WATERWAVES CALCULATION BY NAVIER-STOKES EQUATIONS

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Abstract : N.S.L. program is a finite-difference code for two-dimensionnal flows with a free surface in a vertical plane. Basic equations are Navier-Stokes équations with a simple simulation of turbulent effects by an eddy viscosity coefficient related to the mixing length and the mean velocity gradient. Theses equations are solved in a variable domain in time. The main features of the numerical method are presented. Some comparisons with theoretical solutions give a good validation of the code both in linear and non linear cases. Other examples of application are given.

1. INTRODUCTION

Large computational programs, solving time-dependant Navier-Stokes equations, in two and even three space dimensions, are now well developped for industrial design. Their adaptation to waves problems should be a new and powerful mean of investigation in theoretical and practical studies, because non linear and viscous effects are taken into account.

The N.S.L. program "Navier-Stokes à Surface Libre" presented here, is a first step to this ambitious objective. It solves numerically N.S. equations in a vertical plane - i.e. only for two-dimensionnal flows - with a free surface. The free surface is a moving boundary for the computational domain, and is also an unknown of the problem. Time-varying pressure and velocity fields are the other results of this program for suitable initial and boundary conditions, and any given bottom shape. In order to give a more pratical background to the study, N.S.L. program can be considered as a "numerical wave flume". It allows simulation of all sorts of waves, in deep or in shallow water, with small or large amplitude, except perhaps breaking ones.

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2. MATHEMATICAL FORMULATION

2.1. General equations :

In the fluid, velocities and pressure are related by dynamical and continuity equations :

(1) $\frac{DV}{Dt} + \frac{1}{\rho} \nabla p^* = \mathrm{Div}\ (\nu_T \nabla V)$

(2) $\mathrm{Div}\ V = 0$ $\qquad\qquad (p^* = p + gz)$

The turbulent viscosity coefficient, is given empirically by :

(3) $\nu_T = \nu + K^2 \ell^2\ (2\ Dij.Dij)^{1/2}$

with $Dij = \frac{1}{2}\ (V_i,\ j + V_j,\ i)$

The mixing length l is constant in the fluid except near the bottom, where l is proportionnal to the distance from the bottom.

2.2. Computational domain and boundary conditions

They are summarized on figure 1.

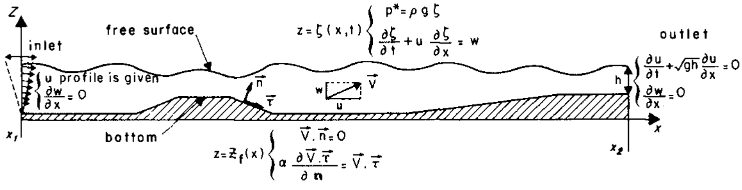


Figure 1 - Boundary conditions - Left b.c. simulates a wave- machine (e.g. $U = U_o(z) \sin \omega t$) - Right b.c. is absorbing if h is small enough to eliminate dispersive phenomena. On the bottom, the second b.c. simulates a tangential stress in a boundary layer.

2.2.1. Sea surface boundary conditions :

The second relation, shown in figure 1, is a partial differential equation which can be used for the free surface determination. It must be solved along the whole ζ-line, with its own boundary conditions. These ζ-boundary conditions, in $x = x_1$ and $x = x_2$, can be deduced from the u-ones, by using the horizontal projection of eq. (1), and the surface relation : $p^* = \rho g \zeta$.

(4) $\frac{\partial \zeta}{\partial x} = \frac{1}{\rho g} \frac{\partial p^*}{\partial x} = -\frac{1}{g} \left(\frac{Du}{Dt} - \mathrm{Div}\ (\nu_T \nabla u)\right) \simeq -\frac{1}{g} \frac{Du}{Dt}$

Theoretically, another boundary condition is required, along the free surface, in order to well pose the problem - a zero tangential stress, for instance-. But this point is ignored in the present, because viscous terms are almost vanishing near the free surface, as long as wind effects are not studied.

