CHAPTER SIXTY FIVE

Steep Unsteady Water Waves: An Efficient Computational Scheme

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A new method for computing the unsteady motion of a water surface, including the overturning of water waves as they break, has been developed. It is based on a Cauchy thoerem boundary integral for the evaluation of multiple time derivatives of the surface motion. The numerical implementation of the method is efficient, accurate and stable.

1. Introduction

Longuet-Higgins and Cokelet [2] first described a means of computing steep surface motions and the overturning of water waves in 1976. Since then their method has been used and modified by various authors ([1] to [8]). A useful discussion of most of the techniques available to date is given by Baker et al [1]. The water motion is assumed to be inviscid and irrotational; the necessary solution of Laplace's equation for the velocity potential is cast into some form of boundary integral; and the free surface boundary conditions are used to advance the calculations in time.

Most of these methods have been found to suffer from a "sawtooth" numerical instability [7], which is usually controlled by some kind of smoothing. They can also be computationally expensive; using an implementation of the method of Longuet-Higgins and Cokelet, New [4] found that about 20 hours of processor time on a Honeywell level 68 (multics system) computer were needed to determine, with reasonable accuracy, the motion of a breaking wave to the point at which the jet strikes the lower surface.

The method described here uses Cauchy's theorem for analytic functions of a complex variable to formulate the boundary-integral equation in a form suitable for efficient solution by iteration. The same equation is used to find extra time derivatives of the surface motion at each step. For a given accuracy the resulting truncated Taylor series is used to estimate the maximum acceptable time-step, and to perform the time stepping.

The method is found to be both stable and very accurate for steep steadily-propagating waves. Unsteady waves can only be checked against other numerical solutions. Comparisons with solutions obtained using the approach of Longuet-Higgins and Cokelet give clear indications that this method is more accurate. Furthermore the program for this method is found to run an order of magnitude faster, for given accuracy.

2. Outline of Method

Incompressible irrotational flow is described by a velocity potential satisfying

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = 0 \tag{1}$$

where

$$\underline{u} = \nabla\phi. \tag{2}$$

It is assumed that the fluid is bounded below by an impermeable flat bottom at $y = -h$. The appropriate boundary condition for ϕ is

$$\phi_y(x,-h,t) = 0. \tag{3}$$

With these conditions governing the behaviour of the body of the fluid it is possible to focus attention on the fluid surface alone; as described in section 4, a boundary-integral equation can be used to summarise the whole fluid's behaviour in terms of its surface properties. In particular the normal and tangential gradients of ϕ at the surface can be determined, knowing only the surface values of ϕ. This gives the velocity $\underline{u}$ from (2). The kinematic condition, that a surface particle moves with its own velocity, shows that the instantaneous movement of the surface is then fully determined; if (X,Y) marks a particle on the surface then the rates of change of X and Y are given by

$$\frac{D}{Dt}(X,Y) = \underline{u} = (\phi_x, \phi_y). \tag{4}$$

To follow the motion of the surface further it is also necessary to know the way in which ϕ varies. Bernoulli's equation gives the rate of change of ϕ on the surface as

$$\frac{D\phi}{Dt} = \frac{u^2}{2} - (P/\rho + gY) \tag{5}$$

where P takes the value of the surface pressure which for many practical purposes may be taken as a constant, the actual value of which makes no difference to the motion of the fluid.

Most existing numerical schemes for following the movement of the fluid surface make use of this formulation with the following basic algorithm.

Given (X,Y,ϕ) on the surface at the time t:

i) solve $\nabla^2\phi = 0$ with $\phi_y(x,-h,t) = 0$ to obtain $\nabla\phi$ on the surface and hence evaluate $D(X,Y,\phi)/Dt$

ii) time-step to obtain (X,Y,ϕ) on the surface at the time $t + \Delta t$, and repeat.

