



A new exponential Chebyshev operational matrix of derivatives for solving high-order ordinary differential equations in unbounded domains

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Abstract

The purpose of this paper is to investigate a new exponential Chebyshev (EC) operational matrix of derivatives. The new operational matrix of derivatives of the EC functions is derived and introduced for solving high-order linear ordinary differential equations with variable coefficients in unbounded domain using the collocation method. This method transforms the given differential equation and conditions to matrix equation with unknown EC coefficients. These matrices together with the collocation method are utilized to reduce the solution of high-order ordinary differential equations to the solution of a system of algebraic equations. The solution is obtained in terms of EC functions. Numerical examples are given to demonstrate the validity and applicability of the method. The obtained numerical results are compared with others existing methods and the exact solution where it shown to be very attractive with good accuracy.

Keywords: Exponential Chebyshev functions, High-order ordinary differential equations, Collocation method.

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1. Introduction

Spectral methods are one of the principal methods for solving differential equations. The main idea of spectral methods is to approximate the solutions of differential equations by means of truncated series of orthogonal polynomials. The most used of spectral methods are tau, collocation, and Galerkin methods [9, 25, 15, 3, 8, 2, 26, 1, 7]. One of the most important orthogonal polynomials is Chebyshev polynomials, which are widely used in numerical analysis. The well-known Chebyshev first kind $T_n(x)$ [12] are orthogonal polynomials on the interval $[-1, 1]$, these polynomials have many applications in numerical analysis, and a

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lot of studies are devoted to show the merits of them in various ways. One of the applications of Chebyshev polynomials is the solution of ordinary differential equations with boundary conditions, with collocation points [27, 14, 13]. Many studies are considered on the interval $[-1, 1]$ in which Chebyshev polynomials are defined. Therefore, this limitation causes a failure of the Chebyshev approach in the problems that are defined on larger domains, especially including infinity. Under a transformation that maps the interval $[-1, 1]$ into a semi-infinite domain $[0, \infty)$, the authors of [4, 5, 6, 16, 17, 24, 28, 20, 21, 22, 18, 19] successfully applied spectral methods to solve problems on semi-infinite intervals. In their studies, the basis functions called rational Chebyshev functions $R_n(x)$, and defined by

$$R_n(x) = T_n\left(\frac{x-1}{x+1}\right). \quad (1)$$

Recently, Koc, and Kurnaz [11] has proposed modified type of Chebyshev polynomials as an alternative to the solutions of the problems given in all real domain. In their studies, the basis functions called exponential Chebyshev (EC) functions $E_n(x)$ are orthogonal in $(-\infty, \infty)$. This kind of extension tackles the problems over the whole real domain. The EC functions are defined as

$$E_n(x) = T_n\left(\frac{e^x - 1}{e^x + 1}\right), \quad (2)$$

where the corresponding recurrence relation is

$$\begin{aligned} E_0(x) &= 1, & E_1(x) &= \frac{e^x - 1}{e^x + 1}, \\ E_{n+1}(x) &= 2\left(\frac{e^x - 1}{e^x + 1}\right)E_n(x) - E_{n-1}(x), & n &\geq 1 \end{aligned} \quad (3)$$

In [23] we introduce a modification form of the operational matrix of the derivatives by processing the truncation in [11]. In this paper we introduce a new operational matrix of derivatives of EC functions and employ it to solve high-order linear ordinary differential equations with variable coefficients in unbounded domain using the collocation method. The high-order linear nonhomogeneous differential equations that considered in this paper is

$$\sum_{k=0}^m P_k(x)y^{(k)}(x) = f(x), \quad -\infty < x < \infty, \quad (4)$$

with the mixed conditions

$$\begin{aligned} \sum_{k=0}^{m-1} \sum_{j=0}^J d_{ij}^k y^{(k)}(b_j) &= \alpha_i, \\ -\infty < b_j < \infty, & \quad i = 0, 1, \dots, m-1, \quad j = 0, 1, \dots, J \end{aligned} \quad (5)$$

where the $P_k(x)$ and $f(x)$ are continuous functions on the interval $(-\infty, \infty)$, d_{ij}^k , b_j and α_i are appropriate constants, or b_j may tends to $\pm\infty$.

