CHAPTER 130

A MATHEMATICAL MODEL FOR SALINITY INTRUSION*

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ABSTRACT

A long term time-dependent mathematical model has been developed for predicting the salinity distributions in the upper York River System, including the tidal portions of the Mattaponi and Pamunkey Rivers.

The method of calculating the longitudinal dispersion coefficient is discussed in detail. The study area and field project are described. The downstream boundary condition was found from a scheme combining a semi-explicit technique and linear extrapolation. The mass-balance equation, averaged over a tidal cycle and solved numerically by the implicit finite difference scheme, provided a reasonable solution and afforded economy in computer time. Field data were compared with the corresponding model results, indicating the general accuracy of the methodology.

INTRODUCTION

The York River System of Virginia includes the Pamunkey and Mattaponi Rivers. The junction of the two rivers forms the York River which is an estuarine river with a 30 mile course from West Point to the Chesapeake Bay near Yorktown, Virginia. The tidal portion of the upper York River serves as a spawning and nursery ground for anadromous commercial and sport fish. The construction of a dam has just been completed on the North Anna River, a main tributary of the Pamunkey, and a second dam is proposed on the Pamunkey River. The effects of these dams will be the regulation and reduction of fresh water flow, thus altering the salinity regime and affecting the existing biota in the estuarine system.

This paper presents a mathematical model developed to study salinity intrusion in the upper York River System. It was used to assess the increased salinity intrusion due to various degrees of reduction of fresh water flow in the Pamunkey River. Predicted salinity distributions aid in regulating flows from the impoundments resulting from the dams.

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MATHEMATICAL FORMULATION

The transport of salt in a roughly sectionally homogeneous estuarine river may be described by the one-dimensional mass balance equation

$$\frac{\partial}{\partial t}(AS) + \frac{\partial}{\partial x}(AUS) = \frac{\partial}{\partial x}\left(AE_s \frac{\partial S}{\partial x}\right) \tag{1}$$

where t is time, x is the distance along river, A is the cross-sectional area, E_s is the dispersion coefficient, U and S are the cross-sectional mean velocity and salinity, respectively. The lateral variation of axial velocity and the transport of salt due to lateral convection and diffusion are not explicitly represented in equation (1), but are lumped into a single dispersion term. The concept of dispersion in a shear flow was first illustrated by Taylor (1953, 1954), both theoretically and experimentally. Aris (1956) gave a rigorous mathematical proof of the dispersion representation of the transport due to interaction between lateral diffusion and velocity shear. Harleman (1971) has given a brief account of the subsequent extensions of the dispersion concept to natural bodies of water.

To describe the long term, such as seasonal, variation of salinity intrusion, a time increment of numerical computation larger than a tidal cycle is desirable. This large time increment can not be applied to equation (1) directly; it has to be applied to the equation averaged over a tidal cycle. Okubo (1964) performed the time average of equation (1) and arrived at

$$\frac{\partial}{\partial t}(\bar{A}\bar{S}) + \frac{\partial}{\partial x}(\bar{A}U_f\bar{S}) = \frac{\partial}{\partial x}\left(E\bar{A}\frac{\partial \bar{S}}{\partial x}\right) \tag{2}$$

where the overbars represent the average over a tidal cycle, and U_f is the velocity due to fresh water discharge Q, given by

$$U_f = \frac{Q}{\bar{A}} \tag{3}$$

E is a dispersion coefficient including the time average of E_s and the effect of transport by oscillating tidal currents.

