CHAPTER SIXTY NINE

PROFILE ASYMMETRY OF SHOALING WAVES ON A MILD SLOPE

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Abstract

From limited experimental evidence, shoaling monochromatic waves seem to possess certain similarities. Specifically, the harmonic composition follows the trend of solitary or cnoidal waves, and the phase difference of harmonics vary with water depth monotonically. Based on the above observation a similarity model is constructed to study the asymmetric properties of shoaling waves. The implication of wave asymmetry on wave breaking is discussed.

Introduction

When the amplitude of a monochromatic wave becomes appreciable, the wave profile becomes asymmetric with respect to the horizontal axis, that is, a higher crest elevation than trough levels (Stokes, 1880). On shoaling water, in addition to the horizontal asymmetry, the wave form is obviously skewed with respect to the vertical axis with a steeper crest front than the crest rear face. Extensive studies have been conducted to study the wave asymmetry, especially for the case of breaking waves (McCowan, 1894; Munk, 1949; Iversen, 1952; Biesel, 1952; Ursell, 1953; Ippen and Kulin, 1955; Eagleson, 1957; Galvin, 1969; Goda, 1970; Hwang, 1982; Kjeldsen, 1981, 1983). Many studies of wave transformation at different stages of shoaling process also presented asymmetric information for nonbreaking waves (Iwagaki, 1968; Iwagaki and Sakai, 1972; Svendsen and Buhr-Hansen, 1978; Flick et al., 1981). The writer is especially interested in the approach of Flick et al. who performed harmonic analysis of the nonlinear shoaling waves. They found that both the amplitude and the phase spectra of a monochromatic shoaling wave vary with water depth in a deterministic fashion from the beach toe up to the breaking point: the amplitude spectrum resembles that of a cnoidal wave of the same height and the phase angle of each harmonic changes monotonically with decreasing water depth. Hwang (1982) applied harmonic analysis on breaking waves at different locations on sloping beaches, a very similar amplitude and phase spectral transformation was observed. It is speculated that certain similarity relationships of shoaling waves exist. The following presents an attempt to construct a similarity model for shoaling waves and to study their asymmetric properties.

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A Similarity Model

Flick et al. (1981) and Hwang (1982) studies wave profiles by Fourier decomposion

$$\eta = \sum_{n=1}^{\infty} a_n \cos(n\sigma t + \phi_n) \qquad (1)$$

where η is the surface fluctuation, a_n the n-th component, σ the angular frequency, t the time coordinate and ϕ_n the phase lag of the n-th component. They found that the variation of a_n with water depth follows closely the Fourier components of solitary or cnoidal waves (Figure 1a), and the phase of each oomponent changes monotonically with water depth (Figure 1b).

The amplitude spectrum of a cnoidal wave is given by (Cayley, 1895)

$$\frac{a_n}{H} = \frac{\frac{8}{3} k^2 h^3 n q^n}{1 - q^{2n}} \qquad (2)$$

where

$$q = \exp\left[-\pi K(1-m)/K(m)\right] \qquad (3)$$

and K(m) is the complete elliptic integral of the second kind. The cnoidal parameter m is related to the wave parameters by the following relationship

$$m = \frac{3\pi^2 Ur}{4K^2(m)} \qquad (4)$$

Ur is the Ursell number defined as

$$Ur = H/k^2h^3 \qquad (5)$$

where H is wave height, k the wave number and h the water depth.

The amplitude spectrum of a solitary wave is given by (Gradshteyn and Ryzhik, 1980)

$$\frac{a_n}{H} = \frac{4n}{3Ur}\left[\sinh \frac{n\pi}{(3Ur)^{1/2}}\right]^{-1} \qquad (6)$$

Eqs. (3) and (6) are plotted on Figure 1a along with experimental data. The difference between two theories is found insignificant in the range shown, and both theories agree with the trend of data reasonably well. In the following, we use Eq. (6) to calculate harmonic amplitudes due to its simplicity.

