CHAPTER 52

THEORY ON FORMATION OF RIP-CURRENT AND CUSPIDAL COAST

By Mikio Hino

Professor, Department of Civil Engineering
Tokyo Institute of Technology
O-okayama, Meguro-ku, Tokyo

SYNOPSIS : The hydrodynamic instability theory is developed on the formation of rip-current and cuspidal coast. The most prefered wave length is shown to be about four times the distance from the shore to the breaker zone. Typical patterns of flow field and bottom configurations are represented. Finally, the theory is compared with field observations.

I INTRODUCTION

Recently two theories have been proposed on the mechanism of formation of rip-currents. Bowen's theory which has been presented in Journal of Geophysical Research in 1969 is based on the forced mechanism caused by standing edge waves which induce the spatially periodic distribution of radiation stress.

On the other hand, in 1972 the author proposed a hydrodynamic instability mechanism. A series of papers on this problem has been published in the Proceedings of Japanese Conference on Coastal Engineering and the Technical Report of Department of Civil Engineering, Tokyo Institute of Technology.

The author's basic idea is as follows; If waves are incident on a straight coastal line and if the water depth is uniform along it, the uniform wave-setup along the shoreline should be formed caused by the radiation stress of incident waves. Such a uniformly long wave-step would be unstable to an infinitesimal disturbance, as if a slender rod compressed axially buckles when a critical compressive stress is exceeded. Moreover, if waves are obliquely incident, the same mechanism as that for sand-wave formation in open channels will operate to form sand-bars caused by longshore currents.

II BASIC EQUATIONS

The motion of water is described by eqs. (1), (2) and (3) in terms of the mean surface elevation η , the vertically averaged horizental velocity components U and V in offshore and longshore directions and water depth below the still water surface h(x),

$$\begin{aligned}\rho\partial[u(h+\eta)]/\partial t &+ \partial[\rho(h+\eta)u^2]/\partial x \\ &+ \partial[\rho(h+\eta)uv]/\partial y + \partial s_{xx}/\partial x + \partial s_{xy}/\partial y \\ &= -\rho g(h+\eta)\partial\eta/\partial x - \rho\tilde{C}u \qquad \cdots \quad (1)\end{aligned}$$

$$\begin{aligned}\rho\partial[v(h+\eta)]/\partial t &+ \partial[\rho(h+\eta)uv]/\partial x \\ &+ \partial[\rho(h+\eta)v^2]/\partial y + \partial s_{xy}/\partial x + \partial s_{yy}/\partial y \\ &= -\rho g(h+\eta)\partial\eta/\partial y - \rho\tilde{C}v \qquad \cdots \quad (2)\end{aligned}$$

$$\begin{aligned}\partial(h+\eta)/\partial t &+ \partial[u(h+\eta)]/\partial x \\ &+ \partial[v(h+\eta)]/\partial y = 0 \qquad \cdots \quad (3)\end{aligned}$$

where $\tilde{C}$ means the frictional coefficient having dimension of velocity, and s_{xx}, s_{yy} and s_{xy} represent the radiation stress tensor introduced by Longuet-Higgins and Stewart (1964),

$$\left.\begin{aligned} & s_{xx} = 3E/2 - Ec^2(\sin\theta/c)^2, \quad s_{xy} = Ec\cos\theta(\sin\theta/c) \\ & s_{yy} = E/2 + Ec^2(\sin\theta/c)^2 \\ & E = \frac{1}{8}\rho g(2a)^2 \\ & (2a) = \gamma(h+\eta) \qquad (0 \leq x \leq L_B) \\ & \tilde{C} = \frac{2}{\pi}C'U_{max} = \frac{2}{\pi}C'(\frac{\gamma}{2}\sqrt{gh}) = C_d\sqrt{gh}, \; C_d = \gamma C'/\pi, \; C' = O(0.01) \end{aligned}\right\} \quad (4)$$

where E expresses energy of wave, c is wave celerity and L_B is the distance from shore to a breaker zone. By Snell's reflection law, ($\sin\theta$ /c) remains constant in course of the propagation of wave.

The last equation to solve the problem is the conservation equation of bottom material transport;

$$\partial h/\partial t = \partial(C_s u)/\partial x + \partial(C_s v)/\partial y \qquad \cdots \quad (5)$$

where C_s is the transport coefficient of bottom material and is considered to be dependent not only on the material but also on the wave characteristics and water depth. However, for the time being it is assumed to be a constant.

Small perturbations are imposed on the steady state. Hence, the variables are expressed as

$$\left.\begin{aligned} U &= u'(x, y, t) \\ V &= V_0(x) + v'(x, y, t) \\ \eta &= \eta_0(x) + \eta'(x, y, t) \\ h &= d(x) + h'(x, y, t) \end{aligned}\right\} \quad (6)$$

where $V_0(x)$ represents the longshore current, $\eta_0(x)$ is the wave setup in an equilibrium state, and $d(x)$ means the initial water depth.