Since the York River model was to be used to predict the long term effect of fresh water reduction on salinity intrusion, the 'slack tide approximation' was chosen. In the model, only the maximum salinity in the tidal cycle, i.e., the salinity at high water slack, was predicted. No attempt was made to predict the salinity variation within a tidal cycle. The variation of parameters within a tidal cycle may be written as

$$A = \overline{A} + A' \tag{4}$$

$$U = \overline{U} + U' \tag{5}$$

$$S = S_h + S' \tag{6}$$

$$E_s = \overline{E}_s + E' \tag{7}$$

where A', U' and E' are the deviation from the respective quantities averaged over tidal cycle, S_h is the salinity at high water slack and S' is the deviation from S_h. Substituting (4), (5), (6) and (7) into equation (1) and averaging over tidal cycle, the equation becomes

$$\frac{\partial}{\partial t}(\overline{A}S_h) + \frac{\partial}{\partial x}(\overline{A}U_f S_h) = \frac{\partial}{\partial x}(\overline{A}E\frac{\partial S_h}{\partial x}) \tag{8}$$

with

$$E = \overline{E}_s + E_t \tag{9}$$

where

$$E_t = -\frac{\overline{S'U'}}{\frac{\partial S_h}{\partial x}} \tag{10}$$

The one-dimensional continuity equation may be written as

$$\frac{\partial}{\partial t}A + \frac{\partial}{\partial x}(AU) = q \tag{11}$$

where q is the lateral fresh water inflow along a unit length of estuary. Averaging over a tidal cycle, equation (11) becomes

$$\frac{\partial}{\partial t}\overline{A} + \frac{\partial}{\partial x}(\overline{A}U_f) = \overline{q} \tag{12}$$

Substituting equation (12) into equation (8), the mass balance equation becomes

$$\frac{\partial}{\partial t} S_h + U_f \frac{\partial}{\partial x} S_h = \frac{1}{\bar{A}} \frac{\partial}{\partial x} \left(\bar{A}E \frac{\partial S_h}{\partial x}\right) - \frac{\bar{q}}{\bar{A}} S_h \qquad (13)$$

FINITE DIFFERENCE APPROXIMATION

Equation (13) was applied to a part of the upper York River System, between transects upstream of the salt intrusion limits in the Pamunkey and Mattaponi Rivers and a transect four miles downstream of their junction in the York River. The inclusion of Mattaponi River in the model was necessary, even if the expected fresh water reduction was to occur only in the Pamunkey River. The Mattaponi River contributes about 35% of the fresh water discharge to the York River. The increased salt intrusion due to fresh water reduction in the Pamunkey will depend on the fresh water discharge in the Mattaponi while the salinity regime in the Mattaponi will be altered by the fresh water reduction in the Pamunkey. The two rivers are a coupled system and can not be separated.

The equation was solved numerically with an implicit finite difference scheme for each of the three rivers. Twenty, fifteen and four transects were chosen for the Pamunkey, Mattaponi, and York rivers respectively, with average distance between transects being about 3 miles. Except for the end transects of the three rivers, equation (13) was approximated by the following finite difference form for each of the transects.

$$\begin{aligned}
&\frac{S_{h,i}' - S_{h,i}}{\Delta t} + \frac{1}{2(\Delta x_{i-1} + \Delta x_i)} \left[U_{f,i}'(S_{h,i+1}' - S_{h,i-1}') + U_{f,i}(S_{h,i+1} - S_{h,i-1})\right] \\
&= \frac{1}{\Delta x_{i-1} + \Delta x_i} \left\{ \frac{1}{\bar{A}_i} \left[\left(\frac{\bar{A}_i E_i + \bar{A}_{i+1} E_{i+1}}{2}\right) \left(\frac{S_{h,i+1} - S_{h,i}}{\Delta x_i}\right) - \left(\frac{\bar{A}_{i-1} E_{i-1} + \bar{A}_i E_i}{2}\right) \right.\right. \\
&\left. \left(\frac{S_{h,i} - S_{h,i-1}}{\Delta x_{i-1}}\right) \right] + \frac{1}{\bar{A}_i'} \left[\left(\frac{\bar{A}_i' E_i' + \bar{A}_{i+1}' E_{i+1}'}{2}\right) \left(\frac{S_{h,i+1}' - S_{h,i}'}{\Delta x_i}\right) \right. \\
&\left.\left. - \left(\frac{\bar{A}_{i-1}' E_{i-1}' + \bar{A}_i' E_i'}{2}\right) \left(\frac{S_{h,i}' - S_{h,i-1}'}{\Delta x_{i-1}}\right) \right] \right\} - \frac{\bar{q}_i}{\bar{A}_i} S_{h,i} \qquad (14)
\end{aligned}$$

where the subscript i designates the quantities at the ith transect, subscripts i-1 and i+1 designate the upstream and downstream transects respectively, the prime quantities are evaluated at the end of the time step Δt, unprimed quantities are evaluated at the beginning of the time step, Δx_i is the distance between the (i-1)th and the ith transects.