2.2.2. Bottom boundary conditions :

The first one traduces the impermeability of a fixed bottom. The second one comes from the idea that the tangential stress

$$(5) \quad \rho \nu_T \frac{\partial(\vec{V}.\vec{\tau})}{\partial n} = \rho u_*^2$$

is prescribed. Its value is calculated by doing the hypothesis that the velocity profile, near the bottom, follows a logarithmic law, for instance :

$$(6) \quad \vec{V}.\vec{\tau} = u_* \left(\frac{1}{K} \text{Log} \frac{y}{d} + 8.5\right)$$

where y is the distance from the bottom.

As ν_T, near the bottom, reduces to

$$(7) \quad \nu_T \simeq K^2 y^2 \frac{\partial \vec{V}.\vec{\tau}}{\partial n}$$

one has, by elimination of u_*, a relation of the form

$$(8) \quad \alpha \frac{\partial \vec{V}.\vec{\tau}}{\partial n} = \vec{V}.\vec{\tau}$$

2.2.3. Lateral boundary conditions :

These conditions are fitted to wave flume simulation : the left one simulates a wave machine and the right one, allows the waves to get out without reflection. The absorbing boundary condition :

$$(9) \quad \frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

must be used in a shallow water situation. In dispersive waves, the celerity c depends on the wave length. It is thus necessary, for irregular waves, to reduce the depth with a mild slope, in order to have only shallow water waves at the boundary, and then a unique celerity $c = \sqrt{gh}$.

3. NUMERICAL METHOD

3.1 Coordinate transformation

Before solving numerically the system (1) (2) and its boundary conditions, a coordinate transformation is done :

$$(10) \qquad \begin{cases} \xi = x \\ \eta = H\ (z - Z_f(x)/(\zeta\ (x,t) - Z_f(x)) \end{cases}$$

This transformation maps the physical variable domain in a fixed one, with boundaries parallel to the coordinate-axis (figure 2).

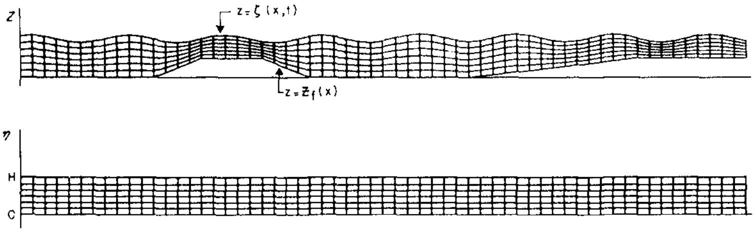


Figure 2 - The semi-curvilinear grid, transformed in a rectilinear one by the transformation : $\eta = H(z-Z_f/(\zeta - Z_f)$.

Equations and boundary conditions are expressed in the new independant variables (ξ, η, t).
Calculations are performed on the transformed domain, by solving a finite difference approximation of the transformed equations. These involve, in each derivative, one of the following factors :

$$(11) \quad \eta_x = -\ (H\ Zf_x + \ (\zeta_x - Zf_x))/(\zeta - Z_f)$$

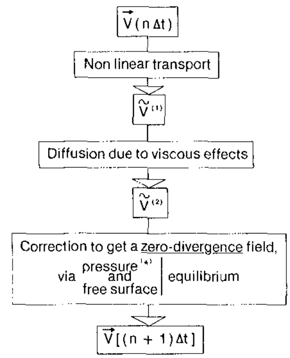
$$(12) \quad \eta_z = H/(\zeta - Z_f)$$

$$(13) \quad \eta_t = -\eta\zeta_t/(\zeta - Z_f)$$

which vary in time (due to the free surface motion) and are evaluated at each time step.

3.2. Algorithm of a time step

A fractionary step method is used [1]. This method is illustrated by the following scheme :



Each time step is splitted in three parts which are solved successively ; the velocity field $V^n = V(n\,\Delta t)$, being known at time $n\,\Delta t$, intermediate fields $\tilde{V}(1)$ and $\tilde{V}(2)$ are computed verifying respectively :

- a transport equation solved by the Characteristics method.

$$(14)\qquad \frac{\tilde{V}^{(1)} - V^n}{\Delta t} + V^n \otimes \nabla V^n = 0$$