Equation (6) is substituted into equations (1), (2), (3) and (5). The 0-th order terms yield the wave setup equations,

$$\partial\eta_0/\partial x = -(3\gamma^2/8)\cos^2\theta/[1+(3\gamma^2/8)\cos^2\theta]\cdot\partial h_0/\partial x \cdots (7)$$

and the velocity distribution of longshore current,

$$V_0(x) = -(5\gamma^2/16C_d)(\cos\theta\cdot\sin\theta_B\cdot\sqrt{h_{0B}+\eta_{0B}})[(h_0+\eta_0)/(h_{0B}+\eta_{0B})]$$
$$\cdot[d(h_0+\eta_0)/dx] \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (8)$$

On the other hand, the first-order perturbation terms give the linearized equation for perturbation variables. Henceforward, variables will be nondimensionalized by the gravitational acceleration g and the distance from shoreline to the breaking zone L_B. For instance $u/\sqrt{gL_B} \rightarrow u$, $x/L_B \longrightarrow x$ and $t\sqrt{g/L_B} \longrightarrow t$. Moreover, primes to express perturbed quantities will be omitted henceforth,

$$\frac{\partial w_i}{\partial t} + A_{i1}\frac{\partial u}{\partial x} + A_{i2}\frac{\partial u}{\partial y} + a_i u + B_{i1}\frac{\partial v}{\partial x} + B_{i2}\frac{\partial v}{\partial y} + b_i v$$
$$+ C_{i1}\frac{\partial \eta}{\partial x} + C_{i2}\frac{\partial \eta}{\partial y} + c_i\eta + D_{i1}\frac{\partial h}{\partial x} + D_{i2}\frac{\partial h}{\partial y} + d_i h = 0 \qquad (9)-(12)$$

$$(i = 1, 2, 3, 4)$$

where $w_1 = u$, $w_2 = v$, $w_3 = \eta$ and $w_4 = h$ and A_{i1}, A_{i2}, a_i etc. are functions of x which are complicated and lengthy; for instance,

$$\left.\begin{aligned} A_{12} &= V_0(x) \\ a_1 &= C_d/\sqrt{h_0+\eta_0} \\ C_{11} &= (3\gamma^2/8)[1-(h_0+\eta_0)(\sin^2\theta/c^2)]+1 \\ C_{12} &= (5\gamma^2/16)\cos\theta(\sin\theta/c)\sqrt{(h_0+\eta_0)} \\ c_1 &= -(3\gamma^2/8)\cdot(\sin^2\theta/c^2)[d(h_0+\eta_0)/dx] \\ D_{11} &= (3\gamma^2/8)[1-(h_0+\eta_0)(\sin^2\theta/c^2)] \\ D_{12} &= C_{12} \\ d_1 &= c_1 \end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned}
a_2 &= V_0'(x) \\
B_{22} &= V_0(x) \\
b_2 &= C_d / \sqrt{h_0+\eta_0} \\
C_{21} &= (5\gamma^2/16)[\cos\theta(\sin\theta/c)]\sqrt{h_0+\eta_0} \\
C_{22} &= 1+\gamma^2/8+(3\gamma^2/8)(\sin\theta/c)^2(h_0+\eta_0) \\
c_2 &= -(C_d/\pi)[V_0(h_0+\eta_0)^{3/2}]+(5\gamma^2/32)\cdot \\
&\qquad [\cos\theta(\sin\theta/c)](h_0+\eta_0)^{-1/2}(h_0'+\eta_0') \\
D_{21} &= C_{21} \\
D_{22} &= C_{22}-1 \\
d_2 &= c_2
\end{aligned}\right\} \quad (14)$$

III SOLUTION OF FULL EQUATIONS

In this section, a method is presented which solves straightforwards the full equations (linear simultaneous partial differential equations) for the four dependent variables, eqs. (9) through (12).

a) Linear ordinary differential equations

The temporal and spatial changes of the perturbation variables are expressed as eq. (15),

$$\left.\begin{aligned}
u(x,y,t) &= U(x)\exp[iky+pt] \\
v(x,y,t) &= V(x)\exp[iky+pt] \\
\eta(x,y,t) &= Z(x)\exp[iky+pt] \\
h(x,y,t) &= H(x)\exp[iky+pt]
\end{aligned}\right\} \quad (15)$$

The perturbations are assumed to be periodic with wave number k in the longshore direction, and to grow exponentially with time. If the real part of p which is to be determined later as an eigen-value problem is positive, the small perturbation is unstable to develop fully into large bottom configuration and strong longshore current system.

Substitution of the above Fourier component decomposition expression into eqs. (9) - (12) gives a system of linear simultaneous ordinary differential equations,

$$\left.\begin{aligned} F_1U(x) \qquad\qquad &+r_1Z'(x)+R_1Z(x)+s_1H'(x)+S_1H(x)=-pU(x)\\ F_2U(x)+G_2V(x)&+r_2Z'(x)+R_2Z(x)+s_2H'(x)+S\ H(x)=-pV(x)\\ f_3U'(x)+F_3U(x)+G_3V(x) \qquad &+R_3Z(x) \qquad\qquad +S_3H(x)=-pZ(x)\\ f_4U'(x) \qquad +G_4V(x) \qquad\qquad & \qquad\qquad\qquad\qquad =-pH(x) \end{aligned}\right. \tag{16}$$

where F_i, f_i, G_i, . . . ($i = 1, 2, 3, 4$) are functions of x and given as follows,