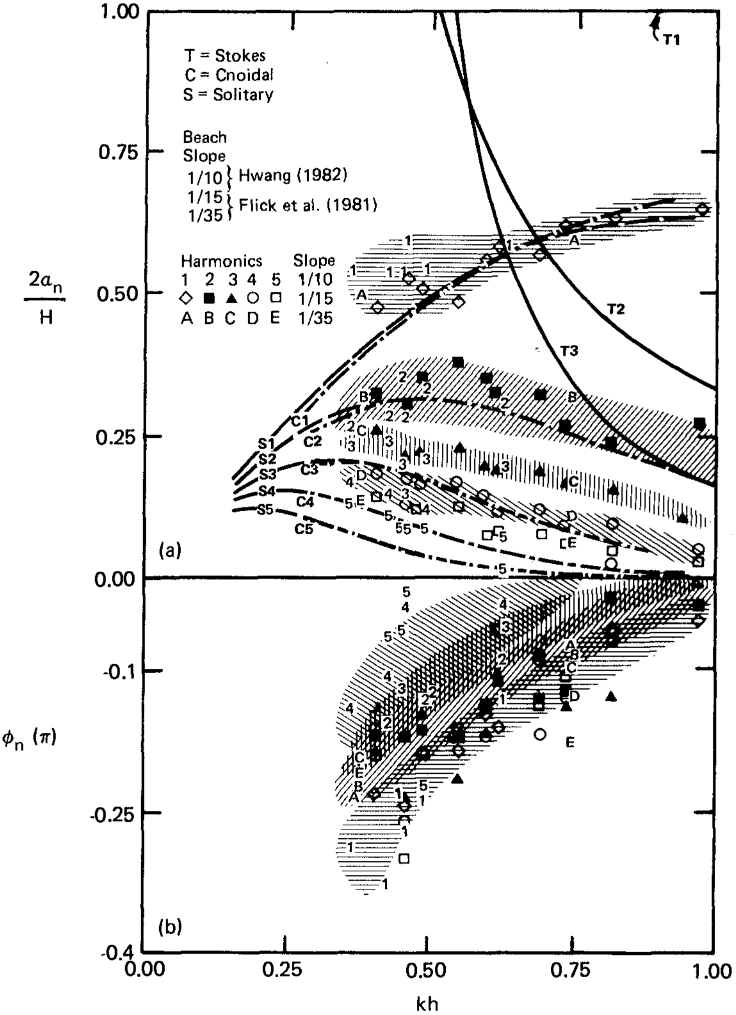


Figure 1 – Variation of (a) amplitude, (b) phase lag, of harmonics vs. kh.

Wave theories developed that considered the bottom slope modulation of the surface profile (Iwagaki, 1968; Chu and Mei, 1970; Iwagaki and Sakai, 1972; Guza and Davis, 1974) generally show poor agreement of the harmonic phase lags. The empirical relationship derived from the data in Figure 1b is used

$$\phi_n = -0.03 - 0.34(kh-1) \qquad (7)$$

where ϕ_n is expressed in π-radian. Alternatively, ϕ_n can be expressed in Ur (Figure 2)

$$\phi_n = -0.05 - 0.052\ \mathrm{Ur} \qquad (8)$$

Calculation shows little difference on the asymmetric properties (discussed in next section) using either equation. Eqs. (7) and (8) did not consider the variation of ϕ_n with bottom slope or the harmonic numbers. From the limited data collected, the effect of bottom slope on ϕ_n is very difficult to distinguish. Trying to establish a complete empirical relationship of ϕ_n from such a small data set is probably not justified. The simplification employed therefore only gives a qualitative description. Eqs. (3) or (6) and (7) or (8) define a wave profile.

Wave Profiles and Asymmetric Factors

Figure 3 shows a few examples of the normalized wave profiles defined by Eqs. (6) and (7). For each profile, fifteen harmonics were included (Eq. 6). Due to the superposition of harmonics, the wave form is asymmetric with respect to the horizontal axis. The wave form is also skewed due to different phase lags among harmonics. The degree of asymmetry (both horizontal and vertical) increases with decreasing depth (or increasing Ur).

There are many different parameters proposed to define the wave asymmetry (Iversen, 1952; Adeyemo, 1968; Iwagaki and Sakai, 1972; Kjeldsen, 1981, 1983, among others). The parameters ε, δ, λ, and μ proposed by Kjeldsen (Figure 4) are adopted for the following discussion. The parameter ε (the crest front steepness) is of special interest to this study since it represents the largest "local" steepness of the wave form and may dominate the wave breaking inception; ε is related to the "global" wave steepness H/L (where L is the wave length) by

$$\varepsilon = \eta'/L' = \frac{\eta'}{H}\,\frac{L}{L'}\,\frac{H}{L} \qquad (9)$$

The factor $\alpha_F = (\eta'/H)(L/L')$ can be interpreted as a "front steepness amplification factor," which defines the ratio between the crest front and the overall steepnesses. For a pure sinusoidal wave, $\alpha_F = 2$. Similarly, a "rear steepness amplification factor" can be defined as $\alpha_B = (\eta'/H)\ (L/L'')$. For nonlinear symmetric waves $\alpha_F = \alpha_B > 2$, and for asymmetric waves $\alpha_F > \alpha_B > 2$.

