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A family of Three-point Methods of Eighth-order for Finding Multiple Roots of Nonlinear Equations

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ABSTRACT. In this paper, two new three-point eighth-order iterative methods for solving nonlinear equations are constructed. It is proved that these methods have the convergence order of eight requiring only four function evaluations per iteration. In fact, we have obtained the optimal order of convergence which supports the Kung and Traub conjecture. Kung and Traub conjectured that the multipoint iteration methods, without memory based on n evaluations, could achieve optimal convergence order 2^{n-1}. Thus, we present new iterative methods which agree with the Kung and Traub conjecture for $n = 4$ Numerical comparisons are included to demonstrate exceptional convergence speed of the proposed methods using only a few function evaluations.

KEYWORDS. Modified Newton method; Root-finding; Nonlinear equations; Multiple roots; Order of convergence; Efficiency index.

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1. Introduction

Solving nonlinear equations is one of the most important problems in numerical analysis. In this paper, we consider iterative methods to find a multiple root α of multiplicity m, i.e., $f^{(j)}(\alpha) = 0$, $j = 0, 1, ..., m-1$ and $f^{(m)}(\alpha) \neq 0$, of a nonlinear equation

$$f(x) = 0, \tag{1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval I and it is sufficiently smooth in the neighbourhood of α. In recent years, some modifications of the Newton method for multiple roots have been proposed and analysed [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. However, there are not many methods known to handle the case of multiple roots. Hence we present two higher order methods for finding multiple zeros of a nonlinear equation and only use four evaluations of the function per iteration. In addition, the new methods have a better efficiency index than the third and fourth order methods given in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11], in view of this fact, the new methods are significantly better when compared with the established methods.

The well-known Newton's method for finding multiple roots is given by

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \tag{2}$$

which converges quadratically [3]. For the purpose of this paper, we use (2) and transform Thukral et al. eighth-order method [12] to construct new eighth-order methods for finding multiple roots of nonlinear equations. In order to construct the new eighth-order method for finding multiple roots we use the new concept applied in [10].

The prime motive of this study is to develop a new class of three-step methods for finding multiple roots of nonlinear equations. The eighth-order methods presented in this paper only use four evaluations of the function per iteration. In fact, we have obtained the optimal order of convergence which supports the Kung and Traub conjecture [13]. Kung and Traub conjectured that the multipoint iteration methods, without memory based on n evaluations, could achieve optimal convergence order 2^{n-1}. Thus, we present new derivative-free methods which agree with the Kung and Traub conjecture for $n=4$. Furthermore, the new eighth-order methods have a better efficiency index than existing the two-step and three-step order methods presented in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. In view of this fact, the new methods are significantly better when compared with the established methods.

The paper is organized as follows. Two new three-point methods of optimal order are constructed in the next section by transforming the Thukral et al. eighth-order method. Convergence analysis is provided to establish the eighth-order convergence. Finally, in section 3, several numerical examples are given to demonstrate the performance of the new methods for different values of b.

2. Development of the methods and convergence analysis

In this section we define new eighth-order method for finding multiple roots of a nonlinear equation. In order to establish the order of convergence of these new methods we state the three essential definitions.

Definition 2.1. Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α. The order of convergence p is given by

$$\lim_{n\to\infty}\frac{x_{n+1}-\alpha}{(x_n-\alpha)^p}=\zeta\neq 0, \tag{3}$$

where ζ is the asymptotic error constant and $p\in\mathbb{R}^+$.

Definition 2.2. Let k be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index [14, 15] and defined as

$$p^{1/k} \tag{4}$$

where p is the order of convergence of the method.

Definition 2.3. Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1). Then the computational order of convergence [16] may be approximated by

$$\mathrm{COC}\approx\frac{\ln\left|(x_{n+1}-\alpha)(x_n-\alpha)^{-1}\right|}{\ln\left|(x_n-\alpha)(x_{n-1}-\alpha)^{-1}\right|}. \tag{5}$$

where $n\in\mathbb{N}$.