$$\left.\begin{aligned} F_1(x) &= C_d/[h_0(x)+\eta_0(x)]^{1/2} + ikV_0(x)\\ f_1 &= G_1 = g_1 = 0\\ R_1 &= -(3\gamma^2/8)(\sin\theta/c)^2(h_0'(x)+\eta_0'(x))\\ &\quad + ik(5\gamma^2/16)[\cos\theta(\sin\theta/c)](h_0(x)+\eta_0(x))^{1/2}\\ r_1 &= (3\gamma^2/8)[1-(\sin\theta/c)^2(h_0(x)+\eta_0(x))]+1\\ S_1 &= R_1\\ s_1 &= r_1-1 \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} F_2(x) &= dv_0(x)/dx\\ G_2(x) &= C_d/\sqrt{h_0(x)+\eta_0(x)} + ikV_0(x)\\ R_2(x) &= (5\gamma^2/32)[\cos\theta(\sin\theta/c)](h_0(x)+\eta_0(x))^{-1/2}\cdot(h_0'(x)\\ &\quad +\eta_0'(x)) - C_d V_0(x)/[2(h_0(x)+\eta_0(x))^{3/2}] + ik[(\gamma^2/8)\\ &\quad +(3\gamma^2/8)(\sin\theta/c)^2(h_0(x)+\eta_0(x))+1]\\ r_2(x) &= (5\gamma^2/16)\cos\theta(\sin\theta/c)\sqrt{(h_0(x)+\eta_0(x))}\\ S_2(x) &= R_2(x) - ik\\ s_2(x) &= r_2(x)\\ f_2(x) &= g_2 = 0 \end{aligned}\right\} \tag{18}$$

$$\left.\begin{aligned} f_3 &= h_0(x)+\eta_0(x)+C_s\\ F_3 &= h_0'(x)+\eta_0'(x)\\ G_3 &= ik[h_0(x)+\eta_0(x)+C_s]\\ R_3 &= ikV_0(x)\\ S_3 &= ikV_0(x)\\ g_3 &= r_3 = s_3 = 0 \end{aligned}\right\} \tag{19}$$

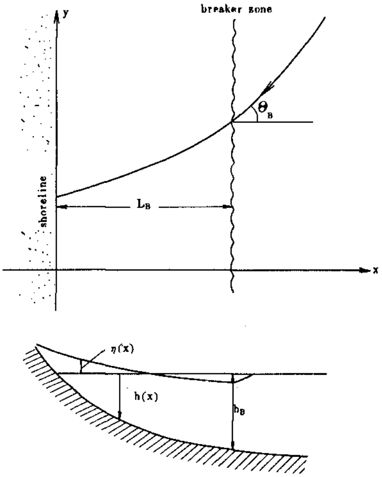


Fig. 1 : Coordinate system and symbols

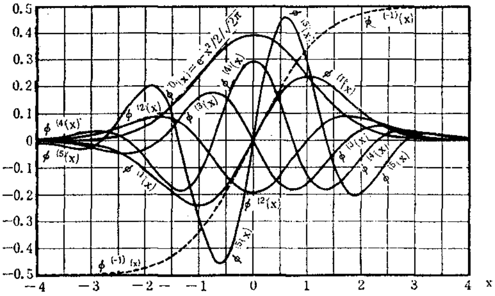


Fig. 2 : Graph of Hermitian polynomial functions mutiplied by exp $(-x^2/2)/n!$

$$\left.\begin{aligned}
&f_4 = -C_s \\
&G_4 = -ikC_s \\
&F_4 = g_4 = R_4 = r_4 = S_4 = s_4 = 0
\end{aligned}\right\} \quad (20)$$

b) Boundary conditions

The boundary conditions imposed on the above system of differential equations are written as

$$\left.\begin{aligned}
&U(x) = 0 && (x = 0) \\
&U(x) \to 0 && (x \to \infty) \\
&Z(x) \to 0 && (x \to \infty) \\
&H(x) \to 0 && (x \to \infty)
\end{aligned}\right\} \quad (21)$$

c) Eigenvalue problem, Determination of p

Equations (16) together with the imposed boundary conditions at s at the coast and the infinity (21) constitute the eigenvalue problem. Two methods of solution may be applied; that is, (a) the finite diference approximation of equations (16) reduces them into a set of simultaneous linear equations which requires the determinant of coefficient matrix to be zero. (b) The dependent variables are expressed by series of Hermitian polynomials which are substituted into equations (16). By applying the minimum weighted residual method, a set of linear equations for the expansion coefficients is obtained which also defines the eignevalue problem.

The unknown functions U, V, H and Z of x, are expanded as the series of Hermitian polynomial functions. Figure 2 shows the Hermitian polynomial functions of the lower order multiplied by $e^{-x^2/2}$. As may be imagined, these curves seem to represent the real bottom topography. As a consequence, it is expected that the Hermitian polynomial series expansion will converge rapidly,

$$\left.\begin{aligned}
U(x) &= \sum_{n=1}^{\infty} \alpha_n e^{-x^2/2} H_{2n-1}(x) \\
V(x) &= \sum_{n=1}^{\infty} \beta_n e^{-x^2/2} H_{2(n-1)}(x) \\
Z(x) &= \sum_{n=1}^{\infty} \gamma_n e^{-x^2/2} H_{2(n-1)}(x) \\
H(x) &= \sum_{n=1}^{\infty} \delta_n e^{-x^2/2} H_{2(n-1)}(x)
\end{aligned}\right\} \quad (22)$$

where H_n's are the Hermite polynomial functions. In order to satisfy not only the boundary condition of U(x) defined at $x = \infty$ but also one at $x = 0$, only the odd order Hermite polynomials are considered in the first expression of eq. (22). While other variables (V(x), Z(x) and H(x)) are expressed as sum of the even order Hermite functions.