- and a diffusion equation solved by an explicit method

$$(15)\qquad \frac{\tilde{V}^{(2)} - \tilde{V}^{(1)}}{\Delta t} - \mathrm{Div}\,(\nu_T\ \nabla \tilde{V}^{(1)}) = 0$$

The definitive, zero-divergence velocity field, at time $(n+1)\,\Delta t$, V^{n+1}, is calculated via a pressure and free surface equilibrium, which is expressed by the following system of equations :

$$\left.\begin{aligned} &(16)\qquad \frac{V^{n+1} - \tilde{V}^{(2)}}{\Delta t} + \frac{1}{\rho}\nabla p^* = 0 \\ &(17)\qquad \mathrm{Div}\ V^{n+1} = 0 \end{aligned}\right\}\ \text{in the domain}$$

$$\left.\begin{aligned} &(18)\qquad p^* = \rho g \zeta \\ &(19)\qquad \frac{\partial \zeta}{\partial t} + u^{n+1}\frac{\partial \zeta}{\partial x} = w^{n+1} \end{aligned}\right\}\ \text{on the free surface}$$

Elimination of V^{n+1} between (16) and (17), give a Poisson equation for pressure

$$(20)\qquad \Delta p^* = \frac{\rho}{\Delta t}\,\mathrm{Div}\ \tilde{V}^{(2)}$$

This equation is solved with Neumann boundary conditions, derived from the normal projection of equation (16), in which V(n+1).n is known, on the bottom and on lateral boundaries. On the free surface, p* verify the Dirichlet boundary condition (18). As ζ is related to V(n+1) by eq. (19), the system (16) to (19) is solved iteratively.
Discretisations, in space and time which are not specified in the formulas (14) to (19), are performed in a staggered grid, as shown in figure 3.

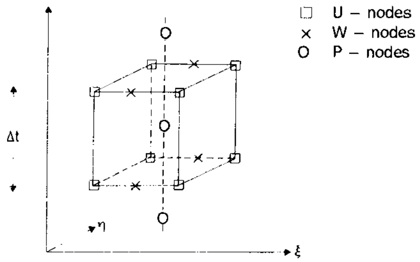


Figure 3 – Numerical staggered grid, in space and time.

Pressure nodes, on this figure, are drawn on three time levels to indicate that, in equation (16), the pressure gradient is splitted on three time steps ; let us write p^* without star :

$$(21) \quad \nabla p \simeq \alpha \nabla p^{n+\frac{3}{2}} + \beta \nabla p^{n+\frac{1}{2}} + \gamma \nabla p^{n-\frac{1}{2}}$$

$$\alpha + \beta + \gamma = 1$$

Wheighting coefficients α, β, γ are choosen as close as possible to 1/4, 1/2, 1/4 respectively ; nevertheless, α must be slightly greater than γ, in order to insure the stability of the numerical scheme(*). Of course, equation (20) is modified in consequence :

$$(22) \quad \Delta p^{n+\frac{3}{2}} = \frac{1}{\alpha} \operatorname{Div} \left[\frac{\rho}{\Delta t} \tilde{V}^{(2)} - \beta \nabla p^{n+\frac{1}{2}} - \gamma \nabla p^{n-\frac{1}{2}} \right]$$

Pressure fields at $(n+\frac{1}{2})\,\Delta t$ and $(n-\frac{1}{2})\,\Delta t$ are assumed to be already calculated, and $p^{n+\frac{3}{2}}$ is the unknown of this equation.

It is thus necessary to prescribe, as initial conditions, not only velocities, but also pressures at two consecutive time levels. Equation (19), involving $\zeta^{n+\frac{3}{2}}$, and $\zeta^{n+\frac{1}{2}}$, is approximated with a centered discretisation.

Simpler methods have been studied, but they have shown a very poor accuracy in the tests presented hereafter.

4. COMPARISONS BETWEEN COMPUTED AND THEORETICAL RESULTS

N.S.L. program has been tested through numerous comparisons between computed and theoretical results, specially in two cases :

- the Stokes linear wave solution for small amplitude waves,
- the solitary wave's solution.

For these comparisons, viscous effects have been neglected.

4.1. Tests for small amplitude waves.

Figure 4 shows the waves calculated in the numerical wave flume, when the left boundary condition simulates a piston-wave-machine with period 4.5 s, and maximum velocity 0.2 m/s ; the right boundary condition is a radiative condition.

