonlinear approach would not be appropriate.

(b) Though near-critical conditions may be appropriate, the simple model presented here, in which only one wavenumber k_c is important, may be unrealistic, because k_c is only a 'first among equals' (Drazin and Reid, 1981), wavenumbers immediately to the sides of k_c having growth rates only infinitesimally smaller than that for the FGM. Thus there will be a group of wavenumbers propagating at group speed $c_g \approx \partial\omega_r/\partial k$ (see §2). In this case, the appropriate analysis is that of the Ginzburg-Landau equation. The stable solutions given by the theory presented here (stable only to *self-interaction*), may no longer be so for other wavelengths (there may be a side-band instability), and disturbances may evolve further to a larger amplitude. Work on this topic is now under way.

It is clear that, whatever their eventual amplitude, instabilities will in some measure modify the mean longshore current profile. The appropriate equation for determining this profile would therefore be an equation like (24). However, some confusion seems to have arisen concerning this new, mean profile: the profile produced by the instabilities is only the mean *component* of a fully developed *finite* disturbance; *the flow is stable to disturbances of wavenumber* k_c, *and it is no longer laminar, and the methods of linear stability can no longer be applied to it.* Therefore, any new, large shears apparent in the new profile should not be interpreted as leading to new instabilities.

This work provides justification for the growth rate scaling of DOT92. The final amplitude is indeed proportional to the growth rate, and the very large value of ℓ for k_2 indicates that the amplitudes will decrease rapidly with increasing k (ω_r).

Finally, two points should be mentioned: (1) The rigid-lid assumption does in fact break down in the weakly nonlinear analysis, but only at $O(\epsilon^3)$. This produces a small correction to the growth rate; (2) Bottom friction is not the only form of damping of the instabilities; turbulence induced by wave breaking will also have a direct effect, and should be examined in future studies.

References

Bowen, A.J. and R.A. Holman, (1989). Shear instabilities of the mean longshore current, 1.Theory. *J. Geophys. Res., 94*, 18023–18030.

Dodd, N., J. Oltman-Shay and E.B. Thornton, (1992). Shear instabilities in the longshore current: a comparison of observations and theory. *J. Phys. Oceanogr., 22*, 62–82.

Dodd, N. (1992). *Weakly-nonlinear stability of shear instabilities in the longshore current*, Institute for Marine and Atmospheric Research Utrecht, report (in preparation).

Drazin, P.G. and W.H. Reid, (1981). *Hydrodynamic stability*, Cambridge University Press, 527pp.

Oltman-Shay, J., P.A. Howd and W.A. Birkemeier, (1989). Shear instabilities of the mean longshore current, 2.Field observations. *J. Geophys. Res., 94*, 18031–18042.

Stuart, J.T., (1960). On the non-linear mechanics of wave disturbances in stable and unstable flows. Part 1. The basic behaviour of plane Poiseuille flow. *J. Fluid. Mech., 9*, 353–370.

Thornton, E.B. and R.T. Guza (1986). Surf zone longshore currents and random waves: field data and models. *J. Phys. Oceanogr., 16* 1165–1178.