Wadden Area

Part 2
BEACH AND SHORELINE PROCESSES

Walcheren Dike

CHAPTER 11
THEORETICAL FORMS OF SHORELINES

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INTRODUCTION.

In previous publications Pelnard-Considère, Bruun and Larras have derived theoretical shore formations. When doing so, it is necessary to idealize the conditions, such as a littoral transport by waves only, unvarying wave characteristics and a simple relation between the angle of wave approach and the littoral transport. Moreover various other simplifications have to be made in order to make it possible to handle the equations.

The question may arise whether results, obtained from such an idealized situation, have any value for practical cases, where the conditions are much more complex and variable. The answer is no when we expect to obtain a true and detailed picture of the development of any particular stretch of coast. Such theoretical exercises can be of real value, however, because they help us to understand why and how certain formations come into being and how they are influenced by certain physical processes. This is the case for instance with such formations as deltas, spits and tombolos.

We cannot say that we really know the function which determines the littoral transport. Up to now one of the simplifications in the mathematical treatment has been the restriction to stay within an area in which the values of α are so small that the transport may be assumed to increase in direct proportion to the increase of the value of α (α being defined as the angle between the wave direction and the direction of the normal on the coast in the point considered). However, experiments indicate that the littoral transport very likely reaches a maximum for a wave angle between 45° and 60°. Interesting phenomena are bound to occur when this maximum is approached.

With this in mind we have tried to introduce a transport function $T = A \sin 2\alpha$, having its maximum when $\alpha = 45^\circ$.

THE MATHEMATICAL TREATMENT.

Considering a stretch of shore of length ds (fig. 1), the quantity of deposited (or eroded) material must be equal to the difference between the quantity of transported material in A and in B. Expressed by mathematical terms, we have:

$$-D\frac{\partial y}{\partial t}\,dt\,dx = \frac{\partial T}{\partial x}\,dx\,dt$$

or

$$\frac{\partial T}{\partial x} + D\frac{\partial y}{\partial t} = 0 \qquad (1)$$

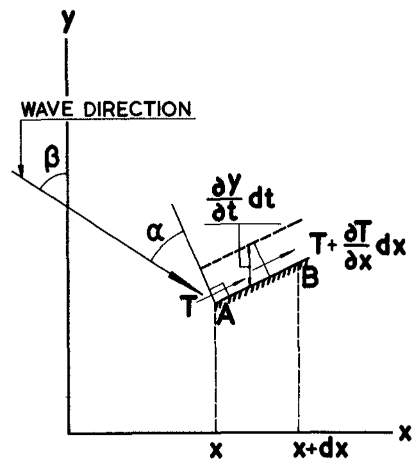


Fig. 1

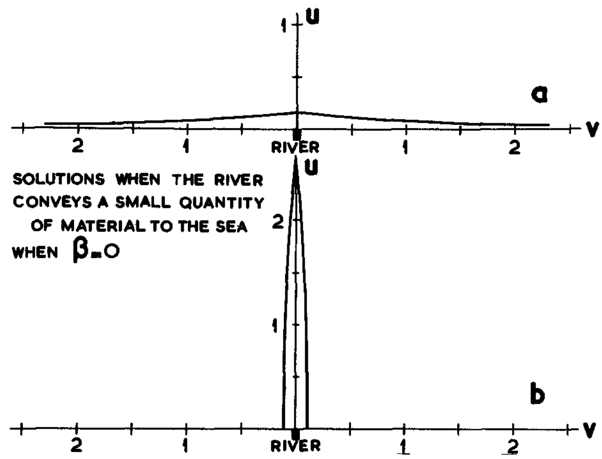


Fig. 2

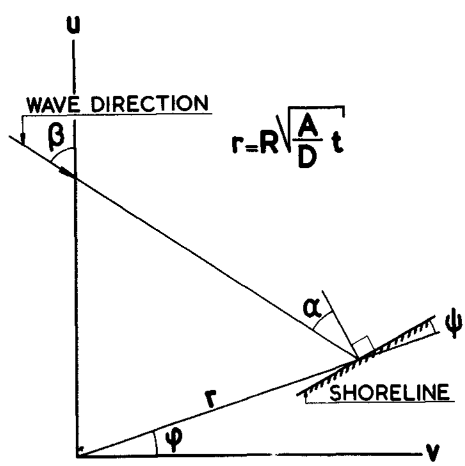


Fig. 3

in which:

T is the transport function
D is the waterdepth
t is the time
x and y are the coordinates

