CHAPTER 46

SEDIMENT TRANSPORT IN RANDOM WAVES AT CONSTANT WATER DEPTH

by

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Abstract

Sediment transport in random waves at constant water depth is analyzed by dividing the flow field into two regions--the internal region and the boundary layer region. The suspension and transport of sediment in these two regions are treated separately. The total transport is then obtained as the total of these regions through matching boundary.

In the bed layer, the load concentration is assumed to be proportional to the specific weight of the sediment and to the probability that the fluctuating lifting force exceeds the weight of the sediment. The bed load is then transported by the secondary flow (which is unidirectional) in the boundary.

In the internal flow region, the sediment suspension is treated as a diffusion problem with the intensity of diffusion to be proportional to the amplitude of fluid particle motion. The transport velocity is assumed to be the same as the mass transport velocity of the wave.

The predominant mode of transport is found to be suspended load. The total transport in a wind-generated wave field can be expressed as a power law of wind speed. For the case tested, the power should be of the order of 4.

1. INTRODUCTION

The analysis presented in this paper applies only to regions well beyond the surf zone where the flow field, though irregular in appearance, can be described in reasonable detail without going into micro-structures. In other words, the water surface variations, the internal flow characteristics, and the pressure field can still be expressed in manageable mathematical expressions in gross terms.

To facilitate calculations of sediment motion, the flow field is divided into two zones. The upper zone, where viscosity plays a negligible role except

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on water mass transport, is termed zone of suspended load. In this zone, sediment is in a state of suspension due to the combined effects of turbulence and orbital motion of fluid mass. The lower zone, which is a thin turbulent boundary layer near the sea bed created by wave motion overhead, is termed bed load layer following Einstein's terminology. In this zone, sediment particles roll, slide, and sometimes jump. As stated by Einstein (1972), the measureable thickness of this turbulent boundary layer is usually rather small; 5 - 10 m.m. are very common values. However, since it is directly adjacent to the sea bed, it could be very effective in sediment transport. The suspended load can be considered to occupy the full water depth. Therefore, although the sediment concentration may be relatively small, the transport rate may still be appreciable. From these observation, sediment transport in both zones need proper attention.

Therefore, the analysis naturally breaks down into investigating the following four parts systematically: 1) Bed Load Concentration, 2) Bed Load Transport, 3) Suspended Load Concentration, and 4) Suspended Load Transport.

The problems are first formulated for monocromatic wave train and the results are then generalized for random waves. The total sediment transport rate is obtained through matching boundary conditions of the two zones. Because of the complicated mathematical formulations, computer programs are developed to facilitate numerical computations.

2. BED LOAD

Concentration--Let the bed-load concentration be expressed as $\bar{C}_o = N_s W$ where N_s is the number of particles per unit volume inside the bed layer and $W = A_1 D^3 \gamma_s$ is the dry weight of sediment particle. Since in a unit volume N_s is inversely proportional to the size of the sediment particle and is proportional to the probability, p, of a sediment particle to be set in motion, we have

$$C_o = A_o \; p \; \gamma_s \qquad (1)$$

where A_o is an empirically determined constant. The probability, p, is equal to the probability that the instantaneous lift exceeds the weight of the particle (Einstein, 1950), i.e.,

$$p = p \; (\bar{L} + L' > W)$$

where $\bar{L}$ is the mean lift force and L' is the fluctuating component of the lift force.

On the further assumption that in a turbulent boundary layer L' is a random process following a normal distribution of zero mean and standard deviation proportional to the mean lift ($\sigma = \eta_o L$, where η_o is a constant of proportionality), Einstein arrived at the following expression for the probability distribution function

$$p = \frac{1}{\sqrt{2\pi}} \int_{B_* \psi - \frac{1}{\eta_o}}^{\infty} e^{-\frac{z^2}{2}} dz \tag{2}$$

where $B_* = \frac{2A_3}{C_L \eta_o}$; D is equivalent diameter of the sediment particle; C_L is the lift coefficient of sediment particle; A_3 is a constant and $\psi = \frac{(\rho_s - \rho)\, gD}{\rho\, u^{-2}}$

is known as the flow intensity function in which u is the amplitude of the oscillatory flow velocity in the direction of wave propagation. Therefore, the problem of determining the bed-load concentration, aside from a number of constants of proportionality, becomes a problem of determining the horizontal velocity in the boundary layer.