The Hermitian series expression eq. (22) is substituted into eq. (16). These equations are multiplied by the n-th order Hermite function and integrated with respect to x from 0 to infinity. This procedure is the so-called moment method (one of minimum weighted residual methods). If we truncate the Hermite expansion at the n-th term, we obtain the 4n × 4n dimensional linear simultaneous equations for the unknown expansion coefficients. These equations are represented in the matrix form as given by eq. (23). This matrix system defines the typical eigenvalue problem of p.

$$\begin{pmatrix} I^{(1)} & O & K^{(1)} & L^{(1)} \\ I^{(2)} & J^{(2)} & K^{(2)} & L^{(2)} \\ I^{(3)} & J^{(3)} & K^{(3)} & L^{(3)} \\ I^{(4)} & J^{(4)} & O & O \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \\ \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix} = -p \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \\ \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \\ \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix} \qquad (23)$$

$$\mathbf{A} \cdot \mathbf{Y} = (-p)\,\mathbf{Y} \qquad \cdots \quad (23\text{ a})$$

where

$$\mathbf{Y} = [\alpha_1, \alpha_2, \cdots \alpha_n, \beta_1, \cdots \beta_n, \gamma_1, \cdots \gamma_n, \delta_1, \cdots \delta_n]^T$$

In this matrix representation, the elements **I**, **J**, **K** and **L** further constitute the submatrix system which are to be determined by the equations given

$$I^{(1)}_{m,n} = \frac{\sqrt{2}}{(2m-1)!\sqrt{\pi}} \cdot \int_0^\infty e^{-x^2/2} H_{2m-1}(x) H_{2n-1}(x) F_1(x) dx \quad \cdots (24)$$

$$K^{(1)}_{m,n} = \frac{\sqrt{2}}{(2m-1)!\sqrt{\pi}} \cdot \int_0^\infty e^{-x^2/2} H_{2m-1}(x) [\{H'_{2(n-1)}(x) - xH_{2(n-1)}(x)\}$$
$$r_1(x) + R_1(x) H_{2(n-1)}(x)] dx \quad \cdots (25)$$

$$L^{(1)}_{m,n} = \frac{\sqrt{2}}{(2m-1)!\sqrt{\pi}} \cdot \int_0^\infty e^{-x^2/2} H_{2m-1}(x) [\{H'_{2(n-1)}(x) - xH_{2(n-1)}$$
$$(x)\} s_1(x) + S_1(x) H_{2(n-1)}(x)] dx \quad \cdots (26)$$

and for i = 2, 3 and 4

$$I^{(i)}_{m,n} = \frac{\sqrt{2}}{(2(m-1))!\sqrt{\pi}} \cdot \int_0^\infty e^{-x^2/2} \cdot H_{2(m-1)}(x) \cdot [-f_i(x) H_{2n}(x) + H_{2n-1}(x) F_i(x)] dx \quad \cdots \quad (27)$$

$$J^{(i)}_{m,n} = \frac{\sqrt{2}}{(2(m-1))!\sqrt{\pi}} \cdot \int_0^\infty e^{-x^2/2} \cdot H_{2(m-1)}(x) H_{2(n-1)}(x) G_i(x) dx \quad \cdots \quad (28)$$

$$K^{(i)}_{m,n} = \frac{\sqrt{2}}{(2(m-1))!\sqrt{\pi}} \cdot \int_0^\infty e^{-x^2/2} \cdot H_{2(m-1)}(x) \cdot [-r_i(x) H_{2n-1}(x) + H_{2(m-1)}(x) R_i(x)] dx \quad \cdots \quad (29)$$

$$L^{(i)}_{m,n} = \frac{\sqrt{2}}{(2(m-1))!\sqrt{\pi}} \cdot \int_0^\infty e^{-x^2/2} H_{2(m-1)}(x) \cdot [-H_{2n-1}(x) s_i(x) + H_{2(n-1)}(x) S_i(x)] dx \quad \cdots \quad (30)$$

The eigenvalue problem posed by eq. (23) is solved numerically applying the Library subroutine supplied for the system HITAC 8800. The results will be presented in the subsequent section.

After inspection of the components of eigenvector coresponding to a maximum eigenvalue, it was found that the maximum of the eigenvector components (that is the coefficients α_n, β_n, γ_n and δ_n) corresponds not necessarily to δ_n's, the coefficients in the series expansion of bottom deformation. That is to say, the instability mechanism may sometimes dominate in the fluid system rather than in the bottom topography. In such a case, it was also found that the longshore celerity of instability propagation which is given by Im(p)/k where Im means the imaginary part is too much higher than that observed in real phenomena. This may be spurious instability caused by the temporally averaged fundamental equation, which necessitated us to proceed to the improved analysis to be described in the next section. In consequence of these discussions, the bottom mode maximum eigenvalue is defined for which the maximal value of 32 components of the eigenvector corresponds to anyone of the coefficients of δ_n. The surface or fluid mode maximum eigenvalue is defined for which the maximum conponent of the corresponding eigenvector is either one of α_n, β_n or γ_n.

Several parameters are grouped into the following two parameters,

$$\left.\begin{aligned} \Phi &= (3\gamma^2/8C_d)(dh/dx)_B \\ \Lambda &= C_s/C_d\, h_{0B} \end{aligned}\right\} \quad (31)$$

IV. METHOD OF SOLUTION BASED ON RESPONSE TIME CONCEPT

a) Concept of response time

To solve analytically a system of Eqs. (9) to (12) seems to be formidably difficult. Methods of analysis of the modern fluid dynamics are to attack complicated problems not purely mathematically but to solve them after simplification of original equations through the physical interpretation of the basic equations. This attitude has been established by L. Prandtl when he proposed the concept of boundary layer in 1904.