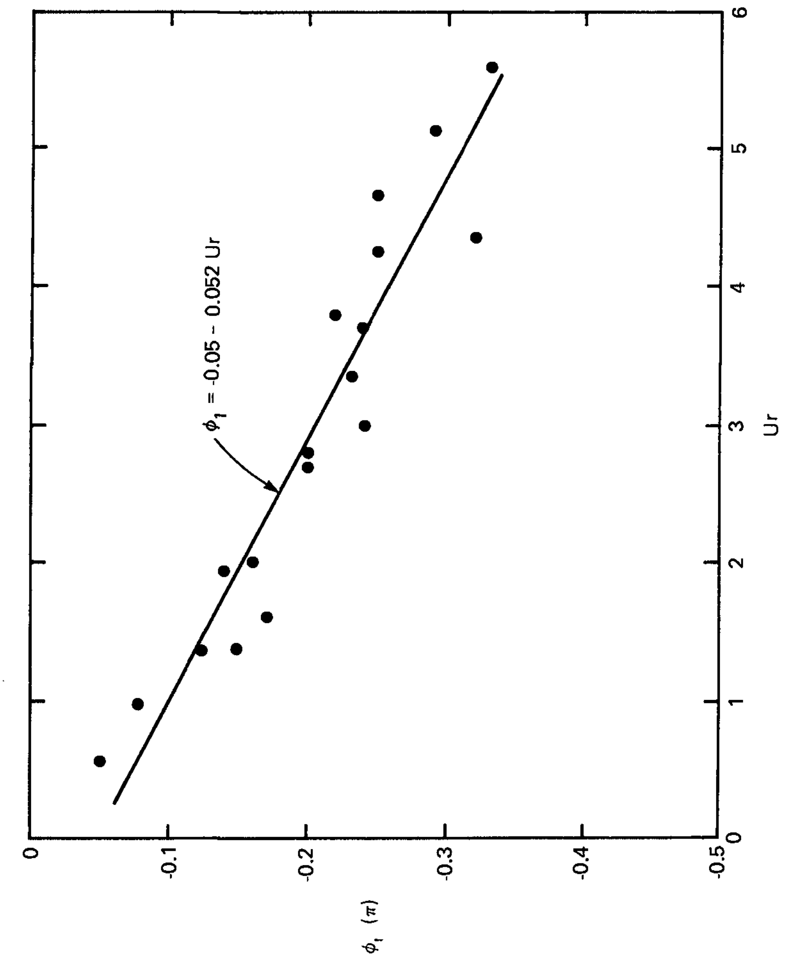


FIGURE 2 —Phase lag of the first harmonic vs. Ur .