We start with the Thukral et al eighth-order iterative scheme, which is presented in [12] and given by

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \tag{6}$$

$$z_n=y_n-\left(\frac{f(x_n)+bf(y_n)}{f(x_n)+(b-2)f(y_n)}\right)\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{7}$$

$$x_{n+1} = z_n - \left(u(t) + \frac{f(z_n)}{f(y_n) - af(z_n)} + 4\frac{f(z_n)}{f(x_n)} \right) \left(\frac{f(z_n)}{f'(x_n)} \right), \tag{8}$$

where $u(t)$ is the arbitrary real function satisfying the conditions

$$u(0) = 1,\ u'(0) = 2,\ u''(0) = 10 - 4b,\ u'''(0) = 72 - 72b + 12b^2, \tag{9}$$

and a and b are real parameters. The function $u(t)$ in (8) can take many forms satisfying the conditions (9). Hence, the following two functions depending on King's parameter b are given by

$$u_1(t) = 1 + 2t + (5 - 2b)t^2 + \left(12 - 12b + 2b^2\right)t^3, \tag{10}$$

and

$$u_2(t) = \frac{5 - 2b - (2 - 8b + 2b^2)t + (1 + 4b)t^2}{5 - 2b - (12 - 12b + 2b^2)t}, \tag{11}$$

where

$$t = \frac{f(y_n)}{f(x_n)}, \tag{12}$$

and satisfy the conditions (9).

The above formula is an optimal eighth-order convergence method and is very efficient in obtaining a simple root of a nonlinear equation. However, in the case of multiple roots the formula loses its eighth-order convergence. For the purpose of this paper we shall introduce new parameters to achieve the optimal eighth-order convergence for obtaining multiple roots.

We apply the new concept recently introduced by Thukral [10] to the formula (8) and obtain the new eighth-order methods for finding multiple roots of nonlinear equations. The transformed form of (8) is given by

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)}, \tag{13}$$

$$z_n = y_n - mt^s \left(\frac{1 + bt^s}{1 + (b - 2)t^s} \right) \left(\frac{f(x_n)}{f'(x_n)} \right), \tag{14}$$

$$x_{n+1} = z_n - m\left(u(t^s) + \left(\frac{f(z_n)}{f(y_n) - af(z_n)} \right)^s + 4\left(\frac{f(z_n)}{f(x_n)} \right)^s \right) \left(\frac{f(x_n)}{f'(x_n)} \right) \left(\frac{f(z_n)}{f(x_n)} \right)^s. \tag{15}$$

In order to obtain the optimal order of convergence of the new methods, we have to modify the $u(t)$ function used in (8). In this case the following two functions depending on King's parameter b are given by

$$u_1(t^s) = 1 + 2t^s + (5 - 2b)t^{2s} + \left(12 - 12b + 2b^2\right)t^{3s}, \tag{16}$$

and

$$u_2(t^s) = \frac{5 - 2b - (2 - 8b + 2b^2)t^s + (1 + 4b)t^{2s}}{5 - 2b - (12 - 12b + 2b^2)t^s}, \tag{17}$$

where $s = m^{-1}$, $n \in \mathbb{N}$, $b \in \mathbb{R}^+$ and provided that the denominators of (13)-(15) are not equal to zero.

Theorem 2.4. Let $\alpha \in I$ be a multiple root of multiplicity m of a sufficiently differentiable function $f : I \subset \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to α and $u_1(t^s)$ is defined by (16) then the order of convergence of the new iterative method defined by (15) is eight.