2. Properties of exponential Chebyshev functions

In this section we list some of the properties of the EC functions.

2.1. Orthogonality of EC functions

The EC functions are orthogonal with respect to the weight function $w(x)$ such that they are orthogonal in the interval $(-\infty, \infty)$ is given by $\sqrt{e^x}/(e^x + 1)$, with the orthogonality condition

$$\int_{-\infty}^{\infty} E_n(x)E_m(x)w(x)dx = \frac{c_m\pi}{2}\delta_{nm}, \quad (6)$$

where

$$c_m = \begin{cases} 2, & m = 0 \\ 1, & m \neq 0 \end{cases}$$

and δ_{nm} is the Kronecker delta function. Also the product relation of EC functions used in the derivative relations is given by

$$E_n(x)E_m(x) = \frac{1}{2}[E_{n+m}(x) + E_{|n-m|}(x)]. \quad (7)$$

2.2. Function expansion in terms of EC functions

A function $f(x)$ well defined over the interval $(-\infty, \infty)$, can be expanded as

$$f(x) = \sum_{n=0}^{\infty} a_n E_n(x), \quad (8)$$

where

$$a_n = \frac{2}{c_n \pi} \int_{-\infty}^{\infty} E_n(x) f(x) w(x) dx.$$

If $f(x)$ in expression (8) is truncated to $N < \infty$ in terms of the EC functions we get

$$f(x) \cong \sum_{n=0}^N a_n E_n(x), \quad (9)$$

also, the (k) th-order derivative of $f(x)$ can be written as

$$f^{(k)}(x) \cong \sum_{n=0}^N a_n (E_n(x))^{(k)}, \quad (10)$$

where $(E_n(x))^{(0)} = E_n(x)$.

2.3. The new operational matrix of derivatives

The new representation of EC functions is presented as follows. The Chebyshev polynomials $T_n(x)$ can be expressed in terms of the power x^n in different forms found in [12], one of them is

$$T_n(x) = \sum_{k=0}^{[n/2]} c_k^{(n)} x^{n-2k}, \quad (11)$$

If we use the expression $v(x) = \frac{e^x - 1}{e^x + 1}$ in the EC functions, we can express it explicitly in terms of powers of $v(x)$ as

$$E_n(x) = \sum_{k=0}^{[n/2]} c_k^{(n)} v^{n-2k}(x), \quad (12)$$

where

$$c_k^{(n)} = (-1)^k 2^{n-2k-1} \frac{n}{n-k} \binom{n-k}{k}, \quad 2k \leq n, \quad (13)$$

from previous relation with simple modification we can define:

if n is even we find

$$E_n(x) = E_{2l}(x) = \sum_{j=0}^l (-1)^{l-j} 2^{2j-1} \frac{2l}{l+j} \binom{l+j}{l-j} v^{2j}(x),$$

if n is odd we can write

$$E_n(x) = E_{2l+1}(x) = \sum_{j=0}^l (-1)^{l-j} 2^{2j} \frac{2l+1}{l+j+1} \binom{l+j+1}{l-j} v^{2j+1}(x).$$

Form above we can deduce general matrix form of EC functions as

$$E(x) = V(x)C^T, \quad (14)$$

where $E(x)$ and $V(x)$ are matrices of the form:

$$E(x) = \begin{bmatrix} E_0(x) & E_1(x) & \dots & E_N(x) \end{bmatrix}, \quad V(x) = \begin{bmatrix} v^0(x) & v^1(x) & \dots & v^N(x) \end{bmatrix},$$

and

$$v^0(x) = 1, \quad v^1(x) = \left(\frac{e^x - 1}{e^x + 1} \right)^1, \quad v^2(x) = \left(\frac{e^x - 1}{e^x + 1} \right)^2, \quad \dots, \quad v^N(x) = \left(\frac{e^x - 1}{e^x + 1} \right)^N,$$