Turning to our problem, the fact is soon appreciable that the response of fluids is quick to the deformation of bottom boundary, while the bottom materials respond very much slowly to the changes in the fluid system. Therefore, the state of fluid system may be considered to be quasi-stationary; that is u, v and η are weakly time-dependent only through a gradual change in the bottom topography h(x,y; t). The concept of response time has already been proposed and applied in the first paper of a series of author's reports on this problem in which the Fourier series expansion is used to express the unknown variables as a preliminary attack.

b) Quasi-stationary solution for fluid system

In the linearized partial differential equations except the last one, i.e.eqs. (9) - (11), the terms of partial differentiation with respect to time are omitted to be solved for a given stationary bottom configuration. The unknown variables u, v and η and the given bottom depth h are expressed in terms of the Hermite polynomials as follows,

$$\left.\begin{aligned}
u(x,y) &= \sum_{n=1}^{\infty} \alpha_n H_{2n-1}(x) e^{-x^2/2} e^{iky} \\
v(x,y) &= \sum_{n=1}^{\infty} \beta_n H_{2(n-1)}(x) e^{-x^2/2} e^{iky} \\
\eta(x,y) &= \sum_{n=1}^{\infty} \gamma_n H_{2(n-1)}(x) e^{-x^2/2} e^{iky} \\
h(x,y) &= \sum_{n=1}^{\infty} \delta_n H_m(x) e^{-x^2/2} e^{iky} \\
&(m = 2(n-1) \text{ or } m = 2n-1)
\end{aligned}\right\} \qquad (32)$$

In the above equations, the function h(x, y) and thus δ_n's are assumed to be given, for the time being; while u, v and η, that is α_n's, β_n's and γ_n's are considered unknown. In order to elucidate clearly the way of thinking, the general expressions of eqs. (32) are simplified, taking only one term of the series but instead multiplying them by $\exp(ik_1 x)$ to compensate for the elimination of other terms, into eq. (33)

$$\left.\begin{aligned}
u(x,y) &= \alpha H_1(x)\, e^{-x^2/2} e^{i(k_1 x + k_2 y)} \\
v(x,y) &= \beta H_1(x)\, e^{-x^2/2} e^{i(k_1 x + k_2 y)} \\
\eta(x,y) &= \gamma Hm(x)\, e^{-x^2/2} e^{i(k_1 x + k_2 y)} \\
h(x,y) &= \delta Hm(x)\, e^{-x^2/2} e^{i(k_1 x + k_2 y)}
\end{aligned}\right\} \qquad (33)$$

Substituting eqs. (33) into eqs. (9) - (11) and applying the moment method (i.e. integrating both sides between $(0, \infty)$ after multiplication by $H_m(x)$, one obtains the linear simultaneous equations for the unknown coefficients, α, β and γ ;

$$\begin{pmatrix} I_1 & J_1 & K_1 \\ I_2 & J_2 & K_2 \\ I_3 & J_3 & K_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = -\delta \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} \tag{34}$$

where

$$\left.\begin{aligned} I_i &= \int_0^\infty [\{ik_1 A_{i1}(x) + ik_2 A_{i2}(x) + a_i(x)\} H_1(x) - A_{i1}(x) H_2(x)] H_m(x) e^{-x^2/2}\, dx \\ J_i &= \int_0^\infty [\{ik_1 B_{i1}(x) + ik_2 B_{i2}(x) + b_i(x)\} H_2(x) - B_{i1}(x) H_3(x)] H_m(x) e^{-x^2/2}\, dx \\ K_i &= \int_0^\infty [\{ik_1 C_{i1}(x) + ik_2 C_{i2}(x) + c_i(x)\} H_m(x) - C_{i1}(x) H_{m+1}(x)] H_m(x) e^{-x^2/2}\, dx \\ L_i &= \int_0^\infty [\{ik_1 D_{i1}(x) + ik_2 D_{i2}(x) + d_i(x)\} H_m(x) - D_{i1}(x) H_{m+1}(x)] H_m(x) e^{-x^2/2}\, dx \end{aligned}\right\} \tag{35}$$

Therefore, the unknown coefficients α , β and γ are obtained

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = -\delta \begin{pmatrix} I_1 & J_1 & K_1 \\ I_2 & J_2 & K_2 \\ I_3 & J_3 & K_3 \end{pmatrix}^{-1} \begin{pmatrix} L_1 \\ L_2 \\ L_2 \end{pmatrix} \tag{36}$$

c) Instability of bottom boundary

If the coefficient δ is considered to be a slowly varing function of time as suggested by the response time concept, the velocity field u and v are also weak functions of time.

Equations (33) together with eq. (36) are substituted into the conservation equation of bottom material transport, eq. (12), to obtain

$$\left.\begin{aligned} H_m(x) \frac{\dot{\delta}(x)}{\delta(x)} &= C_s [\alpha_*(ik_1 H_1(x) - H_2(x)) \\ &\quad + \beta_*(ik_2) H_2(x)] + \varepsilon \end{aligned}\right\} \tag{37}$$

where ε is a residual term and $\alpha_* = \alpha/\delta$, $\beta_* = \beta/\delta$. Applying the moment method, (that is multiplying $H_m(x)$ on both side of the above equation and integrating in the range $(0, \infty)$, one obtains the solution,

$$\delta(t) = Ae^{pt} \tag{38}$$

where A is an integration constant and p is given by

$$p = \frac{\sqrt{2}\ C_s}{\sqrt{\pi}\ m!} \left[\alpha_* \int_0^\infty \{ ik_1\ H_1(x) H_m(x) - H_2(x) H_m(x) \} e^{-x^2/2}\, dx \right. \\ \left. + \beta_* \int_0^\infty \{ ik_2\, H_2(x) H_m(x) e^{-x^2/2}\, dx \} \right] \tag{39}$$

The initial perturbations may be stable or unstable depending on the sign of the real part of p,

$R(p) > 0$: unstable

$R(p) = 0$: neutral

$R(p) < 0$: stable

The maximum value of R (p) has been sought for a family of parameters k_1 and m for the fixed values of the langshore wave number of rip current and cusp k_2 , the incidence angle of waves θ , and the bottom slope dh/dx.