BOUNDARY CONDITIONS

The finite difference approximation of the mass balance equation transforms the differential equation into a system of algebraic equations. In this model, there are three systems of simultaneous equations, corresponding to the three branches of the estuarine river, the Pamunkey, the Mattaponi and the York. These three systems of equations are coupled with a mass balance equation for the element including the confluence. The three transects bounding the confluence are chosen to be so close together that the salinity may be assumed uniform within the circumscribed water body. This leaves two upper and one lower boundary condition to be established to close the whole system of equations. The two furthest upstream transects are located far beyond the salt intrusion limits in the Pamunkey and Mattaponi, hence their boundary conditions may safely be taken as zero salinity. The boundary condition at the downstream end in the York River imposes some difficulty. The technique used is a combination of a semi-explicit scheme and linear extrapolation. The salinity of the downstream boundary at the beginning of a time step is used as a boundary condition to estimate the salinity of other transects at the end of a time step. The boundary condition is refined by linear extrapolation from the estimated salinities at the two transects immediately upstream. The refined boundary condition is then used to calculate the salinity distribution at the new time step.

EVALUATION OF PARAMETERS

Convective Velocity. In this 'slack tide approximation' model, the convective velocity includes only the non-tidal component, which is given by

$$U_f(x,t) = \frac{Q(x,t)}{\bar{A}(x,t)} \tag{15}$$

Q(x,t) is the fresh water discharge from the drainage area upstream of the transect at distance x. This is estimated from the record of stream gauge stations located upstream from the tidal limits. At the ith transect, the fresh water discharge at the mth day is estimated by

$$Q_i(m) = Q_{i-1}(m-n) + I_{i-1,i}\ (m) \tag{16}$$

where $I_{i-1,i}$ is the total lateral fresh water inflow between the (i-1)th and ith transects, and assumed to be proportional to the drainage area increment between the two transects. A delay time of n days is allowed for the discharge Q_{i-1} to travel from (i-1)th transect to ith transect. This travel time is estimated from the average drifting velocity suggested by Pritchard (1958) as

$$U_d = \frac{Q}{\bar{A}} - \beta U_t A_t \tag{17}$$

where U_t and A_t are amplitudes of tidal current and cross-sectional area fluctuations, β is proportional to the correlation coefficient between the variations of tidal velocity and cross-sectional area.

The cross-sectional area averaged over a tidal cycle, $\bar{A}$, is the cross-sectional area corresponding to the fresh water discharge Q. Due to the large volume of average tidal discharge Q_t, $\bar{A}$ is a very weak function of Q except at the transects near tidal limits and at the time of flood. $\bar{A}$ is computed by the hypothetical formula

$$\bar{A} = A_r(1 + \frac{Q}{Q_t})^b \tag{18}$$

where

$$Q_t = \frac{2}{\pi} U_t A_r \tag{19}$$

A_r is the cross-sectional area at zero fresh water discharge, and b is a constant less than unity. It may be inferred from calculations of Gallagher and Munk (1971) on the spectrum of tides in shallow water that A_r should be greater than the cross-sectional area below mean-sea level by less than 1% for the York River system.

Dispersion Coefficient. As shown in equation (9), the dispersion coefficient includes two components: one is $\bar{E}_s$, the time average of dispersion due to shear effect and the other is E_t, the dispersion due to the oscillating tidal current.