In fig. 1 the angle β defines the wave direction with respect to the system of coordinates and the angle α the value of T. Since we take T only depending on α, it is sufficient to differentiate T to one variable. Substituting $T = A \sin 2\alpha$, the equation (1) can be worked out as:

$$\frac{4A\cos 2\beta}{D} \cdot \frac{1 + 2\tan 2\beta \frac{\partial y}{\partial x} - \left(\frac{\partial y}{\partial x}\right)^2}{\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^2} \frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t} \qquad (2)$$

In order to obtain a set of solutions suitable to our purpose, we shall confine ourselves to such coastlines as remain similar in shape whereas the scale of this shape is varying in the course of time. Hence we introduce new variables u and v, proportional to y and x respectively and we require that there exists a functional relation between u and v independant of the time t. An important property of this type of solutions derives from the fact that the transport along the coast depends on its direction only. Since the direction in a point of the coastline is independant of the scale, the transports at the end of a coastline segment remain constant, so that the volume behind this segment must increase proportional to t. Some of these solutions, therefore, are suitable more in particular to describe the development of a delta of a river which charges the shore with material at a constant rate. By the substitution:

$y = u\sqrt{t}$ $\qquad$ $x = v\sqrt{t}$

the differential equation (2) reduces in the requisite way to

$$\frac{4A\cos 2\beta}{D} \frac{1 + 2\tan 2\beta \frac{du}{dv} - \left(\frac{du}{dv}\right)^2}{\left[1 + \left(\frac{du}{dv}\right)^2\right]^2} \cdot \frac{d^2u}{dv^2} + v\frac{du}{dv} - u = 0 \qquad (3)$$

provided the depth is taken as a constant, which means that the seabed is horizontal.

This equation can be simplified somewhat further for a few special cases. First we consider the case in which $\tan 2\beta$ and $\frac{du}{dv}$ are very small in comparison with unity. The equation (3) then yields:

$$a\frac{d^2u}{dv^2} + v\frac{du}{dv} - u = 0$$

in which

$$a = \frac{4A\cos 2\beta}{D}$$

The solution of this equation is:

$$u = C_1 \left(e^{-\frac{v^2}{2a}} + \frac{v}{a} \int_{\infty}^{v} e^{-\frac{v^2}{2a}} dv \right)$$

when the shore at time t = 0 is a straight line. In a slightly differ form this solution has already been presented several times in previo publications.

Likewise we can consider a second case where tan 2β is very sma and $\frac{du}{dv}$ so large that we can now neglect the first and second terms o the numerator and the first term of the denominator of the fraction i (3). This equation turns now into:

$$-a \frac{\frac{d^2u}{dv^2}}{\left(\frac{du}{dv}\right)^2} + v \frac{du}{dv} - u = 0$$

of which the solution is:

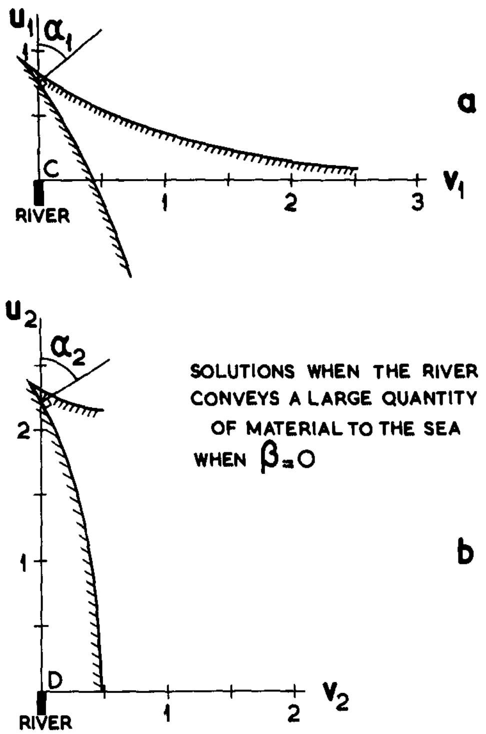
$$v = C_1 \left(e^{\frac{u^2}{2a}} - \frac{u}{a} \int_{0}^{u} e^{\frac{u^2}{2a}} du \right)$$

also when the shore at time t = 0 is a straight line. Figure 2 shows t diagrammatical representation of these two special cases in which β do not differ much from zero while the quantity of material that the rive conveys to the sea is small in comparison with the transport of the coastal regime.

It can be shown that the numerator of the fraction in the equatio (3) becomes zero in the point where $\alpha = 45°$. This means that in that point either $v\frac{du}{dv} - u$ must be zero or $\frac{d^2u}{dv^2}$ becomes infinite. The first solution corresponds to the trivial case of a straight line through th origin. The second solution means that the variation in the gradient o the tangent divided by the variation in v, is infinite. That indicates an abrupt change in the direction of the coastline at the point in question. From a physical point of view, however, we cannot have a sudden jump in the quantity of material transported. Hence, such a poi must be a cusp.