Analogous to the laminar case (Schlichting, 1960), Kalkanis (1964) proposed the following expression for the horizontal velocity component in a fluctuating furbulent boundary layer.

$$u = u_o \{1 + f_1^2 - 2f \cos f_2\}^{1/2} \cos(wt + \theta) \tag{3}$$

in which $f_1 = 0.5\, e^{-\frac{133}{a\beta D} y}$, $f_2 = 0.5\, (\beta y)^{2/3} \approx 0.3\beta y$, $\theta = \tan^{-1} \frac{f_1 \sin f_2}{1 - f_1 \cos f_2}$, and $\beta = \sqrt{\frac{\omega}{2\gamma}}$. Therefore, the bed-load concentration in a monocromatic waye train can be determined through Eqs. (1), (2), and (3). For, random wayes, the velocity is expressed by

$$u = \sum_i u_{oi} \left[\cos(k_i x - \omega_i t + \varepsilon_i) - f_{1i} \cos(k_i x - \omega_i t + f_{2i} + \varepsilon_i)\right] \tag{4}$$

where $\sum_i U_{oi} = \int_0^\infty S(\omega) d\omega$ and $S(\omega)$ the spectral density function of wave energy.

Transport

Within the boundary layer of a wave field, the first order effect is to relieve the sediment particles of all or part of their weight so as to bring them into a state of incipeint equilibrium. At this state, if there is no other incidental current, the net transport will result only from higher order effect.

Now treat the mass transport velocity as the second order effect such that, to the second order, the velocity field inside the boundary can be expressed as:

$$(u + U_e)\ \hat{e}_x + (v + V_e)\ \hat{e}_y$$

where U_e and V_e are the second order velocity components and are of the order of εu and εv, respectively. The ε is a perturbation parameter of the order of ak for deep water wave and hk for shallow water wave. Substituting the above expression into the boundary layer equation in the horizontal direction yields,

$$\frac{\partial(u + U_e)}{\partial t} + \frac{\partial(u + U_e)^2}{\partial x} + \frac{\partial(u + U_e)(v + V_e)}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2}{\partial y^2}(u + U_e) \qquad (5)$$

To the second order, the pressure inside the boundary layer is equal to that at the outer edge of the boundary layer, the mean momentum equation inside the boundary layer reduces to a balance between shear stress and Reynolds stress:

$$\frac{\partial(\overline{uv})}{\partial y} = \nu\frac{\partial^2 \overline{U}_e}{\partial y^2} \qquad (6)$$

From Eq. (3) or (4) the vertical velocity component inside the boundary layer can be obtained through continuity relation. The shear stress $\overline{uv}$ can then be solved through the fact that u and v are orthogonal functions. The second order horizontal velocity $\overline{U}_e$ is then solved from Eq. (6) when the following boundary conditions are specified:

(1) At the outer edge of the boundary layer, the mean shear stress is negligible.

$$\frac{\partial U_e}{\partial y} = 0 \qquad \text{at} \qquad y = \infty$$

(2) At the bottom, the velocity is zero

$$\overline{U}_e = 0 \qquad \text{at} \qquad y = 0$$

The solution obtained is in the Eulerian reference frame. The true mass transport velocity, which is a Lagrangian property, is obtained through transformation of coordinate (Longuet-Higgins, 1953) by

$$\overline{U} = \overline{U}_e + \left(\int u\,dt\right)\frac{\partial u}{\partial x} + \left(\int v\,dt\right)\frac{\partial u}{\partial y} \tag{7}$$

The solution is (Liang and Wang, 1973)

$$\begin{aligned}
\overline{U} = \sum_i \frac{0.5\, k_i\, u_{oi}^2}{2\nu(E_i^2 + 0.09\beta_i^2)} \{&-e^{-E_i y} \sin 0.3\beta_i y\left[E_i y + \frac{1.5\, E_i^2 - 0.225\beta_i^2}{E_i^2 + 0.09\beta_i^2}\right] \\
&-e^{-E_i y} \cos 0.3\beta_i y\left[0.3\beta_i y + \frac{1.2\,\beta_i\, E_i}{E_i^2 + 0.09\beta_i^2}\right] \\
&+\frac{0.15\,\beta_i}{2\,E_i}\left(e^{-2E_i y} - 1\right) + \frac{1.2\,\beta_i\, E_i}{E_i^2 + 0.09\beta_i^2}\} \\
+ \sum_i \frac{k_i\, u_{oi}^2}{2\,\omega_i} \{&1 - e^{-E_i y} \cos 0.3\beta_i y + 0.25\, e^{-2E_i y} \\
&+ 0.5\, e^{-E_i y}\, y\left[E_i \cos 0.3\beta_i y + 0.3\beta_i \sin 0.3\beta_i y\right]\} \\
+ \sum_i \frac{0.5\, k_i\, u_{oi}^2}{2\omega_i(E_i^2 + 0.09\beta_i^2)} \{&0.5(E_i\, e^{-E_i y})^2 - 0.5(0.3\beta_i e^{-E_i y})^2 \\
&+ 0.5\, e^{-E_i y} \cos 0.3\beta_i y\left[-E_i^2 + (0.3\beta_i)^2\right] \\
&- 2(0.5)E_i 0.3\beta_i\, e^{-E_i y} \sin 0.3\beta_i y\}
\end{aligned} \tag{8}$$