For a homogeneous estuarine river with a large width to depth ratio, Harleman (1971) suggested that

$$E_s = 77\, n\, h^{5/6} |U| \tag{20}$$

where n is the Manning friction coefficient, h is the hydraulic mean depth. If equation (20) is substituted into the dispersion term in equation (1) and averaged over a tidal cycle, it is determined that

$$\overline{E}_s = 77\, n\, \overline{h}^{5/6}\, \overline{|U|} \tag{21}$$

to the first order approximation of two assumed small parameters. The parameters are the ratio of depth fluctuation to averaged depth $\overline{h}$ and the deviation of phase angle θ from $\pi/2$, where the angle θ is the phase between tidal current and tidal height. Equation (21) needs to be modified for use in case the estuarine river is not well mixed. No attempt was made to modify this expression for the present model for the following two reasons: first, the model was formulated primarily for predicting the effect of fresh water reduction on salinity intrusion (an estuary usually tends to be better mixed as the fresh water discharge decreases); second, since the empirical data for the dispersion coefficient in the 'slack tide approximation' model are much larger than E_s, it was expected that the modification would not change the total dispersion coefficient appreciably.

For this upper York River model, a simple dimensional argument was used to formulate the dispersion due to the oscillating tidal current. Dimensionally, the dispersion coefficient may be written as

$$E_t = \alpha u \ell$$

where u and ℓ are the velocity and length scale of the transport mechanism involved, α is a coefficient of order of unity. The apparent choice of the velocity scale would be the amplitude of oscillating tidal current U_t. There are several possible choices of length scale. The tidal excursion seems to be the obvious one. In most estuarine rivers of the Chesapeake Bay, including the York River System, the amplitude of the tidal current averages about 1.5 fps, which gives an excursion of 20,000 ft, and $u\ell$ roughly equal to 100 square miles per day, which is an order of magnitude larger than empirical values. Furthermore, if the tidal current is uniform throughout every cross-section of the river, the salt transported upstream during the flood tide will be carried downstream to the original longitudinal position in the ebb tide, even if some may have been diffused laterally or vertically. Thus, neglecting the fresh water flow and longitudinal turbulent diffusion, the same amount of salt will return to the original transect after a complete tidal cycle, resulting in no dispersion regardless of the tidal excursion. It is the non-uniformity of the tidal current within a cross-section which induces longitudinal dispersion.

Saline water is carried upstream faster in the mid-channel and part of it diffuses vertically or laterally. That diffused out of mid-channel will not be carried downstream to the original longitudinal position because of slower currents outside the mid-channel. Therefore, after a complete tidal cycle, salt originally in one transect will be spread out to other transects, resulting in longitudinal dispersion. For a straight estuarine river with large width-to-depth ratio, Holley et. al. (1970) showed that the time scale of lateral mixing due to turbulent diffusion is much larger than a tidal cycle while that of vertical mixing is much smaller. Therefore, the depth will be the choice of length scale. The depth averages 20 ft. for the upper York River and gives

$$u\ell \approx 0.1 \text{ mi}^2/\text{day}$$

an order of magnitude smaller than empirical data. In reality, in an estuarine river with large curvatures, secondary flows always exist; the time scale of lateral mixing may have the same order of magnitude as the vertical one. In this case, the choice of length scale would be the characteristic length of the cross-section such as the square root of the cross-sectional area. In this upper York River model, E_t was computed as:

$$E_t = \alpha U_t \sqrt{A}$$

where the coefficient α was adjusted until the model output agreed with 1970 field survey data, which gave $\alpha = 2.5$. The tidal velocity and cross-sectional area were calculated from field measurements.

FIELD MEASUREMENTS

Intensive hydrographic surveys were carried out in October, 1969. A total of 37 transects were occupied. Each transect had between one and four stations, depending on river width. Distance between transects averaged three miles. Transects located near sharp bends were positioned at least four river widths from the bend to insure representative measurements for the reach.

Salinity, temperature and velocity measurements were obtained at hourly intervals for twenty five consecutive hours at each station. Sampling depths were at six-foot increments from surface to bottom. Bathymetry of each transect was obtained with a recording sonic depth sounder.

Cross-sectional averages of the longitudinal component of velocity were calculated and plotted as function of time over 25 hours which gave the time variation of current for two consecutive tidal cycles. The tidal amplitude was calculated as the average of the maximum flood and ebb currents. Cross-sectional areas were determined by planimetry of the bottom profile data collected from sounding.