3. Extended Approach

Equations (1) and (3) can be successively differentiated with respect to time to yield the following equations and bottom conditions:

$$\begin{aligned} \nabla^2\phi_t &= 0 \quad \text{with} \quad \phi_{ty}(x,-h,t) = 0 \\ \nabla^2\phi_{tt} &= 0 \quad \text{with} \quad \phi_{tty}(x,-h,t) = 0 \\ &\text{etc.} \end{aligned} \tag{6}$$

Clearly ϕ_t, ϕ_{tt} and all similar Eulerian derivatives of

ϕ satisfy Laplace's equation and the bottom condition just as ϕ does. In the same way therefore it is possible to solve for the gradients of these derivatives and obtain the corresponding Eulerian rates of change of velocity, for which they are the appropriate potential functions. This can be seen by differentiating (2). Extension of the kinematic condition (4) to obtain $D^2(X,Y)/Dt^2$ etc. is also straightforward. Similarly, differentiation of Bernoulli's equation (5) gives the corresponding Lagrangian rates of change of ϕ on the surface.

In order to determine these higher derivatives it is necessary to obtain ϕ_t, ϕ_{tt} etc. on the surface. By using Bernoulli's equation once again and expressing the result in terms of Lagrangian, rather than Eulerian derivatives of pressure, these can be found in terms of known variables. For example,

$$\phi_{tt} = - [\underline{u}.\ (\underline{u}_t + \frac{D\underline{u}}{Dt}) + \frac{DP}{Dt}/\rho + g\frac{DY}{Dt}]. \qquad (7)$$

Thus the Eulerian derivatives of ϕ are given once P, DP/Dt etc. are specified. For a constant surface pressure all Lagrangian derivatives of P at the surface would simply be zero.

The algorithm for calculating the surface motion can thus be extended in the following way

Given (X,Y,ϕ) on the surface at the time t:

i) solve $\nabla^2\phi = 0$ with $\phi_y(x,-h,t) = 0$ to obtain $\nabla\phi$ on the surface and evaluate both $D(X,Y,\phi)/Dt$ and ϕ_t

ii) similarly use ϕ_t to obtain both $D^2(X,Y,\phi)/Dt^2$ and ϕ_{tt}

iii) proceed in the same way to calculate up to a chosen order of time derivative

iv) time-step, using Taylor series in X, Y and ϕ with some backward differencing, to obtain (X,Y,ϕ) on the surface at the time $t + \Delta t$, and repeat.

A scheme using calculations up to D^3/Dt^3 in this algorithm has been implemented numerically.

Some immediate advantages are found with this approach. Firstly, it is possible to take larger time steps for a given accuracy. Secondly, the derivatives are calculated for the same surface profile, (X,Y). This means that the solutions are all obtained using exactly the same boundary-integral kernels. Both of these result in an improvement in numerical efficiency and hence shorter computer running times. It is also useful to have more detailed information calculated about the surface motion at each time step.

4. Integral Equation

For a variety of reasons it was decided to use Cauchy's integral theorem as the means of solving Laplace's equation. Its use leads to a fairly simple formulation in which the singularities in the integral kernels are easily taken into account. This contrasts with the more cumbersome logarithmic singularity found with approaches which use Green's identity. In addition the Cauchy formulation is soluble by iteration which can be a quick method of solution.

(i) Cauchy's Integral Theorem

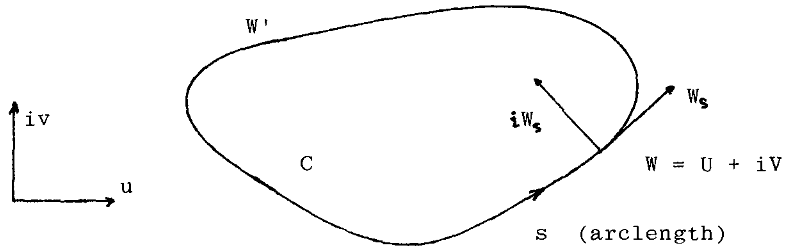

Figure 1: Closed contour

For ϕ satisfying Laplace's equation within the contour C, illustrated in figure 2, the complex potential gradient, $\phi_u - i\phi_v$, is an analytic function of the variable, $w = u + iv$. The following useful expression of the principal value form of Cauchy's integral theorem is then satisfied.