and C is $(N+1) \times (N+1)$ matrix given by

$$C = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & 0 & 0 & \dots & 0 & 0 \\ -\begin{pmatrix} 1 \\ 1 \end{pmatrix} & 0 & 2\begin{pmatrix} 2 \\ 0 \end{pmatrix} & 0 & \dots & 0 & 0 \\ 0 & -\frac{3}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix} & 0 & 2^2\begin{pmatrix} 3 \\ 0 \end{pmatrix} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^l \begin{pmatrix} l \\ l \end{pmatrix} & 0 & (-1)^{l-1} 2^{\frac{2l}{l+1}} \begin{pmatrix} l+1 \\ l-1 \end{pmatrix} & 0 & \dots & 2^{2l-1} \begin{pmatrix} 2l \\ 0 \end{pmatrix} & 0 \\ 0 & (-1)^l \frac{2l}{l+1} \begin{pmatrix} l+1 \\ l \end{pmatrix} & 0 & (-1)^{l-1} 2^{2\frac{2l+1}{l+2}} \begin{pmatrix} l+2 \\ l-1 \end{pmatrix} & \dots & 0 & 2^{2l} \begin{pmatrix} 2l+1 \\ 0 \end{pmatrix} \end{bmatrix}. \quad (15)$$

In this case, we are going to use the last row for odd values of $N = 2l + 1$, otherwise previous one will be the last row of matrix C ($N = 2l$). Now, from (14) we can obtain the k^{th} derivative of the matrix $E(x)$ as:

$$\begin{aligned} E^{(0)}(x) &= V(x)C^T, \\ E^{(1)}(x) &= V^{(1)}(x)C^T, \\ E^{(2)}(x) &= V^{(2)}(x)C^T, \\ &\vdots \end{aligned}$$

then, the k^{th} derivative of the matrix $E(x)$ defined as:

$$E^{(k)}(x) = V^{(k)}(x)C^T. \quad (16)$$

Then, Eq. (16) represents the new operational matrix of derivatives of the EC functions.

3. Fundamental matrix relations

Let us first assume that the solution $y(x)$ of Eq. (4) can be expressed in the form (9). If $y(x)$ is truncated Chebyshev series in terms of EC functions as

$$y(x) \cong \sum_{n=0}^N a_n E_n(x), \quad (17)$$

then $y(x)$ and its derivative $y^{(k)}(x)$ can be represented in the matrix forms as

$$[y(x)] = E(x)A, \quad (18)$$

and

$$\left[y^{(k)}(x) \right] = E^{(k)}(x)A, \quad k = 0, 1, 2, \dots, m \leq N, \quad (19)$$

where

$$E^{(k)}(x) = [(E_0(x))^{(k)} \ (E_1(x))^{(k)} \ \dots \ (E_N(x))^{(k)}], \quad A = [a_0 \ a_1 \ \dots \ a_N]^T,$$

and $E_0(x)$, $E_1(x)$, ..., $E_N(x)$ are the EC functions, a_0 , a_1 , ..., a_N are coefficients to be determined in expression (19). Consequently, the derivative of the matrix $E(x)$ defined in (16), and from expression (19), can be obtained as

$$\left[y^{(k)}(x) \right] = V^{(k)}(x)C^T A, \quad (20)$$

where

$$V^{(k)}(x) = \begin{bmatrix} (v^0(x))^{(k)} & (v^1(x))^{(k)} & \dots & (v^N(x))^{(k)} \end{bmatrix}.$$

Now, let us define the collocation points, so that $-\infty < x_i < \infty$, as

$$x_i = Ln \left[\frac{1 + \cos\left(\frac{i\pi}{N}\right)}{1 - \cos\left(\frac{i\pi}{N}\right)} \right], \quad i = 1, \dots, N-1, \quad (21)$$

and at the boundaries ($i = 0, i = N$) $x_0 \rightarrow \infty$, $x_N \rightarrow -\infty$, since the EC functions are convergent at both boundaries $\pm\infty$, namely their values are ± 1 , the appearance of infinity in the collocation points does not cause a loss or divergence in the method. Then, we substitute the collocation points (21) into Eq.(4) to obtain