Water is at rest, at t = 0 ; then waves begin to propagate from left to right with a celerity about 5 m/s. At t = 50 s, five waves have gone out by the right boundary ; the movement is considered established ; celerity, wave lenght and wave amplitudes can be compared to the theoretical values given by the linear first order solution for irrotationnal waves of small amplitude.

(*) This method is a generalization of the one used by HAUGUEL in the numerical model of storm waves in shallow water [2] and [3].

Other tests have been done with a pivot-wave-machine, and with different depths. Comparisons with theoretical values are summarized in table I : the last case presented in this table, corresponds to figure 4 ;

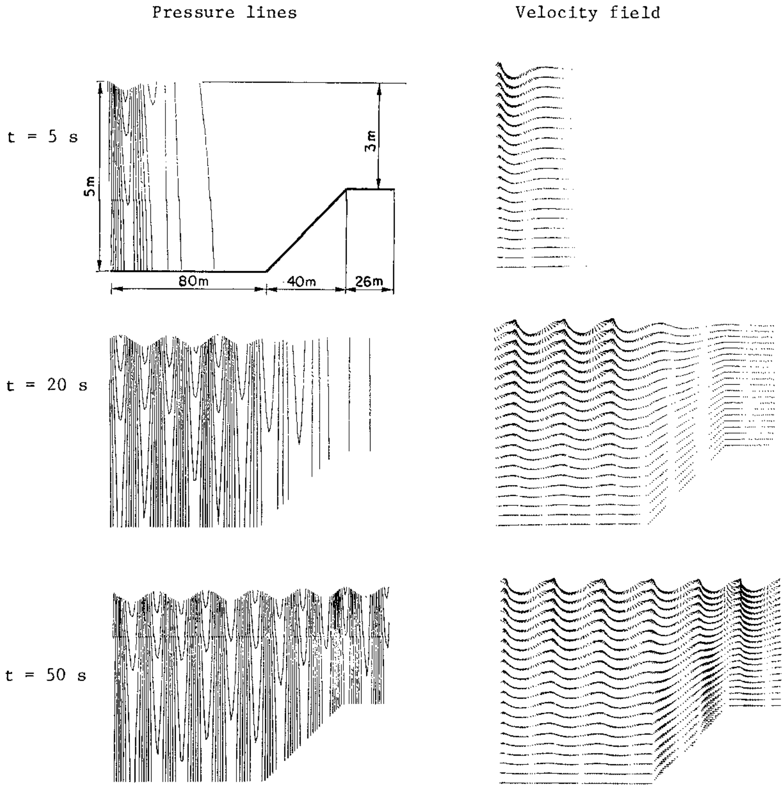
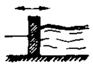
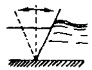
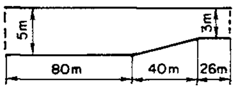


Figure 4 - Waves calculated in the numerical wave flume.
The left b.c. simulates a piston wave-machine (period = 4.5 s, maximum velocity = 0.2 m/s). The right b.c. is a radiative condition.
(Δx = 1 m ; Δt = 0.15 s ; Δz = 0.25 m in the deep part of the flume)

Table 1 : Comparisons between computed and theoretical results for linear waves.

Flume characteristics		Wave-machine characteristics			Computed results and theoretical values(*)				
Depth	length		Period	Velocity maximum	wave length	celerity	wave height	top velocity	$\frac{\text{top V}^{ty}}{\text{bottom V}^{ty}}$
15 m	250 m		4.5 s	0.3 m/s	31.7 m (31.6)	7.05 m/s (7.02)	0.78 m (0.82)	0.58 m/s (0.56)	14 (10)
			4.5 s	0.3 m/s (on surface)	31.5 m (31.6)	7.0 m/s (7.02)	0.57 m (0.60)	0.45 m/s (0.42)	10 (10)
5 m	250 m		4.5 s	0.2 m/s	26.1 m (26.3)	5.8 m/s (5.84)	0.30 m (0.31)	0.26 m/s (0.26)	1.8 (1.8)
variable 5 m slope 5 % 3 m	80 m 40 m 26 m		4.5 s	0.2 m/s	26.0 m (26.3) 21.5 m (22.1)	5.78 m/s (5.84) 4.78 m/s (4.91)	0.29 m (0.31) 0.27 m (0.32)		



(*) Theoretical values are in parenthesis.