$$\phi_n = \frac{1}{\pi} \oint \mathrm{Im}\left[\frac{W_s}{W'-W}\right] \phi_n' ds' + \frac{1}{\pi} \oint \mathrm{Re}\left[\frac{W_s}{W'-W}\right] \phi_s' ds' . \tag{8}$$

In this equation the arclength derivative, W_s, of the surface position, $W(s)$, is the complex unit tangent, and

ϕ_s and ϕ_n are the tangential and inward normal gradients of ϕ, respectively.

(ii) Application

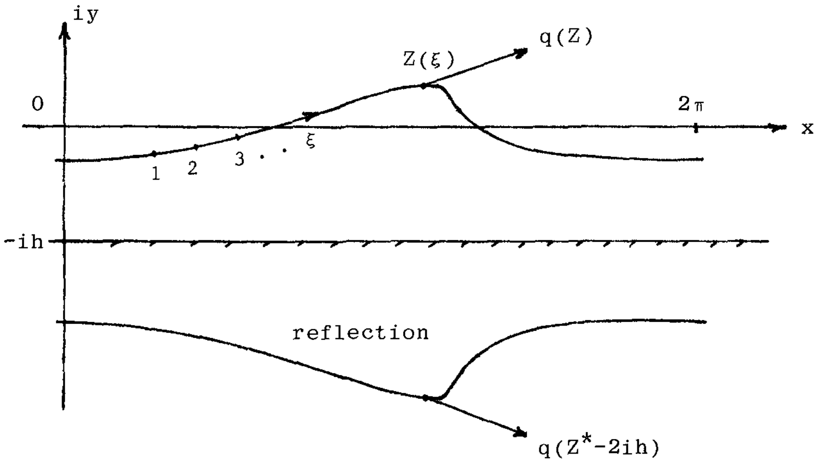

Figure 2: Physical plane

In the complex representation of the physical plane, $z = x + iy$, the complex gradient of the potential becomes

$$q = \phi_x - i\,\phi_y \tag{9}$$

and the bottom condition (3) can be satisfied by adopting the "reflection" condition,

$$q(Z^* - 2ih) = q^*(Z), \tag{10}$$

with the fluid assumed to extend continuously below the bottom to the reflection of the complex surface, Z, in the bottom.

It is useful at this point to assume that the system is periodic in x so that only a finite length of surface need be considered. No generality is lost in taking the spatial period to be exactly 2π. The periodic surface $Z(\xi)$ in the z plane is then transformed to a closed contour $W(\xi)$ in the w plane by the conformal transformation,

$$w = e^{-iz} = u + iv. \tag{11}$$

Applying the formula (8) to this contour and the transformation of the reflected contour leads to the following integral equation

$$\phi_\nu = \frac{1}{\pi} \oint \mathrm{Im} \left[\frac{W_\xi}{W-W'} + \frac{W_\xi}{W-e^{-2h}/W'^*} \right] \phi'_\nu d\xi' + \frac{1}{\pi} \oint \mathrm{Re} \left[\frac{W_\xi}{W-W'} - \frac{W_\xi}{W-e^{-2h}/W'^*} \right] \phi'_\xi d\xi' \tag{12}$$

where ϕ_ξ is simply the derivative of ϕ with respect to the parameter ξ along the surface, while ϕ_ν is the outward normal gradient of ϕ scaled by $|W_\xi|$. Supposing that (12) can be solved for ϕ_ν, the complex potential gradient of ϕ in the physical plane is then given by

$$\phi_x + i\phi_y = \frac{W^* W_\xi}{|W_\xi|^2} (\phi_\xi + i\phi_\nu). \tag{13}$$