$$\sum_{k=0}^m P_k(x_i) y^{(k)}(x_i) = f(x_i), \quad i = 0, 1, \dots, N \quad (22)$$

the system (22) can be written in the matrix form as

$$\sum_{k=0}^m \mathbf{P}_k \mathbf{Y}^{(k)} = \mathbf{F}, \quad (23)$$

where

$$\mathbf{P}_k = \begin{bmatrix} P_k(x_0) & 0 & \dots & 0 \\ 0 & P_k(x_1) & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \dots & P_k(x_N) \end{bmatrix}, \quad \mathbf{F} = [f(x_0) \ f(x_1) \ \dots \ f(x_N)]^T.$$

By putting the collocation points x_i in (20), we have the system

$$\left[y^{(k)}(x_i) \right] = V^{(k)}(x_i)C^T A, \quad i = 0, 1, \dots, N$$

or

$$\mathbf{Y}^{(k)} = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ \vdots \\ y^{(k)}(x_N) \end{bmatrix} = V^{(k)}C^T A, \quad (24)$$

where

$$V^{(k)} = \begin{bmatrix} V^{(k)}(x_0) \\ V^{(k)}(x_1) \\ \vdots \\ V^{(k)}(x_N) \end{bmatrix} = \begin{bmatrix} (v^0(x))^{(k)}_{x=x_0} & (v^1(x))^{(k)}_{x=x_0} & \dots & (v^N(x))^{(k)}_{x=x_0} \\ (v^0(x))^{(k)}_{x=x_1} & (v^1(x))^{(k)}_{x=x_1} & \dots & (v^N(x))^{(k)}_{x=x_1} \\ \vdots & \vdots & \ddots & \vdots \\ (v^0(x))^{(k)}_{x=x_N} & (v^1(x))^{(k)}_{x=x_N} & \dots & (v^N(x))^{(k)}_{x=x_N} \end{bmatrix}.$$

Consequently, from the matrix forms (23) and (24), we obtain the fundamental matrix equation for Eq. (4) as

$$\sum_{k=0}^m \mathbf{P}_k V^{(k)} C^T A = \mathbf{F}, \quad (25)$$

next, we can obtain the corresponding matrix forms for the conditions (5) as follows, using the relation (16) for $x = b_j$ we have the fundamental matrix equation corresponding to the mixed conditions (5) as

$$\sum_{k=0}^{m-1} \sum_{j=0}^J d_{ij}^k V^{(k)}(b_j) C^T A = [\alpha_i], \quad (26)$$

$$-\infty < b_j < \infty, \quad i = 0, 1, \dots, m-1, \quad j = 0, 1, \dots, J$$

4. Collocation method

The fundamental matrix (25) for Eq. (4) corresponding to a system of $(N+1)$ algebraic equations for the $(N+1)$ unknown coefficients $a_0, a_1, \dots, a_N$. We can write Eq. (25) shortly as

$$\mathbf{W} A = \mathbf{F} \quad \text{or} \quad [\mathbf{W}; \mathbf{F}], \quad (27)$$

so that

$$\mathbf{W} = (w_{pq}) = \sum_{k=0}^m \mathbf{P}_k V^{(k)} C^T, \quad p, q = 0, 1, \dots, N$$

we can obtain the matrix form for the mixed conditions by means of (26) as

$$U_i A = [\alpha_i] \quad \text{or} \quad [U_i; \alpha_i], \quad i = 0, 1, \dots, m-1 \quad (28)$$

where

$$U_i = \sum_{k=0}^{m-1} \sum_{j=0}^J d_{ij}^k V^{(k)}(b_j) C^T.$$

To obtain the solution of Eq. (4) under the given condition by replacing the rows matrices (28) by the m rows of the matrix (27) we have required augmented matrix