where $E_i = \dfrac{133\ \sinh(k_i h)}{a_i \beta_i D}$

3. SUSPENDED LOAD

Concentration

Generally, sediment suspension in a fluid media is treated as a diffusion-dispersion process. This line of approach is followed here. The diffusion equation takes the following form:

$$\frac{\partial C}{\partial t} + \nabla \cdot (C\underset{\sim}{V}_s) = \nabla \cdot (e_m \nabla C) \tag{9}$$

where C is sediment concentration (Mass/volume); V_s is the particle velocity vector and e_m is the diffusion coefficient. For suspended sediment in a wave field, the concentration and velocity can both be divided into three components:

$$C = \overline{C}(x,y) + C_w(x,y,t) + C'(x,y,t)$$

$$\underset{\sim}{V}_s = \overline{\underset{\sim}{V}} + \underset{\sim}{V}_w + \underset{\sim}{V}'$$

Here, $\overline{C}$ and $\overline{\underset{\sim}{V}}$ are the mean values, C_w and $\underset{\sim}{V}_w$ are the wave induced components, and C', $\underset{\sim}{V}'$ are the turbulent fluctuations. Substituting C and V into the diffusion equation and taking the time average yields

$$\nabla \cdot (\overline{C}\ \overline{\underset{\sim}{V}} + \overline{C_w \underset{\sim}{V}_w} + \overline{C'\underset{\sim}{V}'}) = \nabla \cdot (e_m\ \nabla\overline{C}) \tag{10}$$

Integrating and assuming $C = 0$, $\nabla C = 0$ at free surface the following equation is arrived at:

$$\overline{C\underset{\sim}{V}} = e_m\ \nabla\overline{C} - \overline{C_w\underset{\sim}{V}_w} - \overline{C'\underset{\sim}{V}'} \tag{11}$$

The right hand side of the above equation represents the effects of molecular diffusion, wave agitation and turbulent diffusion. In the usual context of the diffusion process (Hinze, 1962) it is further assumed that

$$\overline{C_w\underset{\sim}{V}_w} = -e_w\ \nabla\overline{C}\ ; \quad C'\underset{\sim}{V}' = e'\ \nabla\overline{C}$$

where e_w and e' are, respectively, the diffusion coefficients of wave motion and turbulence. Since molecular diffusion is usually negligible, the task of solving Eq. (11) rests on the estimation of e_w and e'. Following Prandtl's mixing theory, these coefficients are assumed to be proportional to certain characteristic velocities and mixing lengths. For sediment suspension in a wave field, it is reasonable to assume that the characteristic velocities for both e_w and e^{*} are proportional to the amplitude of the vertical velocity component of the wave field. Furthermore, if mixing lengths are treated as constant in the flow field, a commonly adopted assumption, the diffusion equation becomes

$$\overline{CV} = \sigma V_m \frac{d\bar{c}}{dy} \qquad (12)$$

where σ is a constant of proportionality and Vm is the amplitude of the vertical velocity component. For small amplitude waves it is expressed as

$$Vm = \frac{a\omega}{\sinh ky} \sinh ky$$

where a is the wave amplitude; w is the angular frequency; k is the wave number, h is the water depth and y is the vertical coordinate measured from sea bottom, as shown below:

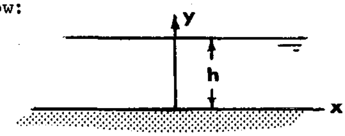

Figure 1. Definition Sketch

For regular waves, the solution of Equation (12) is:

$$\frac{\bar{C}}{\bar{C}_r} = \left[\frac{\tanh \left(\frac{ky}{2}\right)}{\tanh \left(\frac{ky_r}{2}\right)} \right]^R \qquad (13)$$

in which y_r is a reference level where $\bar{C} = \bar{C}_r$ and $R = \frac{G \sinh kh}{\sigma \; k \; a \; \omega}$ where G is the mean particle settling velocity.