Proof. Let α be a multiple root of multiplicity m of a sufficiently smooth function $f(x)$, $e = x - \alpha$ and $\widehat{e} = y - \alpha$ where y is defined in (13). Using the Taylor expansion of $f(x)$ and $f(y)$ about α, we have

$$f(x_n) = \left(\frac{f^{(m)}(\alpha)}{m!}\right) e_n^m \left[1 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots\right], \tag{18}$$

$$f'(x_n) = \left(\frac{f^{(m)}(\alpha)}{(m-1)!}\right) e_n^{m-1} \left[1 + \left(\frac{m+1}{m}\right) c_1 e_n + \left(\frac{m+2}{m}\right) c_2 e_n^2 + \cdots\right], \tag{19}$$

where $n \in \mathbb{N}$ and

$$c_k = \frac{m! f^{(m+k)}(\alpha)}{(m+k)! f^{(m)}(\alpha)}. \tag{20}$$

Moreover by (13), we have

$$y_n = e_n - m\frac{f(x_n)}{f'(x_n)} = e_n - e_n\left[1 - \frac{c_1}{m} e_n + \frac{(m+1)c_1^2 - 2mc_2}{m^2} e_n^2 + \cdots\right]. \tag{21}$$

The expansion of $f(y_n)$ and $f(z_n)$ about α are given as

$$f(y_n) = \left(\frac{f^{(m)}(\alpha)}{m!}\right) \widehat{e}_n^{\,m} \left[1 + c_1 \widehat{e}_n + c_2 \widehat{e}_n^{\,2} + c_3 \widehat{e}_n^{\,3} + \cdots\right]. \tag{22}$$

By using (18) and (22), we get

$$\begin{aligned}\left(\frac{f(y_n)}{f(x_n)}\right)^s &= e_n \left(\frac{c_1}{m}\right) \left[1 + \left(2mc_2 - (m+2)c_1^2\right) m^{-1} e_n^2\right. \\ &\left. + 2^{-1}\left(\left(2m^2 + 7m + 7\right) c_1^3 - 2m(3m+7) c_1 c_2 + 6m^2 c_3\right) m^{-3} e_n^3 + \cdots\right]\end{aligned} \tag{23}$$

Since from (14) we have

$$z_n = y_n - mt^s \left(\frac{1 + bt^s}{1 + (b-2)t^s}\right) \left(\frac{f(x_n)}{f'(x_n)}\right), \tag{24}$$

$$f(z_n) = \left(\frac{f^{(m)}(\alpha)}{m!}\right) d_n^m \left[1 + c_1 d_n + \cdots\right]. \tag{25}$$

As before we expand $f(z_n)$, $\left(f(z_n) f(x_n)^{-1}\right)^s$ and substituting $u_1(t^s)$ expressions in (15). After simplification we obtain the error equation

$$\begin{aligned} e_{n+1} = &- 24^{-1} m^{-7} c_1 \left(mc_1^2 + c_1^2 - 2mc_2 + 4bc_1^2\right) \left(24b^3 c_1^4 + 48rb^2 c_1^4 - 192b^2 c_1^4\right. \\ &+ 24rmbc_1^4 + 24rbc_1^4 - 12mbc_1^4 + 276bc_1^4 - 48rmbc_1^2 c_2 + 24mbc_1^2 c_2 - 78mc_1^4 \\ &- 7m^2 c_1^4 + 3rm^2 c_1^4 - 407c_1^4 + 3rc_1^4 + 6rmc_1^4 + 24m^2 c_1^2 c_2 + 156mbc_1^2 c_2 - 12rm^2 c_1^2 c_2 \\ &\left.- 12rmc_1^2 c_2 - 12m^2 c_1 c_3 + 12rm^2 c_2^2 - 12m^2 c_2^2\right) e_n^8 + \cdots \end{aligned} \tag{26}$$

The error equation (26) establishes the eighth-order convergence of the new method define by (15). □

Theorem 2.5. Let $\alpha \in I$ be a multiple root of multiplicity m of a sufficiently differentiable function $f : I \subset \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficiently close to α and $u_2(t^s)$ is defined by (17) then the order of convergence of the new iterative method defined by (15) is eight.