(iii) Numerical Solution

In order to solve equation (12) numerically it is now convenient to identify ξ as a point label parameter with $X(\xi)$, $Y(\xi)$ and $\phi(\xi)$ defined on the surface for integral values of ξ. Derivatives with respect to ξ can be estimated using either central difference or Fourier series methods. By considering a Taylor expansion for W' the singular part of the integral kernels in (12) can be seen to be of the form

$$\frac{W_\xi}{W-W'} = \frac{1}{\xi-\xi'} + \frac{W_{\xi\xi}}{2W_\xi} + O(\xi'-\xi). \tag{14}$$

Since $(\xi - \xi')^{-1}$ is real this makes it clear that only the kernel comprising the real part of $W_\xi / (W - W')$ is genuinely singular, and since

$$\lim_{\xi'\to\xi} \frac{\phi'_\xi - \phi_\xi}{\xi-\xi'} = -\phi_{\xi\xi} \tag{15}$$

the effect of this singularity in the integral is easily determined. It can thus be shown that the formula

$$\fint \mathrm{Re}\left[\frac{W_\xi}{W-W'}\right]\phi'_\xi d\xi' \approx \sum_{\xi'\neq\xi} \mathrm{Re}\left[\frac{W_\xi}{W-W'}\right]\phi'_\xi + \mathrm{Re}\left[\frac{W_{\xi\xi}}{2W_\xi}\right]\phi_\xi - \phi_{\xi\xi} \tag{16}$$

can be used to provide a numerical estimate for this integral. Taking account of the other non-singular components of (12) the integral kernels were expressed in matrix form. Given any potential function $\phi(\xi)$ on the surface, the derivatives ϕ_ξ and $\phi_{\xi\xi}$ were estimated numerically and the matrix equations were solved iteratively for ϕ_ν . $\nabla\phi$ was then obtained using equation (13).

5. Accuracy, Stability and Speed

The resulting scheme for following the free surface motion was tested in a number of cases and was found to show no sign of the "sawtooth" numerical instability. Figure 3 shows one such test using 32 points to follow the propagation of a steep steady surface wave in deep water (with $ak = 13/32$ or about 92% of the highest wave) for ten wave periods. It is interesting to note the particle drift of nearly twice the wavelength, λ , with the triangle Δ marking the same particle. The results are just as should be expected to within an accuracy of about $\lambda/10^4$. This accuracy reflects an accuracy in the phase speed for the calculated wave of about 0.005%.

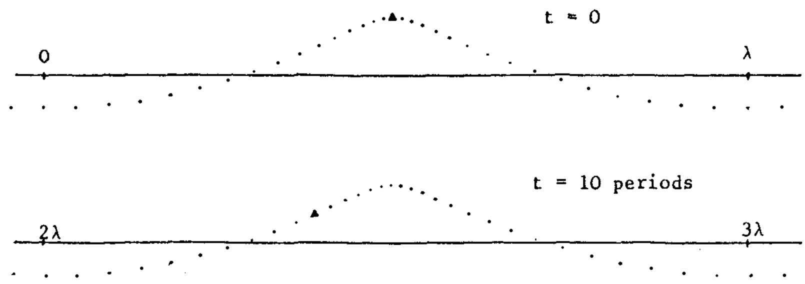

Figure 3: Steep surface wave (ak = 13/32) after 10 wave periods using 32 points.

On a Honeywell level 68 (multics system) computer the program took about 2 minutes of C.P.U. time per wave period to execute this calculation, a running time which could have been reduced with fewer points and/or larger time steps at the sacrifice of some accuracy. The examples shown in figures 4 to 6 below each took about 20 to 30 minutes of C.P.U. time to calculate, and may be compared with the 20 hours, or so, experienced by New [4].