V. THEORETICAL RESULTS AND COMPARISON WITH FIELD OBSERVATIONS

a) Stability diagram

Figures 3 a) and b) show examples from results of this eigenvalue problem obtained by the methods described in III and IV, respectively, (Hino (1973 a and b)). In these figures, the abscissa is the wave number k in the direction of shoreline, and the ordinate is the real part of the maximum eigen values. It is evident that the most unstable mode appears at k = 1.6, that is the wave length in longshore direction of rip current and cusp is

$$L = 2\pi / k \cong 4 \tag{40}$$

Since unit of length is chosen as a distance from shoreline to breaker zone, the most prefered wavelength L_r of rip and cusp is about four times the distance from the shore to the breaker zone L_B ,

$$L_r \cong 4\ L_B \tag{40a}$$

b) Theoretical prediction of flow fields and bottom topography

Simultaneously with the determination of eigenvalue p, the eigen vectors which are coefficients of the Hermite series are determined. Therefore, the rip current and cuspidal bottom system are reconstructed from eq. (22), and shown in figures 4, 5 and 6. Figure 4 illustrates the results of theoretically determined flow field, bottom topography for the case of normal wave incidence. The cellular flow field develops and the so-called rip channel is to be formed. Figures 5 and 6 are also the theoretical results for an oblique wave incidence. It is noted that the longshore current becomes to meander and the sand bars to develop.

c) Comparison with field observation

Figure 7 cited from the report on field experiments (Public Works

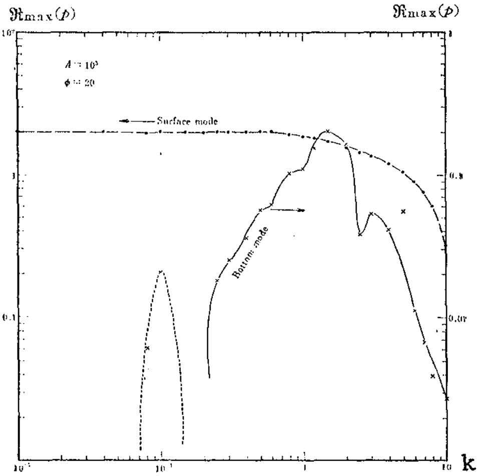


Fig. 3 a) : An example of the instability-curve determined by the method described in III.

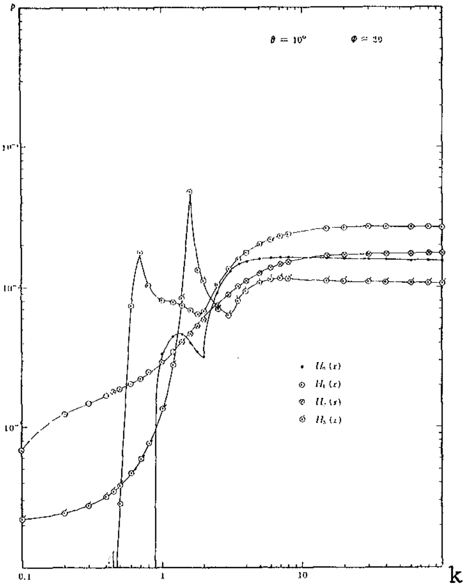


Fig. 3 b) : An example of the instability-curve determined by the method described in IV.

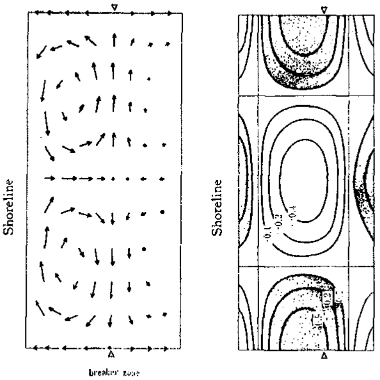


Fig. 4 : The cellular pattern of secondary current (left) and perturbation in bottom topography - the shaded areas are scoured - (right), for normal wave incidence, k=1.6 , Φ=2, $\Lambda=10^3$.

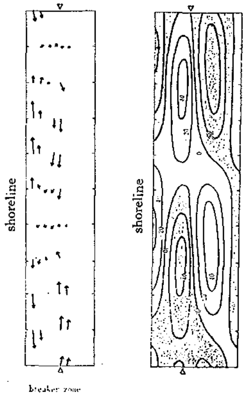


Fig. 5 : The cellular partern of secondary current (left) and the accompanied perturbation in bottom topography - the shaded regions are scoured (right); $\Theta=10^{\circ}$, k=0.6, Φ=2, $\Lambda=10^3$.

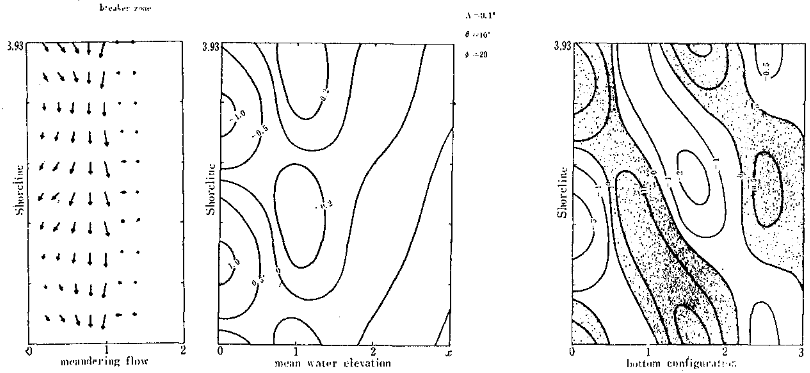


Fig. 6 : The meandering flow pattern composed of the basic longshore current and secondary perturbation flow (left), the perturbation in mean water elevation (middle) and the perturbation in bottom topography - the shaded areas are scoured (right) ; $\Theta=10^{\circ}$, k=1.6, Φ=20, $\Lambda=10^3$.