Proof. Based on $u_2(t^s)$ and substituting the appropriate expressions in (15), we obtain the error equation

$$\begin{aligned} e_{n+1} = & -24^{-1}m^{-7}c_1(2b-5)^{-1}\left(mc_1^2+c_1^2-2mc_2+4bc_1^2\right)\left(307c_1^4+60m^2c_2^2+35m^2c_1^4+390mc_1^4\right. \\ & -15rc_1^4+60m^2c_1c_3-120m^2c_1^2c_2-780mc_1^2c_2+48rmb^2c_1^4-108rmbc_1^4+6rm^2bc_1^4+24rm^2bc_2^2 \\ & +60rm^2c_1^2c_2+60rmc_1^2c_2+1262bc_1^4-792b^2c_1^4-96rmb^2c_1^2c_2+216rmbc_1^2c_2-24rm^2bc_1^2c_2 \\ & +192mbc_1^2c_2-96mbc_1^4+72b^3c_1^4-24mb^2c_1^4-14m^2bc_1^4-24m^2bc_2^2+96rb^3c_1^4-192rb^2c_1^4 \\ & \left.-114rbc_1^4-15rm^2c_1^4-30rmc_1^4-60rm^2c_2^2+48mb^2c_1^2c_2+48m^2bc_1^2c_2-24m^2bc_1c_3\right)e_n^8+\cdots \end{aligned} \tag{27}$$

The error equation (27) establishes the eighth-order convergence of the new method defined by (15).

□

3. Application of the new optimal order iterative method

The present eighth-order methods given by (15) are employed to solve nonlinear equations with multiple roots. To demonstrate the performance of the new eighth-order methods, we use ten particular nonlinear equations. We shall determine the consistency and stability of results by examining the convergence of the new iterative methods. The findings are generalised by illustrating the effectiveness of the eighth-order methods for determining the multiple root of a nonlinear equation. Consequently, we give estimates of the approximate solution produced by the methods considered and list the errors obtained by each of the methods. The numerical computations listed in the tables were performed on an algebraic system called Maple. In fact, the errors displayed are of absolute value and insignificant approximations by the various methods have been omitted in the following tables.

Remark 3.1. The new eighth-order method requires four function evaluations and has the order of convergence eight. To determine the efficiency index of the new method, we shall use the Definition 2.2. Hence, the efficiency index of the new method given by (15) is $\sqrt[4]{8} \approx 1.682$ whereas the efficiency index of the fourth-order and third-order methods is given as $\sqrt[4]{6} \approx 1.565$ $\sqrt[4]{4} \approx 1.414$ and $\sqrt[3]{3} \approx 1.442$ respectively. We can see that the efficiency index of the new eighth-order method is better than the other similar methods.

Remark 3.2. The test functions and their exact root α are displayed in Table 1. The difference between the root α and the approximation x_n for test functions with initial estimate x_0, and $a = 0$ are displayed in tables. In fact, x_n is calculated by using the same total number of function evaluations (TNFE) for all methods. In the calculations, 12 TNFE are used by each method. Furthermore, the computational order of convergence (COC) are displayed in tables. From the tables, we observe that the COC perfectly coincides with the theoretical result. However, this is the case when initial approximations are reasonably close to the sought zeros.

Remark 3.3. For finding simple root different methods were formed from the King's method (7). The special cases are; when $b = 0$ Ostrowski method is obtained, when $b = 1$ Kou's method [17] and when $b = 2$ Chun's method [18], and further eighth-order methods were developed [12, 19]. Hence, for different values of b we list the new eighth-order methods for finding multiple roots in the following tables.

TABLE 1. Test functions and their roots.