The others are performed in a longer horizontal wave flume ; in these cases, results are analyzed before the occurence of perturbations comming from reflection on the right boundary. Computed values are very similar to theoretical ones, indicated in parenthesis.

In these tests, the computing time is about 200 s of CPU for 50 s of physical time, on a CRAY 1.

4.2. Solitary wave tests

One is shown on figure 5 : on the left boundary (x = 0 m), the analytical value of the time-varying u-component is imposed ; w-component and free-surface level are computed results. The wave appears progressively in the domain of calculation, and propagates without deformation, like the theoretical solution [3] drawn on the same figure. The good quality of this result, insure that non linearities are well calculated.

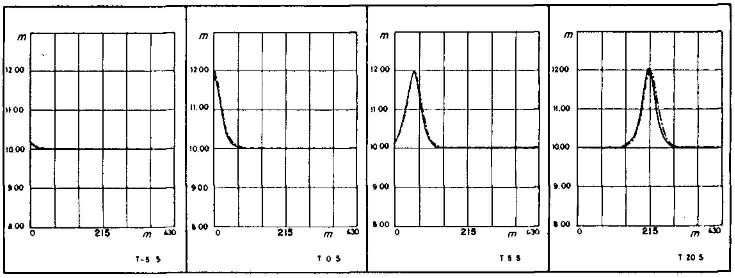


Figure 5 - Free surface of a solitary wave at t = -5 s, 0 s, 5 s, 20 s.
——— Theoretical solution -·-·- Computed solution by NSL program.

5. PRESENTATION OF SOME APPLICATIONS

As an example of the program possibilities, figure 6 shows how the solitary wave is transformed when it propagates on a rise. A similar case will be studied both numerically and experimentally. The numerical model is then able to give informations on the waves deformation by the bathymetry and then to test the validity of different integrated wave theories.

Another application, presented on figure 7, is specially devoted to the simulation of viscous effects ; it is the calculation of a permanent flow over a dredged trench. Drawings show the beginning of the flow : free surface remains nearly horizontal, and transient recirculations can be seen near the bottom. This flow has been studied experimentally in the Delft Hydraulic Laboratory [4], and it will be possible to use velocity measurements, in order to fit the parameters occuring in the turbulence simulation of the mathematical model : mixing length and bottom roughness. In the next future, the model will allow the study of viscous effects on propagating waves.