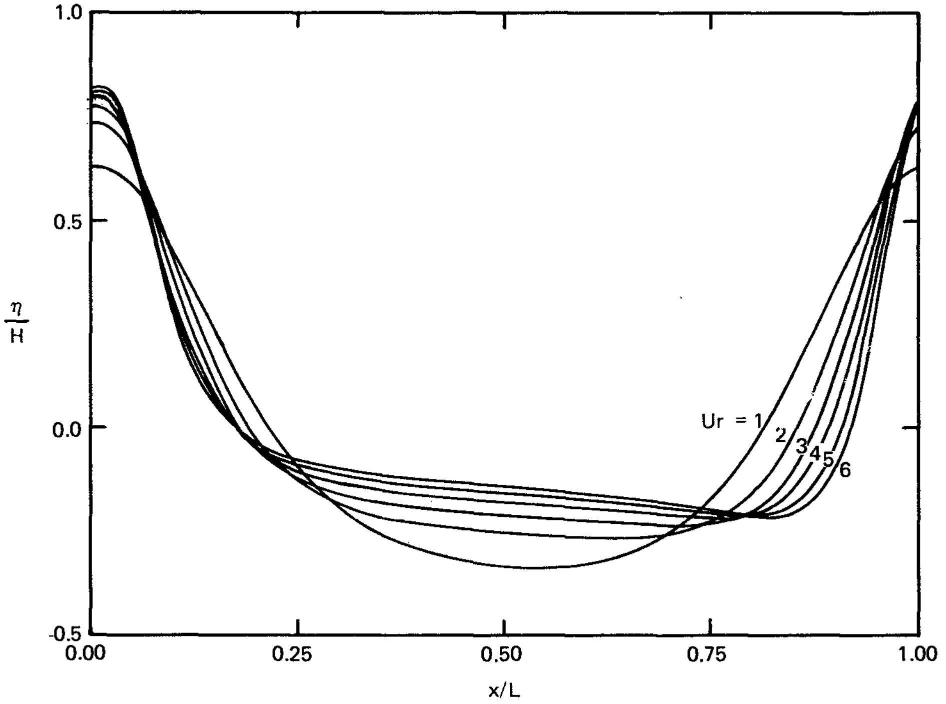
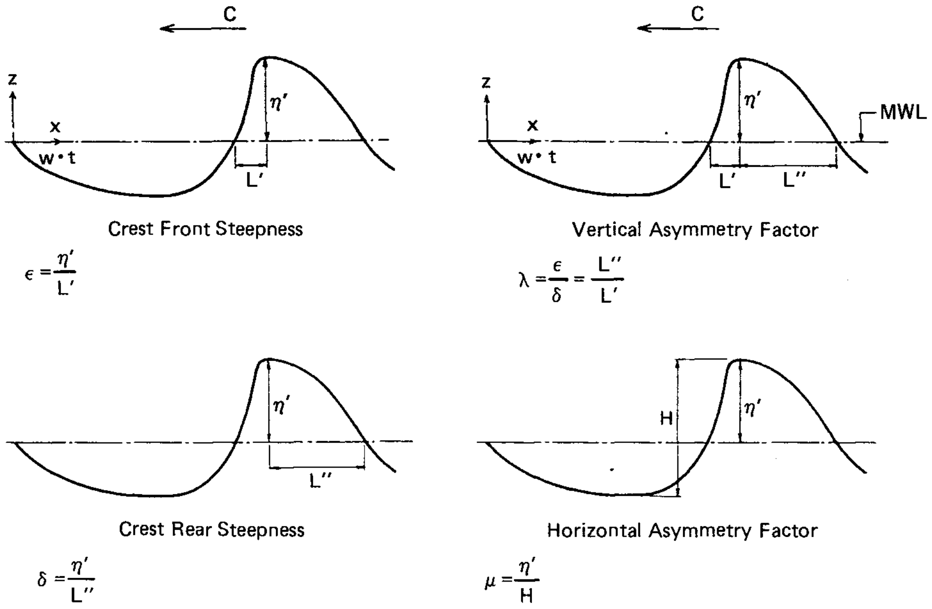


Figure 3 — Normalized wave profile. Waves propagate toward left.



Crest Front Steepness

$\epsilon = \frac{\eta'}{L'}$

Vertical Asymmetry Factor

$\lambda = \frac{\epsilon}{\delta} = \frac{L''}{L'}$

Crest Rear Steepness

$\delta = \frac{\eta'}{L''}$

Horizontal Asymmetry Factor

$\mu = \frac{\eta'}{H}$

Figure 4 — Definitions of asymetric parametres (reproduced from Kjeldsen and Myrhaug, 1979).

Fig. 5 plots the asymmetry factors α_F and α_B with respect to the dimensionless water depth kh. Both vertical and horizontal asymmetry increase toward shallower water. The degree of asymmetry at any given depth (kh) increases with wave nonlinearity (H/h). The horizontal asymmetry μ is found to be smaller than the corresponding symmetric wave of the same amplitude spectrum (compare the solid and the dashed curves of H/h=0.5, Fig. 5b), at the lowest kh range calculated, a reduction of 6% was shown. The vertical asymmetry factor λ is identically one for a symmetric wave. The slope amplification factors at the crest front α_F and crest rear α_B are plotted in Fig. 5c and d. As mentioned before, $\alpha_F=\alpha_B=2$ for a sinusoidal wave. Both α_F and α_B increase with wave nonlinearity, in which case, the crest region becomes narrower and trough region is elongated. For a symmetric wave $\alpha_F=\alpha_B$. When waves become asymmetric, the crest front becomes steeper and the crest rear becomes milder (compare dashed and solid curves for the case H/h=0.5 in Fig. 5c and d).