For random wave systems, the suspended sediment distribution function can be shown as equal to

$$\frac{\bar{C}}{C_r} = \sum_i \left[\frac{\mathrm{Tanh}\, \frac{k_i y}{2}}{\mathrm{Tanh}\, \frac{k_i y_r}{2}} \right]^{R_i} \qquad (14)$$

where $R_i = \dfrac{G \sinh k_i h}{\sigma k_i a_i \omega_i}$ and $\sum\limits_i a_i^2 = \int_0^\infty S(\omega) d\omega$. For shallow water, both Equations (13) and (14) reduce to power law.

Transport--In a wave field, it has been shown by Longuet-Higgins (1953) that the field equation for mass transport, correct to the second order, takes the following form:

$$\nabla^4 \Psi \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}\right) \int \nabla^4 \psi_s \, dt + \nabla^4 \int \frac{\partial \psi_s}{\partial y} dt \frac{\partial \psi_s}{\partial x} \tag{15}$$

where Ψ is the mass-transport velocity stream function, u and v are the first order velocity components, and ψ_s is first order stream function. In the present case of random wave field, the first order velocity components and stream functions are, respectively:

$$\begin{matrix} u \\ v \end{matrix} = \sum_i U_{oi} \frac{\cosh_i k_i y}{\sinh k_i y} \cos(k_i x - \omega_i t + \Sigma_i)$$

$$\psi_s = - \sum_i \frac{a_i \, i}{k_i} \frac{\sinh k_i y}{\sinh k_i h} \cos(k_i x - \omega_i t + \Sigma_i)$$

where Σ is the random phase angle. Substituting the above equations into Equation (15), we have

$$\nabla^4 \Psi = \nabla^4 \left(\sum_i \frac{u_{oi}^2}{4\omega_i} \sinh 2 k_i y\right) \tag{16}$$

Let

$$\psi(0) = 0 \tag{17}$$

then the requirement of zero net mass transport in the whole field yields

$$\psi(h) = 0 \tag{18}$$

at the outer edge of the interior region. Furthermore, at the bottom of the interior region (y = 0), the horizontal velocity must be matched with the outer edge velocity of the boundary layer; i.e.,

$$\left.\frac{\partial \psi}{\partial x}\right|_{y=0} U_\infty$$

where $\overline{U}_\infty$ can be deduced from Equation (18). With the boundary conditions so defined, solution of ψ can be obtained from Equation (16). The mass transport velocity can then be obtained through differentiation. The solution is

$$\overline{U} = \sum_i \frac{a_i^2 \omega_i k_i}{4 \sinh^2 k_i h} \left[2 \cosh 2k_i y + \frac{3 \sinh 2 k_i h}{2k_i h} \left(\frac{y^2}{h^2} - 2 \frac{y}{h}\right) \right]$$

$$+ \sum_i \frac{U_{oi}^2 k_i}{4\nu(E_i^2 + 0.09\beta_i^2)} \left[\frac{1.2\beta_i E_i}{E_i^2 + 0.09\beta_i^2} - \frac{0.3}{4} \frac{\beta_i}{E_i} \right]$$

$$\left[\frac{3}{2} \left(\frac{y^2}{h^2} - 2 \frac{y}{h}\right) + 1 \right] \tag{20}$$

4. RESULTS

Before examining the case of sediment transport in random waves, it is instructional to compare our results with some of the existing information which, unfortunately, is limited to monocromatic wave trains.

Behattacharya (1971) studied the sediment suspension in shoaling waves experimentally. Those data obtained near horizontal bed were compared with Equation (13) and the value of σ was found to be 5.15 m. (ρ_s/ρ = 2.83)(Fig. 2). The mass transport velocity for a regular wave train in a constant water depth is illustrated in Fig. (3) with a roughness of D = 0.0005 to .00015. The theoretical results of Longuet-Higgins (1953) and Hwang (1970), and the experimental results of Russell(1957), are also shown in the same figure to offer some comparisons.

For random waves, the case illustrated here assumes the following input condition: water depth (h) = 7.5 m., density ratio (γ_s/γ) = 2.83, viscosity (ν) = 0.976 x 10^{-6} m^2/sec., mean sand grain size (D) = 0.65 m.m. The random wave is generated in accordance with the deepwater wave spectrum of Pierson and Moskowitz (1964).

For the bed load, Table 1 shows the concentration for various wind speed: Figure 4 illustrates the mass transport velocity distribution inside the boundary layer. For the suspended load, Figure 5 plots the concentration for various wind speeds; the mass transport velocities are shown in Figure 6.