Functions	m	Roots	Initial Estimate
$f_1(x) = (x^3 + x + 1)^7$	$m = 7$	$\alpha = -0.68232...$	$x_0 = -0.9$
$f_2(x) = \left(xe^{x^2} - \sin(x)^2 + 3\cos(x) + 5\right)^4$	$m = 4$	$\alpha = -1.207647...$	$x_0 = -1.2$
$f_3(x) = \left((x-1)^{10} - 1\right)^9$	$m = 9$	$\alpha = 0$	$x_0 = 0.01$
$f_4(x) = (\exp(x) + x - 20)^{95}$	$m = 95$	$\alpha = 2.842438...$	$x_0 = 3.5$
$f_5(x) = (\cos(x) + x)^{15}$	$m = 15$	$\alpha = -0.739085...$	$x_0 = -1$
$f_6(x) = \left(\sin(x)^2 - x^2 + 1\right)^{200}$	$m = 200$	$\alpha = 1.404491...$	$x_0 = 1.7$
$f_7(x) = \left(e^{-x^2} - e^{x^2} - x^8 + 10\right)^{30}$	$m = 30$	$\alpha = 1.239417...$	$x_0 = 1.3$
$f_8(x) = (6x^5 + 5x^4 - 4x^3 + 3x^2 - 2x + 1)^{55}$	$m = 55$	$\alpha = -1.572484...$	$x_0 = -2$
$f_9(x) = (\tan(x) - e^x - 1)^{11}$	$m = 11$	$\alpha = 1.371045...$	$x_0 = 1.4$
$f_{10}(x) = (\ln(x^2 + 3x + 5) - 2x + 7)^{31}$	$m = 31$	$\alpha = 5.469012...$	$x_0 = 6$

TABLE 2. Comparison of iterative methods (14)

f_i	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$
f_1	0.559e-55	0.325e-42	0.582e-38	0.157e-35	0.679e-34
f_2	0.630e-53	0.271e-54	0.190e-60	0.459e-55	0.977e-54
f_3	0.433e-95	0.169e-79	0.263e-74	0.346e-71	0.549e-69
f_4	0.619e-83	0.279e-65	0.441e-60	0.467e-57	0.573e-55
f_5	0.358e-73	0.448e-69	0.200e-66	0.173e-64	0.562e-63
f_6	0.421e-46	0.364e-38	0.980e-35	0.118e-32	0.327e-31
f_7	0.227e-62	0.134e-47	0.523e-43	0.214e-40	0.124e-38
f_8	0.379e-19	0.176e-12	0.218e-10	0.261e-9	0.122e-8
f_9	0.349e-70	0.325e-45	0.341e-40	0.172e-37	0.102e-35
f_{10}	0.170e-124	0.912e-123	0.253e-121	0.436e-120	0.526e-119

TABLE 3. COC of various iterative methods (14)

f_i	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$
f_1	4.0000	3.9998	3.9992	3.9989	3.9984
f_2	4.0500	3.3624	4.4909	4.1145	4.0693
f_3	3.9999	3.9999	4.0001	4.0001	4.0001
f_4	3.9999	3.9999	4.0001	4.0000	3.9999
f_5	3.9999	3.9999	3.9998	4.0000	4.0000
f_6	3.9999	3.9992	3.9983	3.9976	3.9967
f_7	4.0000	3.9999	4.0000	4.0000	3.0000
f_8	3.9798	3.9014	3.8292	3.7653	3.7094
f_9	4.0000	3.9999	3.9996	3.9993	3.9991
f_{10}	3.9999	3.9999	4.0000	4.0000	4.0000

TABLE 4. Comparison of various iterative methods (15)

f_i	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$
f_1	0.337e-324	0.301e-287	0.118e-264	0.445e-248	0.222e-238
f_2	0.211e-183	0.148e-173	0.260e-169	0.665e-166	0.854e-164
f_3	0.620e-617	0.349e-572	0.348e-543	0.328e-520	0.497e-506
f_4	0.134e-235	0.718e-193	0.829e-173	0.573e-159	0.257e-151
f_5	0.101e-509	0.131e-496	0.244e-482	0.843e-470	0.107e-460
f_6	0.470e-179	0.359e-162	0.955e-150	0.174e-140	0.101e-134
f_7	0.302e-368	0.471e-325	0.279e-300	0.522e-282	0.244e-271
f_8	0.221e-256	0.802e-226	0.850e-207	0.473e-193	0.527e-185
f_9	0.983e-381	0.380e-304	0.566e-277	0.105e-257	0.578e-247
f_{10}	0.515e-924	0.386e-917	0.160e-909	0.326e-902	0.518e-896