$$[\mathbf{W}^*; \mathbf{F}^*] = \begin{bmatrix} w_{00} & w_{01} & \dots & w_{0N} & ; & f(x_0) \\ w_{10} & w_{11} & \dots & w_{1N} & ; & f(x_1) \\ \dots & \dots & \dots & \dots & ; & \dots \\ w_{N-m,0} & w_{N-m,1} & \dots & w_{N-m,N} & ; & f(x_{N-m}) \\ u_{00} & u_{01} & \dots & u_{0N} & ; & \alpha_0 \\ u_{10} & u_{11} & \dots & u_{1N} & ; & \alpha_1 \\ \dots & \dots & \dots & \dots & ; & \dots \\ u_{m-1,0} & u_{m-1,1} & \dots & u_{m-1,N} & ; & \alpha_{m-1} \end{bmatrix}, \quad (29)$$

or the corresponding matrix equation

$$\mathbf{W}^* A = \mathbf{F}^*,$$

we always found that the rank $(\mathbf{W}^*) = \text{rank} [\mathbf{W}^*; \mathbf{F}^*] = N+1$ then we can write

$$A = (\mathbf{W}^*)^{-1} \mathbf{F}^*. \quad (30)$$

Thus, the coefficient $a_i, \quad i = 0, 1, \dots, N$ are uniquely determined by Eq. (29), (30)

5. Test examples

Now, we take some test examples to examine our method, the numerical computations are carried out by the Mathematica.7.0.

Example 5.1. Let us consider the following boundary value problem with variable coefficients

$$y'' + \frac{e^x}{1+e^x}y = \frac{2e^x(e^x-1)(2e^x-1)}{(1+e^x)^4}, \quad x \in (-\infty, \infty) \quad (31)$$

with the boundary conditions $y(x) \rightarrow 0$ when $|x| \rightarrow \infty$. The fundamental matrix for the pervious equation is

$$\{\mathbf{P}_0 V^{(0)}(C^T)^0 + \mathbf{P}_1 V^{(1)}(C^T)^1 + \mathbf{P}_2 V^{(2)}(C^T)^2\} A = \mathbf{F},$$

and

$$P_0(x) = \frac{e^x}{(1+e^x)}, \quad P_1(x) = 0, \quad P_2(x) = 1, \quad f(x) = \frac{2e^x(e^x-1)(2e^x-1)}{(1+e^x)^4}.$$

For $N = 4$ the collocation points are $x_0 \rightarrow \infty$, $x_1 = Ln[3 + 2\sqrt{2}]$, $x_2 = 0$, $x_3 = Ln[3 - 2\sqrt{2}]$, $x_4 \rightarrow -\infty$. And $\mathbf{P}_1$ is the zero matrix, $\mathbf{P}_2$ is the identity matrix.

The matrices $V^{(0)}$, $V^{(1)}$, $V^{(2)}$, $\mathbf{P}_0$ are in the form

$$V^{(0)} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{-1}{\sqrt{2}} & \frac{1}{2} & \frac{-1}{2\sqrt{2}} & \frac{1}{4} \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix}, \quad V^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{3}{8} & \frac{1}{2\sqrt{2}} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & \frac{-1}{2\sqrt{2}} & \frac{-3}{8} & \frac{-1}{2\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{4\sqrt{2}} & \frac{-1}{8} & 0 & \frac{1}{8} \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{-1}{4\sqrt{2}} & \frac{-1}{8} & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{P}_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4}(2 + \sqrt{2}) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4}(2 - \sqrt{2}) & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the augmented matrix for the conditions with $N = 4$ is:

for $x \rightarrow \infty$, $y = 0$ give $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ ;0 \end{bmatrix}$,

and when $x \rightarrow -\infty$ and $y = 0$ give $\begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ ;0 \end{bmatrix}$.

After the augmented matrices of the system and condition are computed, we obtain the solution

$$A = \begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 \end{bmatrix}.$$

Then the solution is

$$y(x) = \frac{1}{4}E_0(x) - \frac{1}{4}E_2(x),$$

or

$$y(x) = \frac{2e^x}{(1+e^x)^2},$$

which is the exact solution of the boundary value problem (31).