For the purpose of model verification, a series of slack water surveys have been conducted since August, 1970. Salinity, as well as temperature, dissolved oxygen and biochemical oxygen demand, were measured at local slack water before ebb tide or slack water before flood tide. One station on each transect was sampled with measurements made three feet below the surface and three feet above the bottom.

RESULTS AND DISCUSSION

A mathematical model for salinity intrusion has been developed for the upper York River System. The model is based on the one-dimensional mass balance equation averaged over a tidal cycle. In the model, only the salinity at high water slack is predicted. No attempt is made to predict the salinity variation within a tidal cycle. The objective of this model is to simulate the long term variation of salinity intrusion and to assess the increased salinity intrusion due to various degrees of fresh water reduction in the Pamunkey River.

Figure 1 shows the comparison of the model output with field data of the York and Pamunkey Rivers. The slack water run data of August 14, 1970 was used as the initial condition of the model. Saline water intrudes further upstream in the dry season as indicated by the model output and the field data of late September and mid-November. Figure 2 shows the same comparison for the Mattaponi River. The agreement is not so good as that for the York and Pamunkey, particularly the comparison on November 15, which was four days after a large increase in fresh water discharge. It was observed that the model failed to yield satisfactory results with very high fresh water discharge, because the model responds too slowly to sudden large increases in fresh water discharge. Once it responds to flood conditions, all of the saline water is flushed out of the modeled portion of the estuary. Even after the flood recedes, the salt will not return because of the scheme for setting up the downstream boundary condition.

Figure 3 shows a sample of increased salinity intrusion due to fresh water reduction. With the dam on the North Anna River completed, one of the proposed fresh water flow regulations is a minimum of 40 cfs discharge from the reservoir during the dry season. Using the 1968-1969 fresh water discharge record as input, the solid curves show the salinity distributions with natural discharge and the dashed curves show the salinity distributions with the proposed regulation. In addition to the salinity increase at a particular location, it is also possible to follow the upstream movement of a particular isohaline. The ecologist is usually more interested in this increased intrusion distance for given salinities, e.g. the 5‰ isohaline. Figure 4 shows the similar comparison for the Mattaponi River. Since only the discharge in the Pamunkey is regulated, the effect of the fresh water reduction decreases with distance upstream from the river's junction.

The model was developed to predict the effect of fresh water reduction on salt intrusion. The critical time of this effect is the dry season when the saline water intrudes furthest upstream. Therefore, the present model was verified with field data taken in the dry season and the constants in the model adjusted accordingly. It has been mentioned that the model failed to yield reasonable results for high fresh water discharge and caution should be taken in the application of the model.

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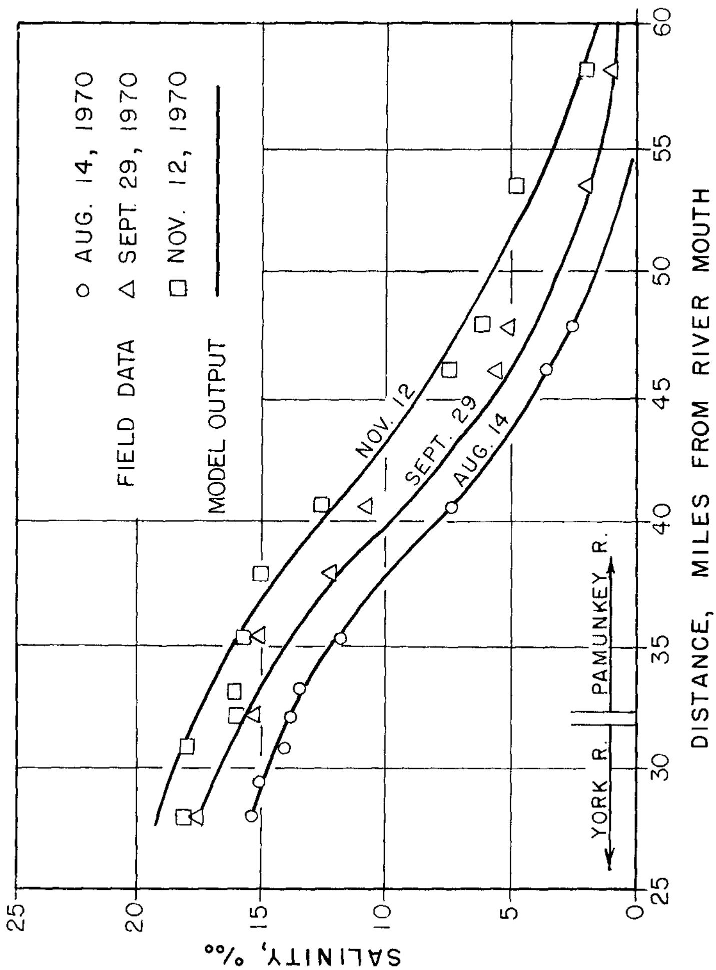