The total transport of sediment (the summation of bed load and suspended load) is shown in Figure 7.

Bed-Load Concentration

TABLE 1

$h = 7.5^{m}$, $\rho s/\rho = 2.83$, $\nu = 0.976 \times 10^{-6}\ m^2/sec$, $D = 0.65^{mm}$

$s(\omega) = (0.0081\ g^2/\omega^3)\ EXP\ [-0.74\ (g/v\omega)^4]$

v (knots)	ψ	P	C_o (PPM)
20	0.227	0.723	1201.324
25	0.079	0.882	1465.064
30	0.035	0.913	1516.925
35	0.018	0.923	1533.957
40	0.010	0.928	1541.190

Although the wave spectrum chosen here may not be compatible with the finite depth case, the results illustrated above revealed a number of interesting facts:

1) The bed load is of minor importance in sediment transport if the wave is the only factor under consideration.

2) The suspended load concentration increases rapidly with increasing wind velocity. However, the gradient of the suspended load in the water column decreases as wind speed increases.

3) For small wind velocity (or small waves) the shape of the mass transport velocity distribution in the vertical direction is similar to that obtained in the laboratory; the velocity possesses forward components at the surface and bottom layers with return flow in the middle. As the wind speed increases,the forward velocity component at the surface layer diminishes and eventually becomes return flow, thus, the wave mass transport advances with the wave celerity at the lower half of the water column and returns from the upper half.

4) The direction of the sediment transport is with the direction of wave propagation; the reate of transport can be approximated by a power law of wind speed; i.e.,

$$Q_S = A(V)^n$$

where A is a constant. For the case studied, the value of n is approximately equal to 4.

Acknowledgment

This work was supported by the Office of Naval Research, Geography Programs, under contract No. N00014-A0407.

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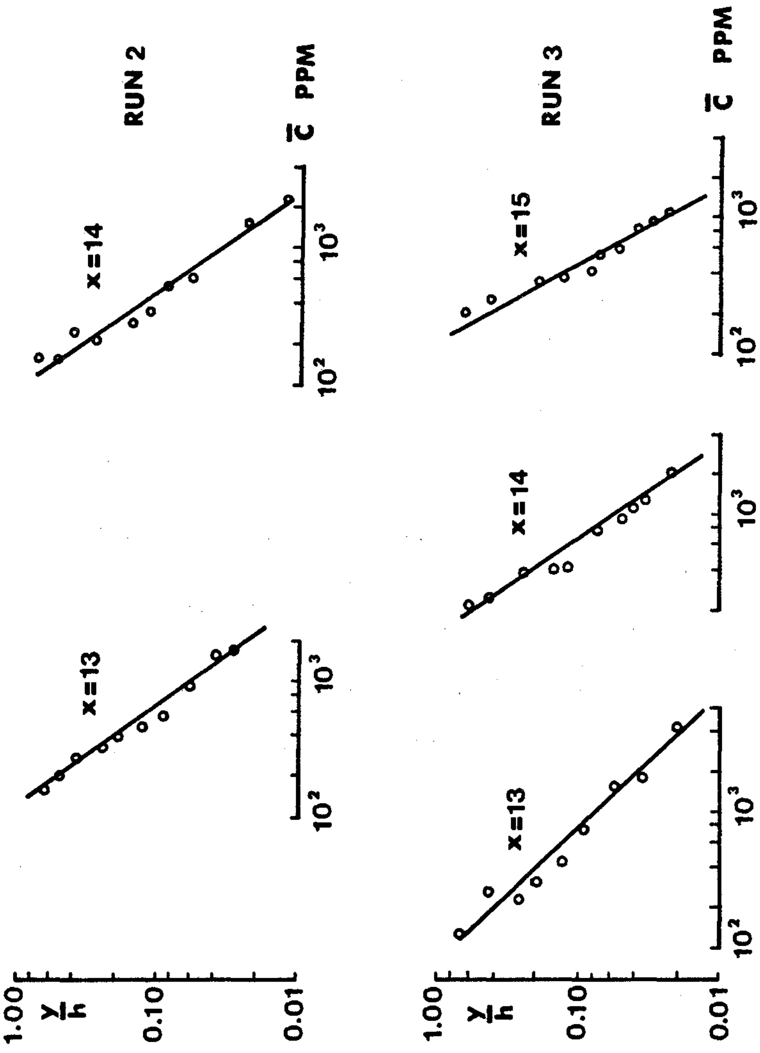

Figure 2 Comparison between experimental data and theoretical prediction for σ = 5.15 m.