Example 5.2. Consider the following equation [23], [10]

$$Ly = f(x), \quad x \in (-\infty, \infty) \quad (32)$$

where the operator $L = \left[\frac{d^2}{dx^2} - 1 \right]$ and the boundary condition $y(x) \rightarrow 0$ when $|x| \rightarrow \infty$. The exact solution given in [10] by Fourier transform as

$$y(x) = F^{-1} \left[\frac{-1}{1 + \omega^2} F[f(x)] \right],$$

where F and F^{-1} the Fourier and inverse Fourier transform operators, and the function $f(x)$ is well defined with respect to the Fourier transform conditions. We apply our present method to Eq. (32), by taking $f(x) = -2\text{sech}^3(x)$. Table. 1 show comparing the approximate and exact solution at different N , and $x \in [-3, 3]$. The computing of the norms L_2 and L_∞ given in Table.2 where

$$L_2 = \sqrt{h \sum_{i=0}^I \left(y_{\text{Exact}}^i - y_{\text{Approximat}}^i \right)^2}, \quad L_\infty = \text{Max} \left| y_{\text{Exact}}^i - y_{\text{Approximat}}^i \right|.$$

Figure.1 show the approximate and exact solution at different N , and $x \in [-10, 10]$. And Figure. 2 show that the error function at different N and $x \in [0, 3]$.

Table.1 comparing the approximate and exact solution in example 5.2

x	Exact solution	EC with $N = 10$	Absolute error	EC with $N = 16$	Absolute error
-3	0.0993279	0.0992278	1.0×10^{-4}	0.0993274	5.13535×10^{-10}
-2.5	0.163071	0.163049	2.18×10^{-5}	0.163071	1.35492×10^{-7}
-2	0.265802	0.265983	1.8×10^{-4}	0.265803	7.49985×10^{-7}
-1.5	0.425096	0.425182	8.63×10^{-5}	0.425095	1.00225×10^{-6}
-1	0.648054	0.647794	2.6×10^{-4}	0.648055	1.0313×10^{-6}
-0.5	0.886819	0.886899	7.98×10^{-5}	0.886818	7.33901×10^{-7}
0	1	1.0002	2.02×10^{-4}	1.0	4.2298×10^{-7}
0.5	0.886819	0.886899	7.98×10^{-5}	0.886818	7.33901×10^{-7}
1	0.648054	0.647794	2.6×10^{-4}	0.648055	1.0313×10^{-6}
1.5	0.425096	0.425182	8.63×10^{-5}	0.425095	1.00225×10^{-6}
2	0.265802	0.265983	1.8×10^{-4}	0.265803	7.49985×10^{-7}
2.5	0.163071	0.163049	2.18×10^{-5}	0.163071	1.35492×10^{-7}
3	0.0993279	0.0992278	1.0×10^{-4}	0.0993274	5.13535×10^{-7}