There are some limitations to the method. It was found that it is not always possible to reliably calculate progressively higher and higher derivatives. This is particularly so in regions of large strain rates, such as under the jet of a breaking wave, where the spreading out of surface particles leads to a reduced ability to resolve the wave profile. Since higher derivatives vary more dramatically in this region they are found to be the first to suffer from a loss of resolution. One can still proceed making use only of lower order derivatives.

Another region where a lack of resolution was encountered is at the tip of a jet such as can be seen in figure 6. The shape of this jet tip reflects a situation in which the point marked with the triangle Δ is being accelerated into the fluid while points nearby are being accelerated outwards. Smoothing was used to control the motion in this case. It is not clear whether this observed tendency reflects a growing numerical instability or whether a genuine physical phenomenon, such as a breaking up of the jet tip, is taking place.

6. Some Examples of Results

For the examples presented below the acceleration due to gravity was given the value unity. Except in cases of zero gravity no generality is lost by this choice.

i) Figure 4 shows the detail of a very small jet formed at the crest of a breaking wave which was generated by suddenly imposing a mean water depth of 2.5 (where the wavelength is 2π) on the otherwise deep water travelling wave of figure 3. After about two wave periods this causes the small jet to form at the crest of the wave. It is interesting to compare the profile at the wave crest with the 120° angle shown by the dotted line in figure 4.

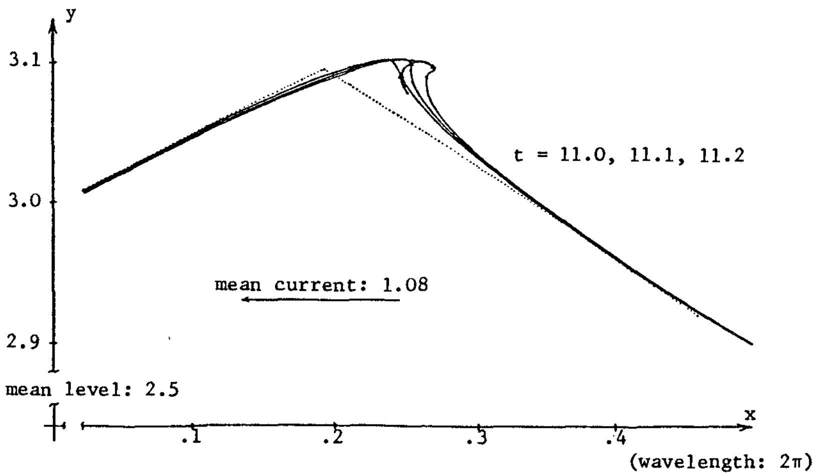

Figure 4: Formation of a "spilling breaker" by introducing finite depth into the wave in figure 3.

ii) By contrast figure 5 shows the development of a very large plunging breaker arising from an initial large amplitude travelling sine wave. Figure 6 shows a detail of the jet tip of this wave taken as far as the computation would allow.

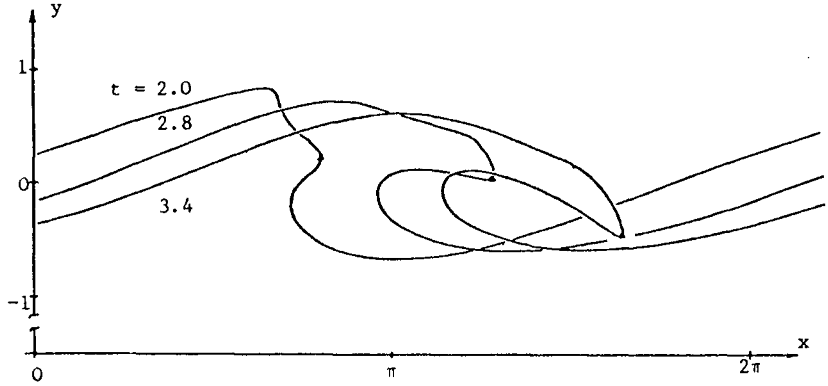

Figure 5: Massive plunging breaker from an initial travelling sine wave of amplitude 0.844, in deep water.