Fig. 6 plots λ, μ, α_F and α_B vs. Ur. Similar conclusions as those of Fig. 5 can be drawn.

Discussion and Conclusion

The amplitude and phase spectra of shoaling waves changes continous with water depth. The former resembles the spectra of either cnoidal or solitary waves. A similarity model was constructed to study the asymmetric properties of shoaling waves using the solitary wave spectrum and an empirical relation of the harmonic phase lags with water depth.

As expected, the resulting wave profiles become more asymmetric both horizontally and vertically as water depth decreases or Ursell number increases. Compared with a symmetric wave of the same amplitude spectrum, the vertically asymmetry factor is always higher but the horizontal asymmetry factor shows a decrease. Due to the skewing of the wave profile, the crest front steepness is considerably greater than the overall wave steepness H/L.

The wave steepness is closely related to the breaking occurance. In the classical studies of limiting (symmetric) Stokes' wave in deep water, the enclosing angle at wave crest was found to be 120° (Stokes, 1880). McCowan (1894) proved that the breaking angle of a wave in shallow water is still 120°. These results were confirmed by laboratory experiments and field observations (Gaillard, 1904, see eg. review of Kinsman, 1965). Michell (1893) showed that the corresponding maximum steepness H/L=1/7 in deep water. When a wave moves into shoaling water, the limiting steepness decreases and becomes a function of both relative depth and the beach slope. Although this subject was studied extensively, the mechanism of wave breaking on a sloping beach is not yet clear. Subsequently, the breaking characteristics were generally expressed in terms of the ratio between the breaking wave height and water depth (H_b/h_b), or the ratio between the wave height at breaking point and at deep water (H_b/H_o). Many empirical formulas were proposed (Iversen, 1952; Galvin, 1969; Goda, 1970 among others). From the above short literature survey, it seems that the breaking angle is one of the rare breaking invariants (another invariant maybe the fluid particle