Table. 2 comparing the L_2 , L_∞ norms

	L_2	L_∞
$N = 8$	2.313×10^{-5}	1.815×10^{-3}
$N = 10$	4.35×10^{-7}	2.60×10^{-4}
$N = 16$	6.89304×10^{-12}	1.0313×10^{-6}

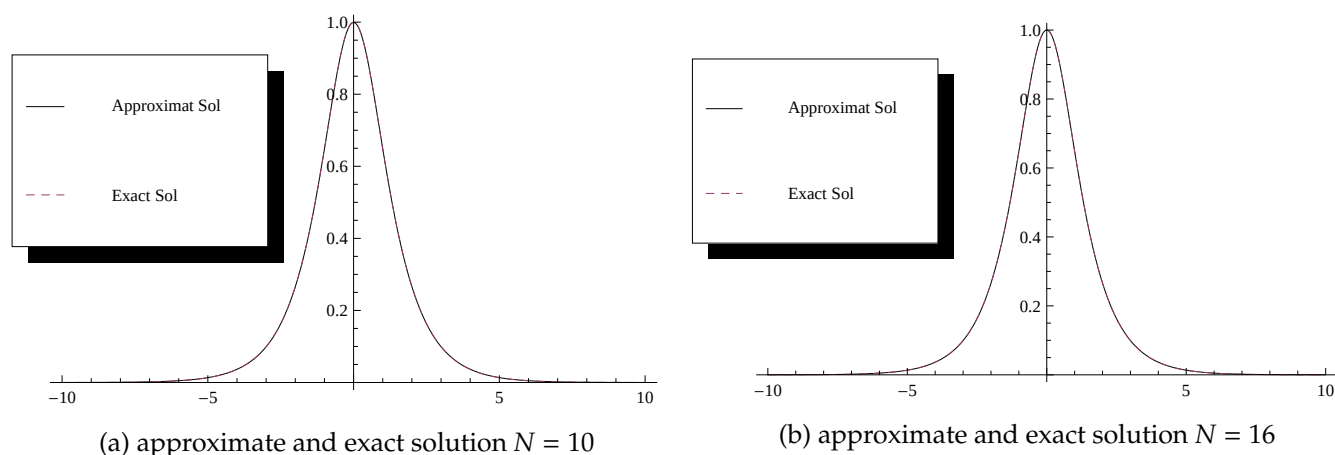
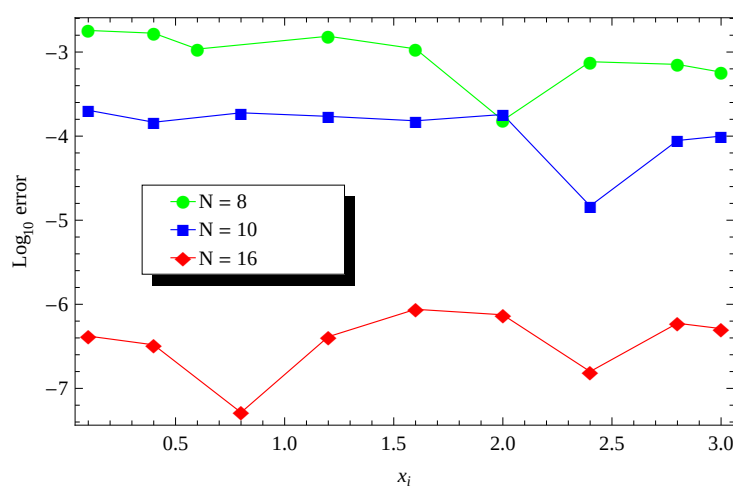


Figure 1: the approximate and exact solutions

Figure 2: error function of example 5.2 at $N = 8, 10$ and 16

Example 5.3. Consider the following differential equation

$$\frac{1}{(1+e^x)^2} y''' - \frac{e^x}{(1+e^x)^2} y'' - \frac{e^{2x}}{(1+e^x)^2} y = h(x), \quad (33)$$

with the mixed conditions, $y(0) = \frac{1}{2}$, $y(x) \rightarrow \frac{1}{2}$ when $x \rightarrow -\infty$, where, $x \in (-\infty, \infty)$ and the function $h(x)$ is in the form

$$-\frac{1}{2(1+e^x)^7} e^x (4 - 79e^x + 129e^{2x} + 10e^{3x} - 2e^{4x} + 13e^{5x} + 5e^{6x}).$$

The fundamental matrix for the pervious equation is

$$\{P_0 V^{(0)}(C^T)^0 + P_1 V^{(1)}(C^T)^1 + P_2 V^{(2)}(C^T)^2 + P_3 V^{(3)}(C^T)^3\} A = F.$$

For $N = 4$, after computing we get

$$A = \begin{bmatrix} 1 & 1 & \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Then the solution is

$$y(x) = E_0(x) + E_1(x) + \frac{1}{2} E_2(x),$$

or

$$y(x) = \frac{1 - 2e^x + 5e^{2x}}{2(1 + e^x)^2}$$

, which is the exact solution of the problem (33).

Example 5.4. Now we consider the following differential equation [23]

$$Ly = g(x), \quad x \in (-\infty, \infty) \quad (34)$$

where the operator $L = \left[\frac{d^3}{dx^3} - 1 \right]$, and the condition is $y(x) \rightarrow 0$ when $|x| \rightarrow \infty$. The exact solution is $y(x) = \text{sech}(x)$, by applying our present method to Eq. (34), the fundamental matrix is

$$\{ \mathbf{P}_0 V^{(0)}(C^T)^0 + \mathbf{P}_1 V^{(1)}(C^T)^1 + \mathbf{P}_2 V^{(2)}(C^T)^2 + \mathbf{P}_3 V^{(3)}(C^T)^3 \} A = F,$$

Table. 3 shows comparing the L_2 and L_∞ norms at different N where $h = 0.1$ and $x \in [-3, 3]$, and Figure. 3 show that the error function at different N , $x \in [0, 3]$.