These results can be obtained from analytical considerations of equation (3). For more general solutions it is necessary to integrate this equation numerically. Then it is more convenient to use polar-coordinates instead of Cartesian coordinates and to introduce two parameters, viz. the angle α mentioned before and the angle ψ between the shoreline and the radius-vector (fig. 3). We obtain the following syst of two simultaneous equations

$$d\Phi = 2\cos 2\alpha \, d\alpha = \frac{1}{2} R^2 d\varphi \qquad (4$$


u1
α1
1
a
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RIVER
1
2
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SOLUTIONS WHEN THE RIVER
CONVEYS A LARGE QUANTITY
OF MATERIAL TO THE SEA
WHEN β=O
2
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v2
RIVER
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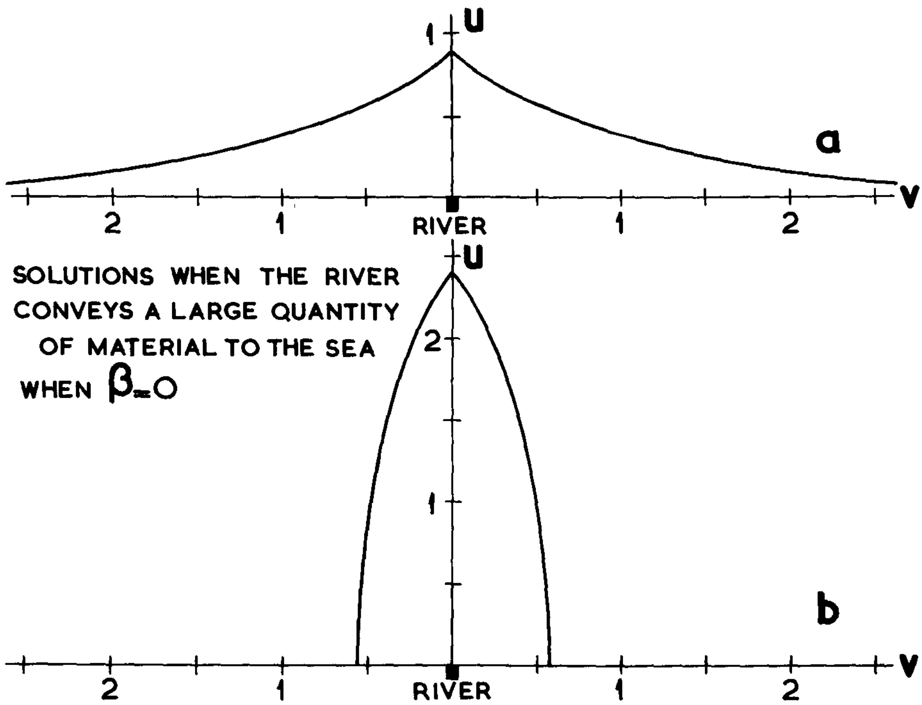
2

Fig. 4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OF MATERIAL TO THE SEA
WHEN β=O
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Fig. 5

$$\tan \psi \, dR = R \, d\varphi \tag{5}$$

in which φ en ψ are linked by

$$\alpha + \varphi + \psi = \beta \tag{6}$$

The equation (4) determines the displacement of the shoreline and corresponds with equation (3). The equations (5) and (6) are geometrical conditions. Q is defined as $\frac{T}{A}$ for which we adopted the function sin 2α.

Solutions are being constructed by means of a computer. By this way of approach it will also be possible, if necessary, to adopt other functions for Q and to alter the assumption that the seabed is horizontal.

In the solutions two integration parameters appear. We intend to produce sets of curves for various values of these parameters. At the moment we write this paper, we are still engaged in pursuing this programme and we have to confine ourselves to show a few preliminary results. Two of the solutions obtained so far with β equal to zero, are shown in the figure 4a and 4b. The curves are solutions which obey the differential equations and each of them has a special value of one of the integration parameters. Mathematically the solutions are correct, but the question is whether physical conditions can be found corresponding to them. This can be done by locating the initial shoreline along the axis v_1 and v_2 and in C and D the mouth of a river conveying a quantity of material corresponding with the angles α_1 and α_2. In this figure 4 we recognize the solutions shown in figure 2. Now, however, the solutions are not restricted to the condition that the ratio between the quantity of material transported by the river to the sea and the quantity which the coastal regime is able to convey, is small. As a matter of fact figure 5 shows the solutions in which α_1 and α_2 are equal to 45°. The ratio mentioned before is then equal to 2.

The work is being continued and further publication of the result is intended.

I wish to thank Dr. Schönfeld of our Department for his aid in the mathematical treatment of the problem.

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