FIGURE 1. SALINITY DISTRIBUTION ALONG PAMUNKEY, 1970.

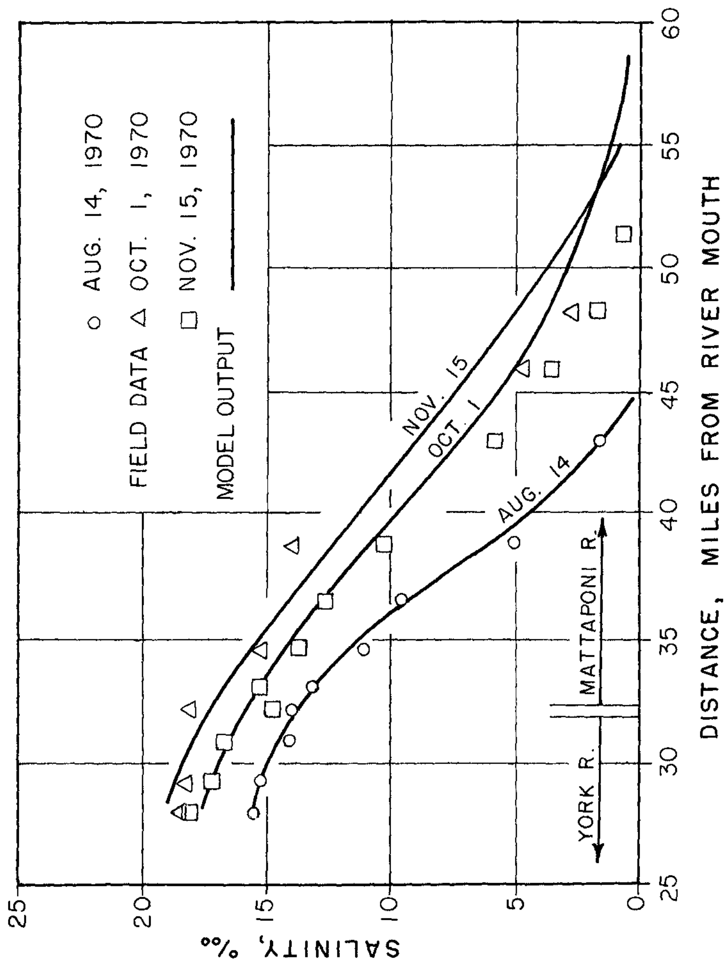

FIGURE 2. SALINITY DISTRIBUTION ALONG MATTAPONI, 1970.