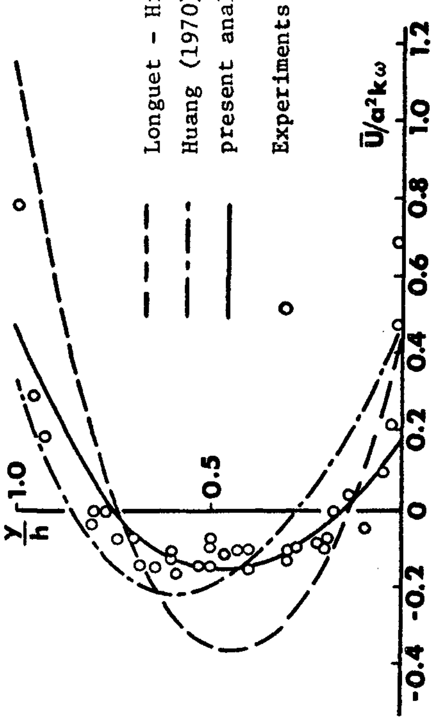

Figure 3 Comparison of mass transport velocities between theories and experiments.

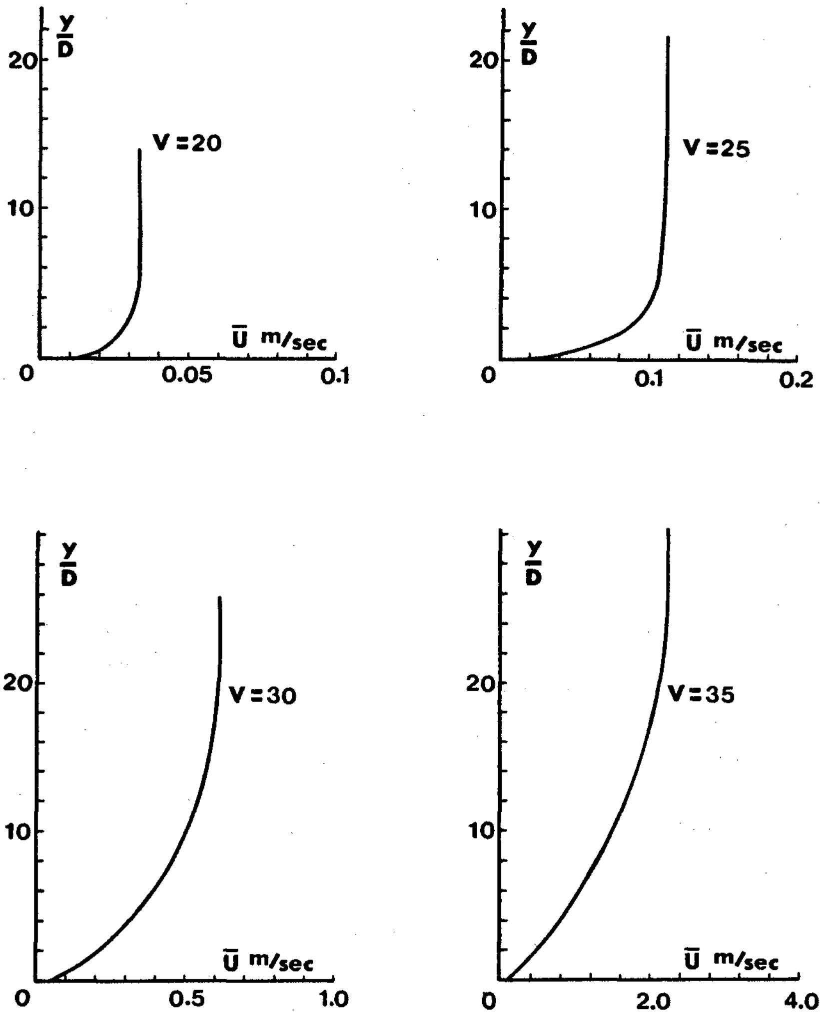

Figure 4

Mass transport velocity distribution inside the boundary layer under a random wave train.