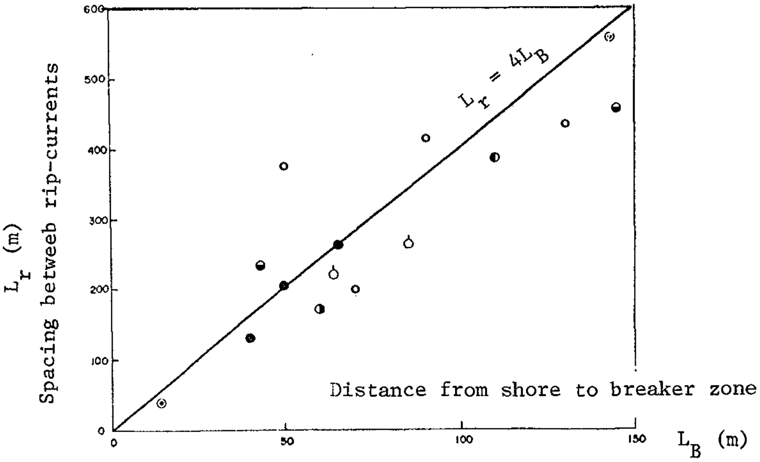


Fig. 7 : Comparison between the theoretical prediction and the field observation (cited from the report of Public Works Research Institute, Ministry of Construction) on the relation of $Lr \cong 4.0\ L_B$, eq. (40 a).

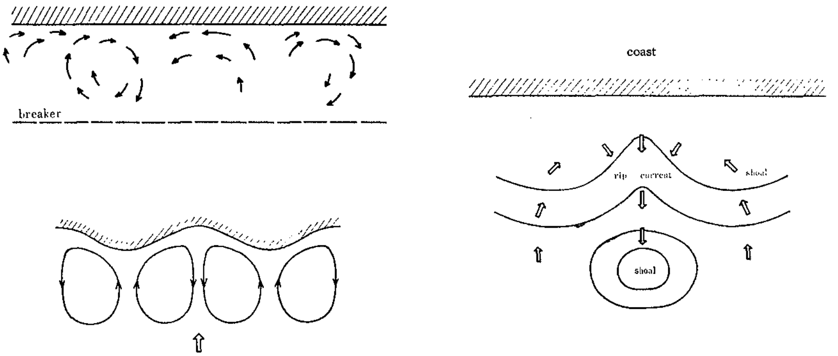


Fig. 8 : Schematic representation of cellular flow patterns (above left) from the papers by Horikawa et al. (1971) and Sonu (1972), and of rip current and rip-channel (right) drawn schematically by the present author based on the field observation by Sonu (1972).

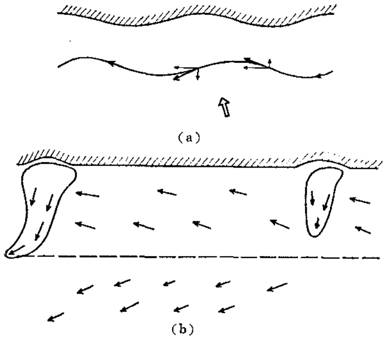
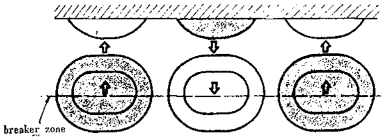
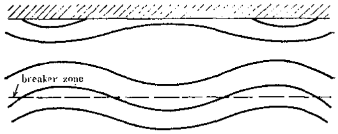


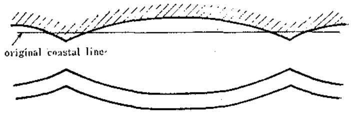
Fig. 9 : Field observation of meandering shore currents for oblique wave incidence



(a) Perturbation in bottom depth



(b) Change in bottom depth contour



(c) Cuspidal coast and crescent bar
(fully developed stage)

Fig. 10 : Theoretically estimated change in bottom depth contour and formation of cuspidal coast and crescent bar.

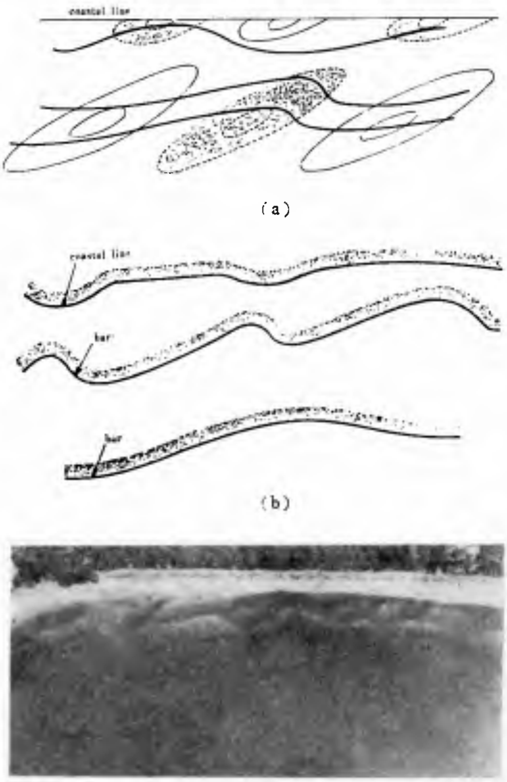


Fig. 11 : Theoretically estimated bottom contour (a) and longshore meandering bars by aerial photography (b)

Research Institute, Ministry of Construction) represents the relationship between the wave length of rip current generation L_r and the distance from shore to breaker zone L_B , showing a good agreement with the theoretical prediction, eq. (40a).