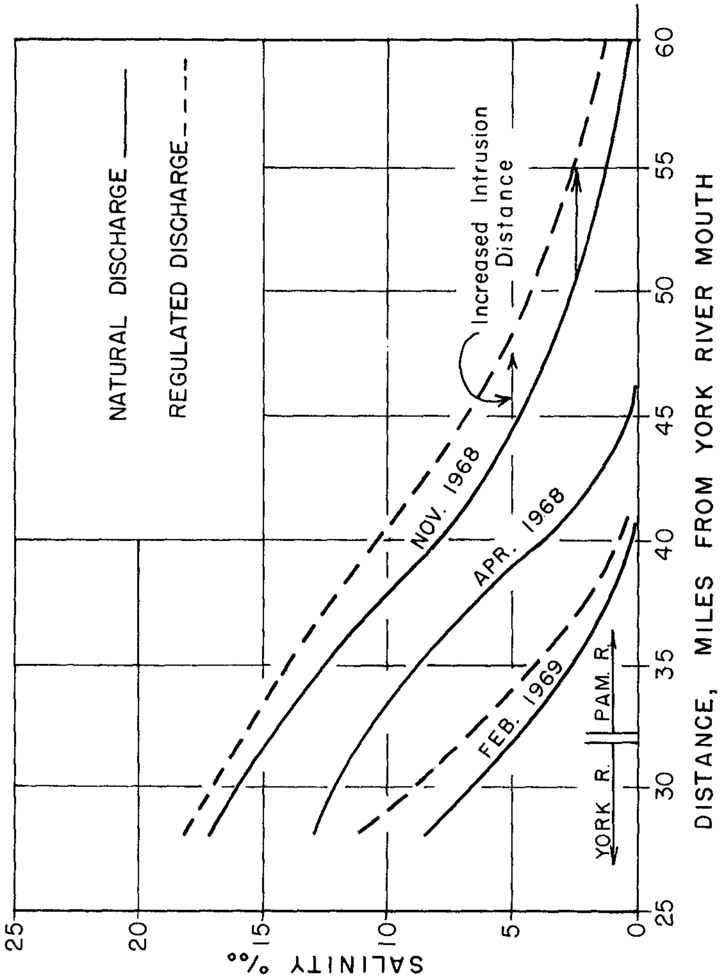

FIGURE 3. 1968-1969 SIMULATED RUN, PAMUNKEY.

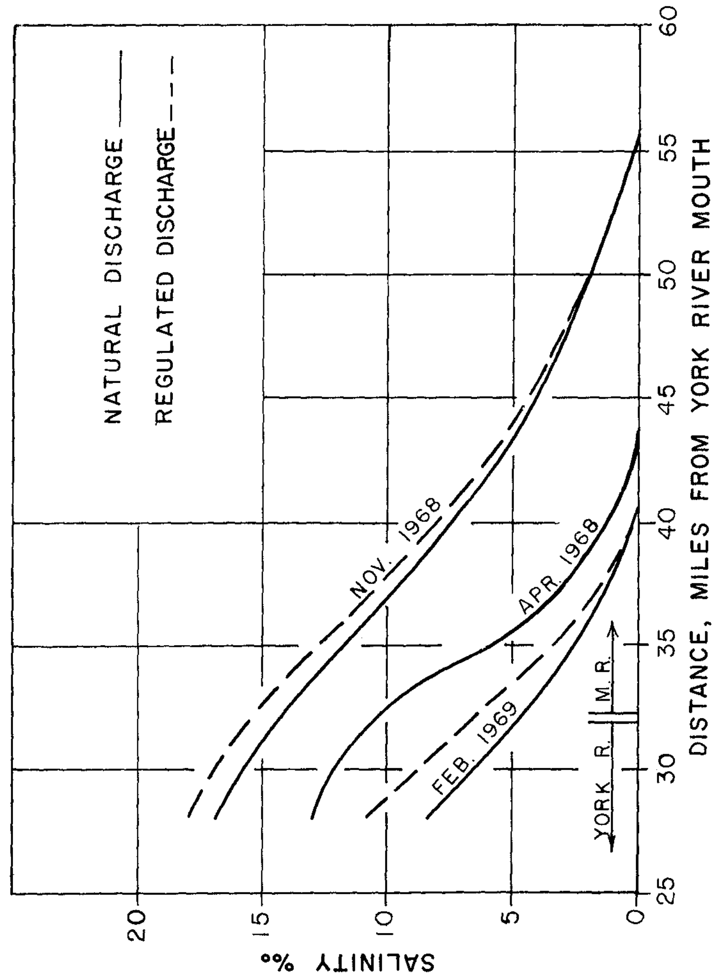

FIGURE 4. 1968-1969 SIMULATED RUN, MATTAPONI.

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