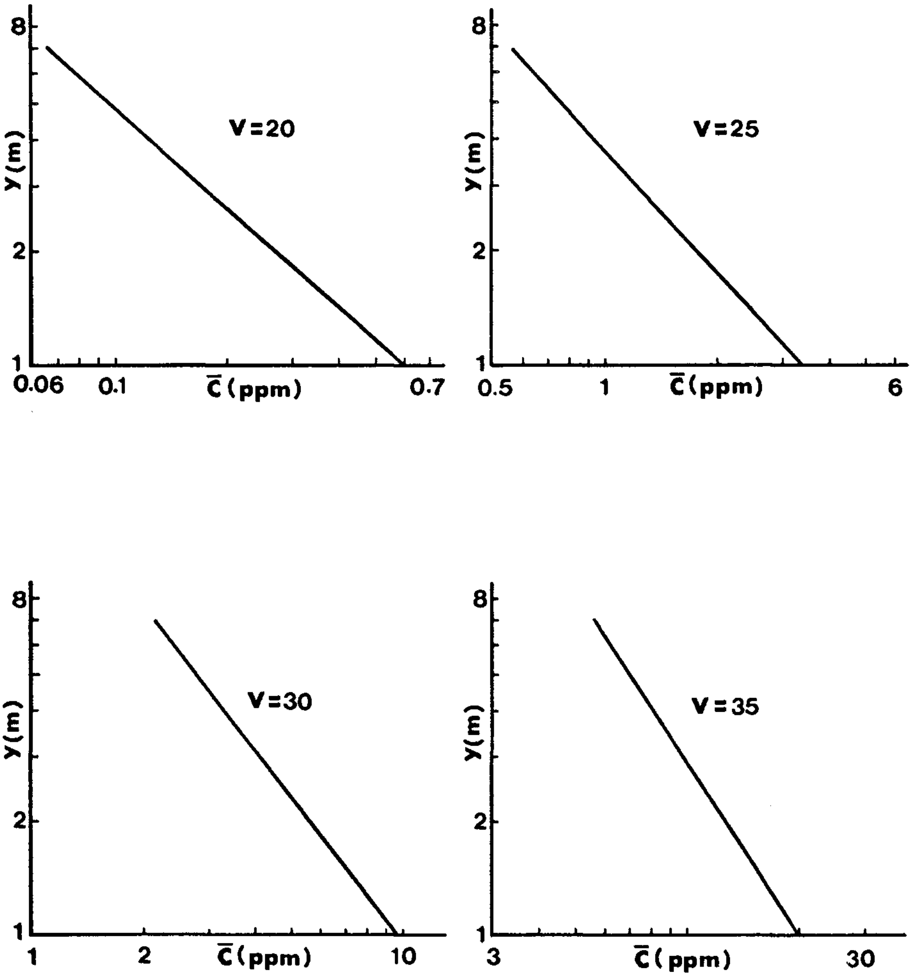

Figure 5 Suspended Sediment Concentration vs. Wind Velocity

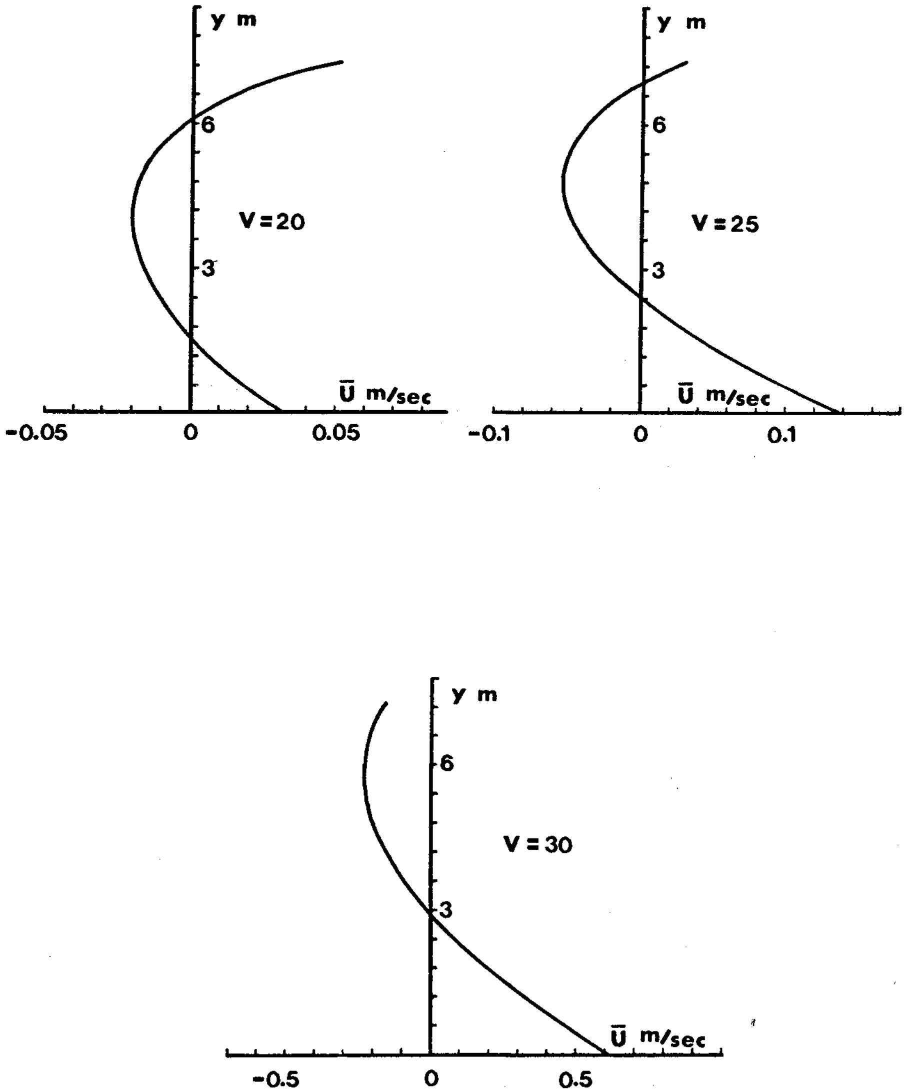