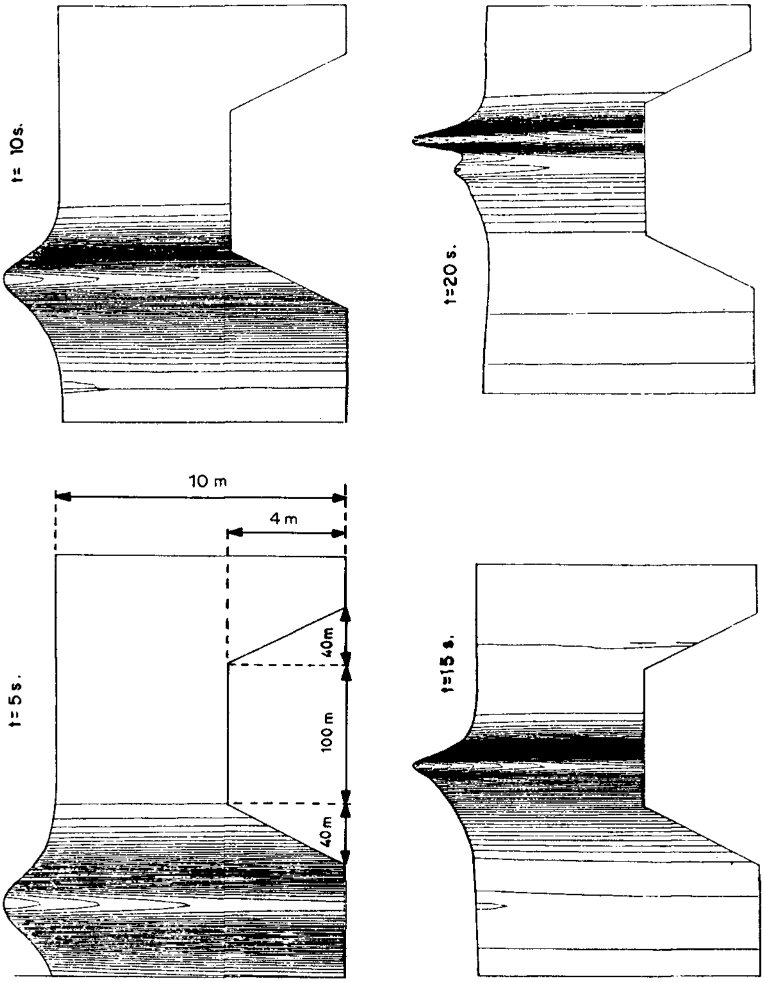


Figure 6 - Solitary wave propagating on a rise
Pressure field p^* (step = 0.05 m of water).

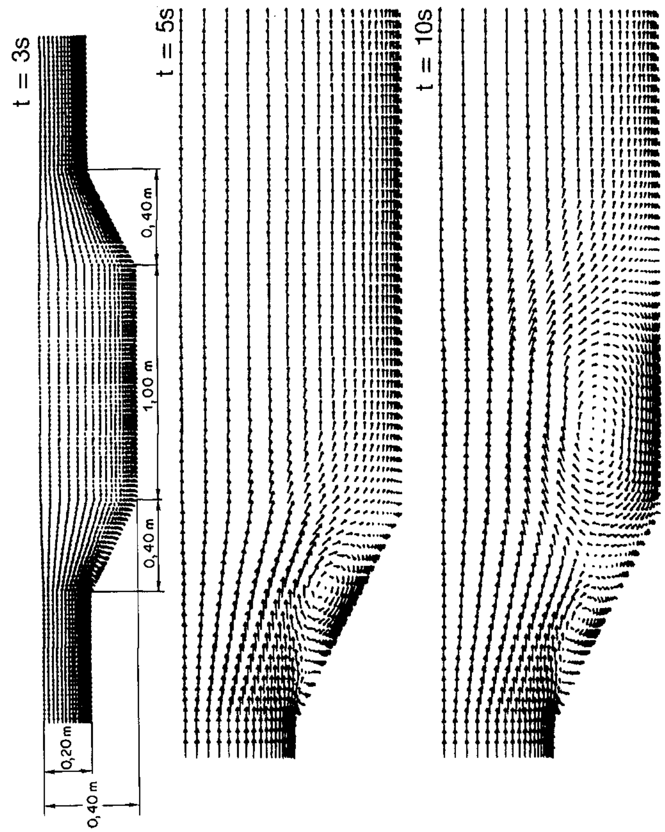


Figure 7 - Flow computation over a dredged trench. Velocity field.

N.S.L. program has also been stated to the study of irregular waves. On figure 8, the calculated heights in three points of a "numerical wave flume" are given. These level variations result from the superposition, at the left boundary of velocity profiles corresponding to three sinusoïdal waves, whose periods are 5 s, 8 s, and 13 s, and height 0.5 m, respectively. The evolution of the sea surface is quite comparable to what is expected in that case, and then, non-linear interactions taken into account in the numerical model, seem to be properly reproduced.

As the space and time steps, in this case, are chosen small enough to get a good precision for shortest wave (Δt = 0.15 s, Δx = 1 m, Δz = 0.75 m in the deep part of the flume), calculations need more computing time than the previous ones : about 800 s of CPU time in the CRAY 1, for 100 s of physical time. But this first test demonstrate the possibility to use the model on irregular conditions with a single simulation.

All these applications have ensured us that the idea of "numerical non linear wave flume" is viable. Further developments and applications will be made to demonstrate this possibility.