Table. 3 comparing the L_2 , L_∞ norms		
	L_2	L_∞
$N = 10$	2.42677×10^{-6}	9.66674×10^{-4}
$N = 14$	8.31182×10^{-10}	1.43335×10^{-5}
$N = 16$	2.08173×10^{-11}	2.38489×10^{-6}

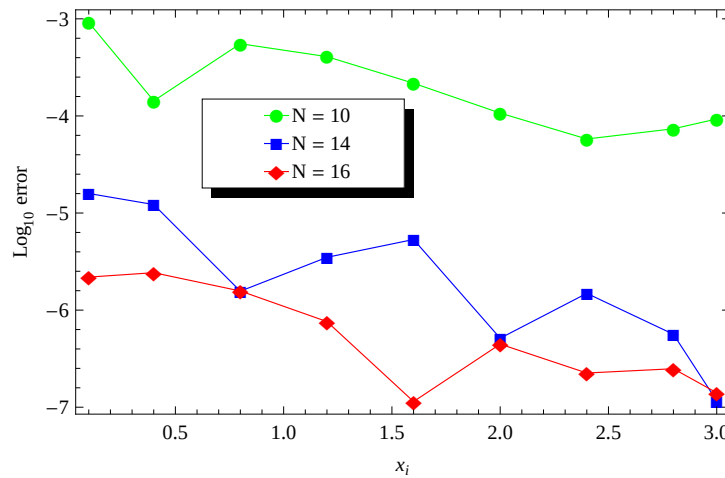


Figure 3: error function of example 5.4 at $N = 10, 14$ and 16

Example 5.5. Consider the following equation [23]

$$y'' - \frac{1}{1 + e^x} y' - \frac{15e^{2x}}{(1 + e^x)^2} y = \frac{e^{2x}}{(1 + e^x)^6}, \quad x \in (-\infty, \infty) \quad (35)$$

where, the boundary conditions $y(x) = 0$ when $x \rightarrow \infty$ and $y(x) = 1$ when $x \rightarrow -\infty$. The fundamental matrix for the pervious equation is

$$\{ \mathbf{P}_0 V^{(0)}(C^T)^0 + \mathbf{P}_1 V^{(1)}(C^T)^1 + \mathbf{P}_2 V^{(2)}(C^T)^2 \} A = F,$$

for $N = 4$, after computing we get

$$A = \begin{bmatrix} \frac{35}{128} & -\frac{7}{16} & \frac{7}{32} & -\frac{1}{16} & \frac{1}{128} \end{bmatrix}.$$

Then the solution is

$$y(x) = \frac{35}{128}E_0(x) - \frac{7}{16}E_1(x) + \frac{7}{32}E_2(x) - \frac{1}{16}E_3(x) + \frac{1}{128}E_4(x),$$

After simplifying we get

$$y(x) = \frac{1}{(1 + e^x)^4},$$

which is the exact solution of the problem (35)

6. Conclusion

In this paper a new exponential Chebyshev (EC) operational matrix of derivatives is investigated. The new operational matrix of derivatives of the EC functions is derived and introduced for solving high-order linear ordinary differential equations with variable coefficients in unbounded domain using the collocation method. The proposed differential equations and the given conditions were transformed to matrix equation with unknown exponential Chebyshev coefficients. On the other hand, the EC functions approach deals directly with infinite boundaries without singularities. This variant for our method gave us freedom to solve differential equations with boundary conditions tends to infinity. In addition, an interesting feature of this method is to find the analytical solutions if the equation has an exact solution that is a rational exponential functions. Illustrative examples are used to demonstrate the applicability and the effectiveness of the proposed technique. The method can be extended for the case of systems of linear differential and integro-differential equations with variable coefficients which is under investigation by the authors.

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