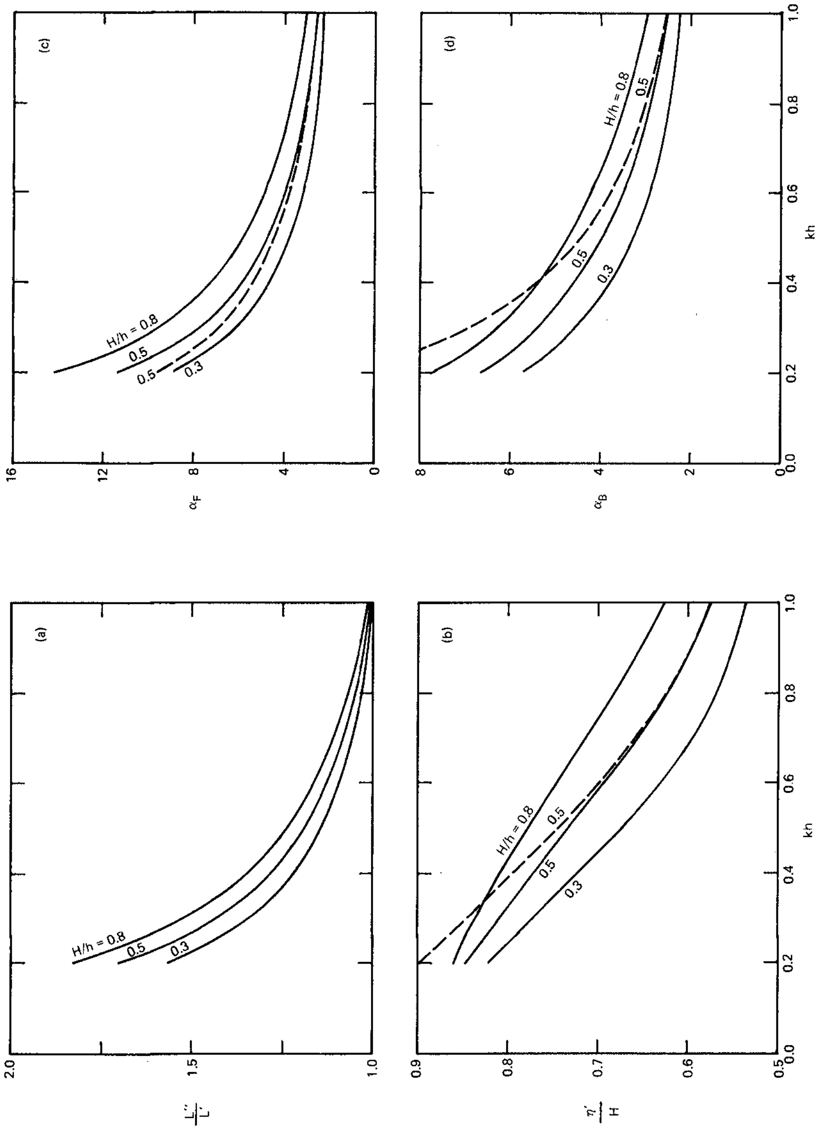


Figure 5 — Calculated asymmetric parameters. (a) Vertical asymmetry, (b) Horizontal asymmetry, (c) Front steepness amplification, and (d) Rear steepness amplification. Dashline corresponds to the case of symmetric waves.

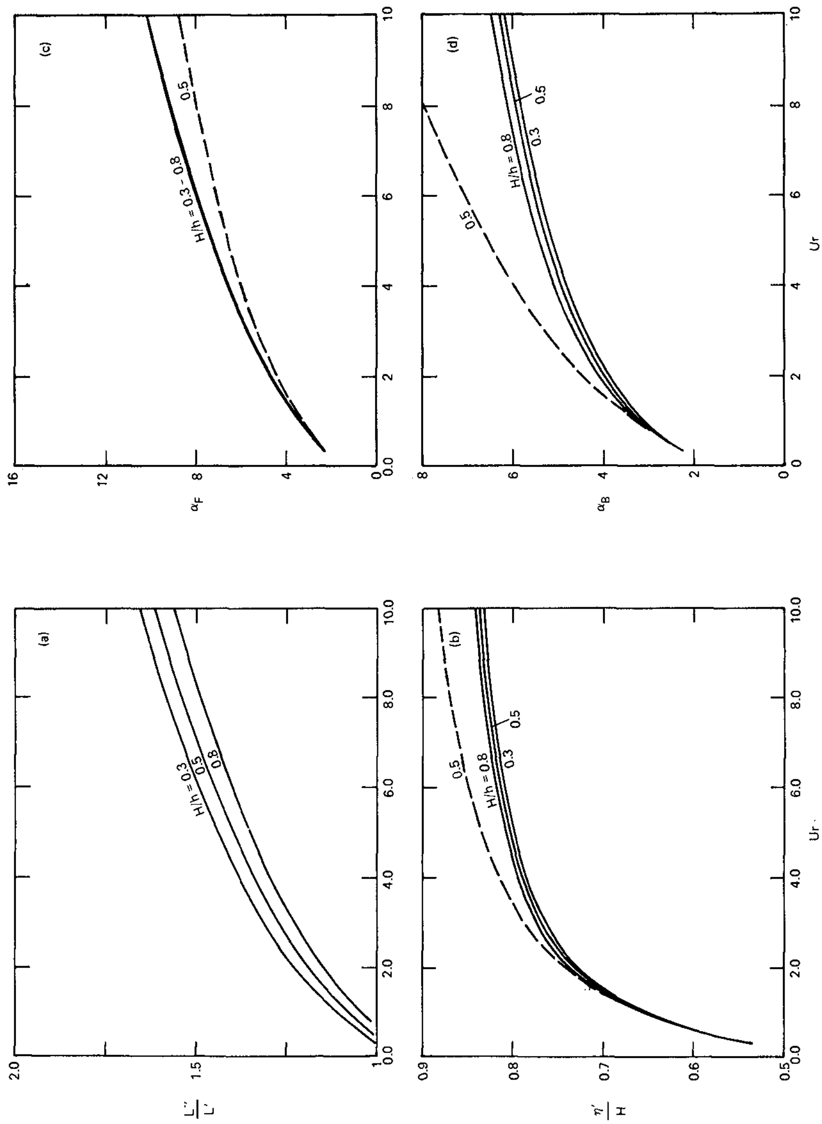


Figure 6 — Same as Figure 5, except plotted against Ursell number Ur

acceleration but this is very difficult to measure). For waves of vertically asymmetric profiles, the breaking angle is more closely related to the crest front and crest rear steepnesses. The crest front steepness is probably an important geometric index for "local" wave steepness. The local wave steepness was used by Longuet-Higgins and Smith (1983) in measurements of breaking probability.

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