TABLE 5. COC of various iterative methods (15)

f_i	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$
f_1	8.0000	8.0001	8.0001	8.0001	8.0001
f_2	8.0001	8.0000	8.0000	8.0000	8.0000
f_3	8.0000	8.0001	8.0001	8.0001	8.0001
f_4	8.0000	8.0000	8.0000	8.0000	8.0000
f_5	8.0000	8.0000	8.0000	8.0000	8.0000
f_6	8.0000	8.0000	8.0000	8.0000	8.0000
f_7	8.0000	8.0000	8.0000	8.0000	8.0000
f_8	8.0000	8.0000	8.0000	8.0000	8.0000
f_9	8.0000	8.0000	8.0000	8.0000	8.0000
f_{10}	8.0000	8.0000	8.0000	8.0000	8.0000

TABLE 6. Comparison of various iterative methods (15)

f_i	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$
f_1	0.108e-371	0.242e-288	0.281e-272	0.369e-212	0.407e-236
f_2	0.946e-201	0.778e-174	0.255e-172	0.106e-163	0.180e-163
f_3	0.347e-680	0.830e-574	0.188e-557	0.197e-501	0.130e-503
f_4	0.273e-291	0.736e-194	0.390e-179	0.243e-35	0.637e-149
f_5	0.159e-521	0.481e-497	0.215e-486	0.355e-461	0.120e-459
f_6	0.214e-194	0.132e-162	0.440e-153	0.150e-24	0.150e-133
f_7	0.147e-425	0.254e-326	0.302e-309	0.210e-251	0.636e-269
f_8	0.271e-291	0.118e-226	0.179e-212	0.255e-88	0.529e-183
f_9	0.128e-513	0.108e-305	0.294e-287	0.625e-214	0.371e-244
f_{10}	0.627e-927	0.290e-917	0.675e-911	0.181e-899	0.125e-895

TABLE 7. COC of the iterative method (15)

f_i	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$
f_1	8.0000	8.0001	8.0001	8.0001	8.0001
f_2	8.0001	8.0000	8.0000	8.0000	8.0000
f_3	8.0000	8.0001	8.0001	8.0001	8.0001
f_4	8.0000	8.0000	8.0000	8.0000	8.0000
f_5	8.0000	8.0000	8.0000	8.0000	8.0000
f_6	8.0000	8.0000	8.0000	8.0000	8.0000
f_7	8.0000	8.0000	8.0000	8.0000	8.0000
f_8	8.0000	8.0000	8.0000	8.0000	8.0000
f_9	8.0000	8.0000	8.0000	8.0000	8.0000
f_{10}	8.0000	8.0000	8.0000	8.0000	8.0000

4. Conclusion

In this paper, we have introduced two new eighth-order iterative methods for solving nonlinear equations with multiple roots. Convergence analysis proves that the new methods preserve their order of convergence. Simply introducing new parameters in (15) we have achieved eighth-order of convergence. The prime motive of presenting these new methods was to establish a higher order of convergence method than the existing third and fourth order methods [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. We have examined the effectiveness of the new methods by showing the accuracy of the multiple roots of several nonlinear equations. After an extensive experimentation, it can be concluded that the convergence of the tested multipoint methods of the eighth order is remarkably fast. Furthermore, in most of the test examples, empirically we have found that the best results are obtained when $b = 0$. The main purpose of demonstrating the new methods for different types of nonlinear equations was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of the new iterative method. We have shown numerically and verified that the new methods converge to the optimal order eight. The major advantages of the new eighth-order iterative method are; first, able to evaluate simple and multiple roots, secondly, simple formula when compared to existing methods containing long expressions of m (see [2]), thirdly, have achieved a first optimal eighth-order method. Finally, we conjecture that new concept introduced by Thukral [10] may be applied to transform the iterative methods used for finding simple root of nonlinear equations.

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