Figure 6

Mass transport velocities under random waves

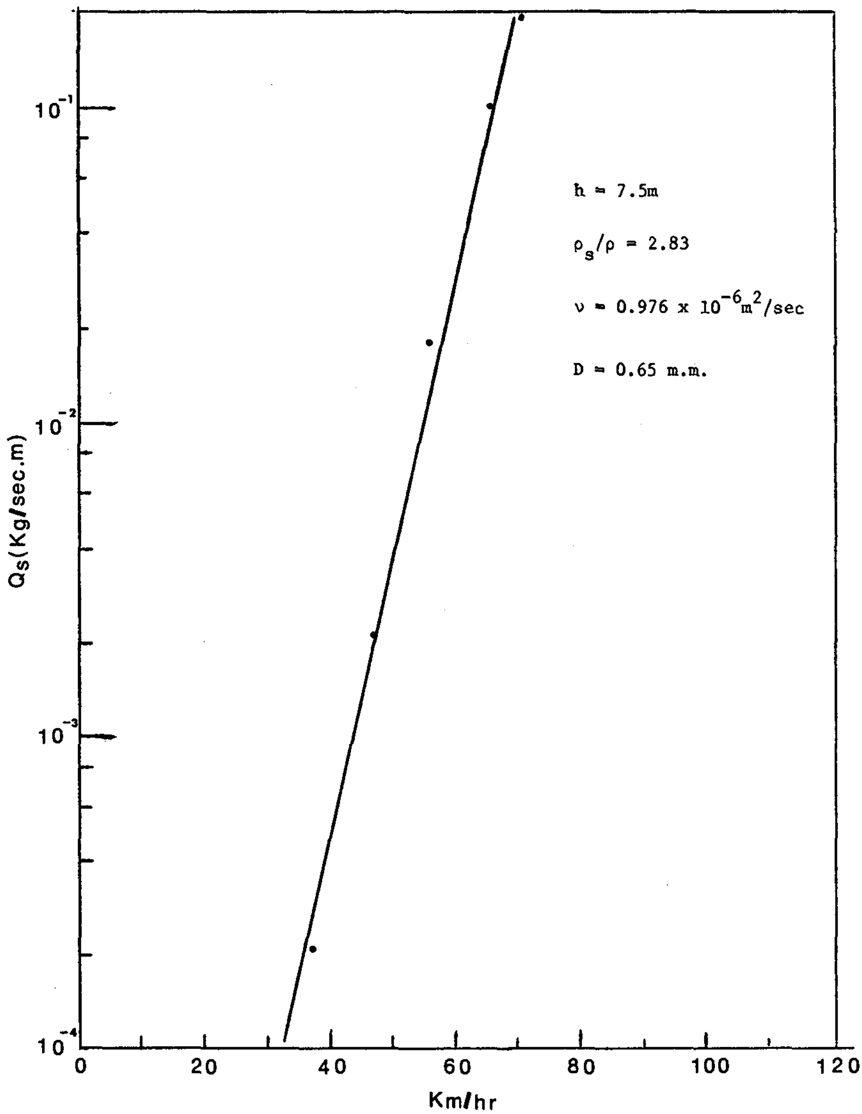

Figure 7 Rate of Sediment Transport vs. Wind Speed