Figure 8 is the schematic representation of results of field observation by Horikawa et al (1971) and Sonu (1972) of longshore current system when waves are incident normally. These figures agree qualitatively with figure 4. In comparing the theory with expriments, it should be remarked that the theory predicts the initial stage of developments which may be treated by linearization, while field data are obtained in general for the fully developed stage where the nonlinear effects predominate.

Figure 9 illustrates schematically the results of field observation performed by Horikawa et al (1971) and Sonu (1972) which are to be compared with the theoretical results, figures 5 and 6.

The process of development of cuspidal coast and crescent bar may be explained by the author's theory. The cyclic perturbation in bottom depth accompanied by periodic shore current system (figure 10a)), if it is superposed on the original bottom configuration, composes undulating contour lines (fig. 10 b)). If the instability mechanism develops further into the nonlinear region, the offshore currents are intensified, the onshore currents becoming weak. At the same time, shorelines will be cyclically excavated and deposited (fig. 10 c)).

Formation of longshore meandering bars (fig. 11 b)) may also be explained by the superposition of perturbed bottom and the original contour line (fig. 11 a)).

VI. CONCLUSION

Formation of the systematic feature of shore current and shore bottom topography is shown to be results of the hydrodynamic instability caused by the radiation stresses. The linearized perturbation equations for the vertically averaged velocity components u and v, the water surface elevation η and the scouring depth of bottom h are derived. The dependent variables are assumed to have functional forms such as $u(x, y, t) = U(x)\exp(iky + pt)$ and so on. Consequently, the fundamental partial differential equations are reduced to the simultaneous ordinary differential equations for variations in the offshore direction. These constitute an eigenvalue problem for the temporal exponential growth rate p under the given boundary conditions.

Two methods of solving the problem are presented. One is the moment method (a procedure of the minimum weighted residual methods) based on the Hermite polynomial expansions of the dependent variables. The other is the quasi-stationary analysis based on the response time concept. Both methods reduce the simultaneous ordinary differential equations to the matrix form of eigenvalue problem.

Results of computation show that the system of shore current and bottom topography is hydrodynamically unstable for a small perturbation.

For a given bottom movability, the temporal exponential growth rate has a peak value at a certain longshore spacing about four times a distance from shoreline to breaken zone.

The cellular flow patterns of perturbed secondary currents as well as the contour maps of bottom scouring and water surface elevation are shown. The several results obtained in this investigation conform with results of qualitative field observations reported by oceanographical geologists and coastal engineers.

ACKNOWLEDGEMENT : The writer wishes to express his thanks to Professor C. J. Sonu, Coastal Studies Institute, Louisiana State University, presently Tetra Tech Inc., by whom he has been stimulated very much during his stay in L.S.U. in May, 1971.

REFERENCES

Bowen, A. J. (1969) : Rip currents, 1. Theoretical investigations, J. Geophy. Res., Vol. 74, No. 23, pp. 5467-5478.

Hino, M. and Hayashi, N. (1972) : Theory on generation of rip currents, Proc. 19th Conference on Coastal Eng., Japan Soc. of Civil Engrs., pp. 93-96. (in Japanese); Coastal Engineering in Japan Vol. 16, 55-60, (in English)

Hino, M. (1972) : Shore-current system and deformation of shore bottom, Preprint of 1972 Fall Conference of Oceanographical Soc. of Japan, pp. 151-152. (in Japanese) ; Also Technical Report No. 13, pp. 99-113, Department of Civil Eng., Tokyo Inst. of Tech., Dec. (in English)

Hino, M. (1973 a) : Hydrodynamic instability theory on the formation of systematic shore current and coastal topography, Part II Solution by Galerkin's procedure based on Hermitian polynomial expansion, Technical Report No. 14, pp. 27-41, Department of Civil Eng., Tokyo Inst. of Tech., June. (in English)

Hino, M. (1973 b) : Theory on generation of rip currents, Part III A simplified theory, Proc. 20th Conference on Coastal Eng., J.S.C.E., pp. 339-344.

Horikawa, K., Sasaki, T., Fukui, W., Igarashi, H. and Kito, H. (1971) : Field observation of shore currents (Part I), Proc. 18th Conference on Coastal Eng. (Japan), Japan Soc. of Civil Engrs., pp. 395-400.

Longuet-Higgins, M. S. (1970) : Longshore currents generated by obliquely incident sea waves, 1., J. Geophy. Res., Vol. 75, No. 33, pp. 6778-6789.

Longuet-Higgins, M. S. and Stewart, R. W. (1964) : Radiation stresses in water waves ; a physical discussion, with applications, Deep-Sea Res., Vol. 11, pp. 529-562.

Public Works Research Institute, Ministry of Construction (1974) : Field observation on wave and coastal currents, Tech. Memo. No. 916.

Sonu, C. J. (1972) : Field observation of nearshore circulation and meandering currents, J. Geophy. Res., Vol. 77, No. 18, pp. 3232-3247, June.

Sonu, C. J. (1972) : Comment on paper by A. J. Bowen and D. L. Inman, 'Edge wave and crescentic bars' J. Geophy. Res.