6. CONCLUSION

The main difficulty to tide over, for a numerical simulation of waves, was to get a good accuracy in propagation phenomena. Numerical models are often affected by excessive damping or phase-shift errors. The cases which have been choosen to test NSL program : - linear cases, as the first-order Stokes wave, and non-linear ones, as the solitary wave, - have lead to an efficient and accurate numerical method.

The other aspect to be tested now, are the turbulence and bottom boundary layer simulation in wave conditions.

N.S.L. program appears already to be an interesting tool for various fundamental studies, as velocities and pressure distributions in random waves, or in waves over immerged obstacles.

Developments of the code are projected to adapt its possibilities to stresses calculations on sharp obstacles and eventually on floating bodies in waves.

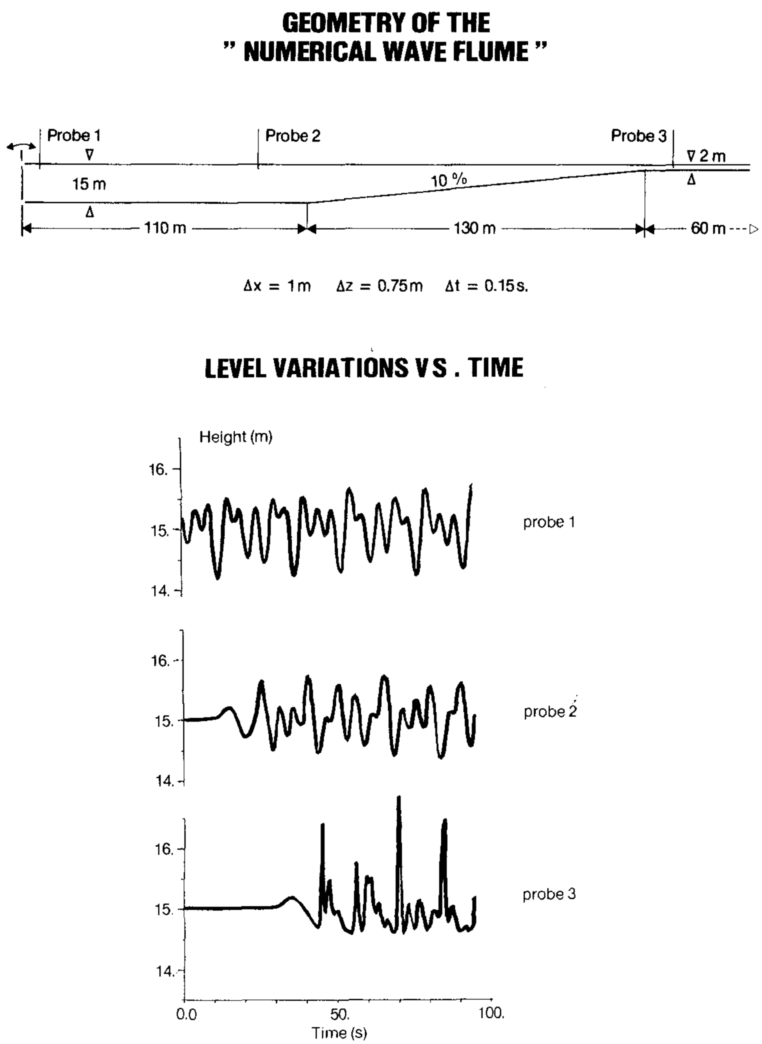


Figure 8 - Calculation of irregular waves.

7. REFERENCES

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Translated by NEPOMIASTCHY P.A. Collection Intersciences A. COLIN - Paris 1962 -

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8. NOTATIONS

x	horizontal coordinate
z	vertical coordinate
t	time
$\vec{V} = (u,w)$	velocity vector
p	pressure ($p^* = p + \rho g z$)
Z_f	bottom level
ζ	free surface level
ν	cinematic viscosity
ν_T	turbulent viscosity
K	Karman's constant (= 0.41)
ℓ	mixing length
ρ	volumic mass
g	gravity constant
y	distance along the normal to the bottom
$\vec{n}, \vec{\tau}$	normal and tangential unit vectors

Div=Divergence = $\frac{\partial}{\partial x} + \frac{\partial}{\partial z}$

∇ Gradient = $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right)$

Δ Laplacian = $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$

Δt time step.
$\Delta x, \Delta z$ space steps