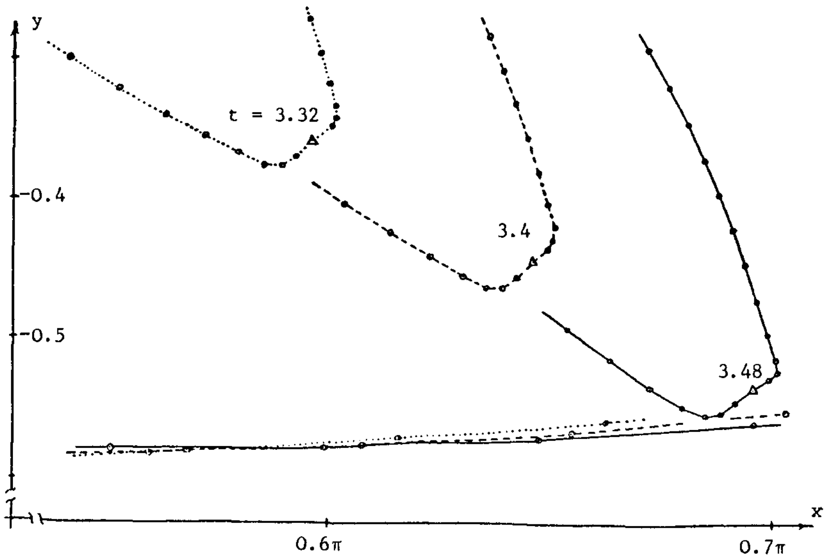

Figure 6: Detail of the tip of the jet in figure 5.

iii) A curious example of surface motion leading to both the formation of a vertical jet and a plunging breaker is shown in figure 7. The initial conditions leading to this are illustrated by the dotted lines, showing a stretching of the surface in one region and a convergence and focussing of the motion of the surface in another.

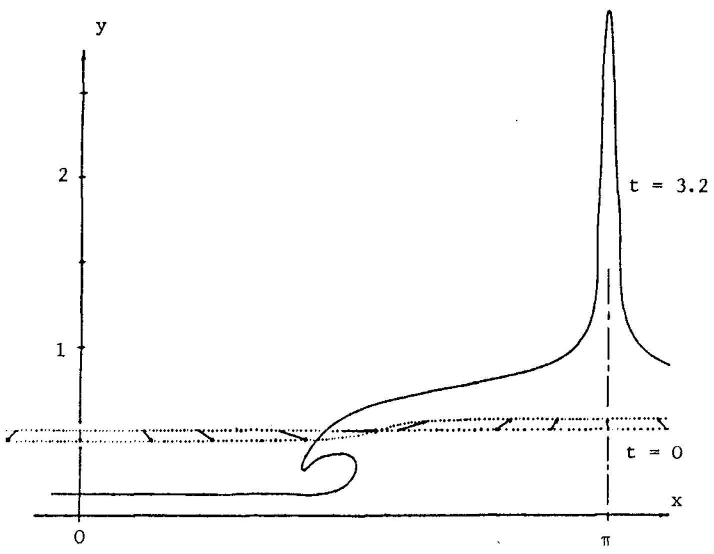

Figure 7: Surface motion arising from the symmetric, periodic, initial conditions in shallow water, of a flat surface with strongly convergent (and divergent) flow, as indicated by the dotted lines.

7. Conclusions

The use of higher derivatives and the use of Cauchy's integral theorem have led to an efficient numerical scheme for calculating the motion of steep surface waves. The efficiency of the program means that it can now be used in a wide variety of ways. It is already well suited to fundamental studies of nonlinear wave phenomena such as breaking. With little further development it could be used for assessing forces on structural elements or vessels which are small compared with the dimensions of the wave. This type of computation could replace the "design wave" concept where steep steady waves are currently used. For this application some measure of breaker "strength" should be developed.

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