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RESEARCH ARTICLE

The Deficient Discrete Quintic Spline Interpolation with Uniform Mesh

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The objective of the paper is to investigate existence, uniqueness and error bounds of deficient discrete quintic spline interpolate matching the given functional values and its first difference at mesh points and second difference at boundary points.

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1. Introduction

Let us consider a mesh on $[0, 1]$ which is defined by

$$0 = x_0 < x_1 < ... < x_n = 1$$

with $x_i - x_{i-1} = P$, for $i = 1, 2, \ldots, n.$ and $P \geq h$, throughout $h > 0$, will be a real number. Consider a real continuous function $S(x, h)$ defined over $[0, 1]$ which is such that its restriction S_i on $[x_{i-1}, x_i]$ is a polynomial of degree 5 or less for $i = 1, 2, \ldots, n$, then $S(x, h)$ defines a discrete quintic spline if

$$D_h^{\{j\}} S_i(x_i, h) = D_h^{\{j\}} S_{i+1}(x_i, h) \quad j = 0, 1, 2, 3. \tag{1}$$

where the difference operator D_h are defined as

$$D_h^{\{0\}} f(x) = f(x), \quad D_h^{\{1\}} f(x) = (f(x + h) - f(x - h))/h$$
$$D_h^{\{2\}} f(x) = \frac{f(x + h) - 2f(x) + f(x - h)}{2h}.$$

Let $B(5, 1, \Delta, h)$ is the class of deficient quintic splines interpolation of deficiency of one, where in $D^*(5, 1, \Delta, h)$ denotes the class of all discrete deficient quintic splines which satisfies the boundary conditions

$$\begin{aligned} D_h^{\{2\}} S(x_0, h) &= D_h^{\{2\}} f(x_0), \\ D_h^{\{2\}} S(x_n, h) &= D_h^{\{2\}} f(x_n), \end{aligned} \tag{2}$$

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Mangasarian and Schumaker [1, 2] introduced discrete splines to find solution of certain minimization problem. For different constructive aspect's of discrete splines, we refer to Schumaker [3], Astor Duris [4] Jia [5], Dikshit and Powar [6] and Rana [7]. Rana and Dubey [8, 9]. The objective of the present paper is to study the existence, uniqueness and convergence properties of deficient discrete quintic spline with uniform mesh point.

We introduce the following interpolating conditions for a given function f

$$\left.\begin{aligned} S(x_i,h) &= f(x_i) \\ D_h^{\{1\}}S(x_i,h) &= D_h^{\{1\}}f(x_i) \end{aligned}\right\} \qquad i = 0,1,\ldots,n \tag{3}$$

and prove the following:

Problem 1: Given $h > 0$, for what restriction on P does there exists a unique $S(x,h) \in B^*(5,1,P,h)$ which satisfies the conditions (2) and (3).

2. Existence and Uniqueness

Let $R(z)$ be a quintic Polynomial on $[0,1]$, then we can show that

$$\begin{aligned} R(z) = R(0)T1(z) + R(1)T_2(z) + D_h^{\{1\}}R(0)T_3(z) + D_h^{\{1\}}R(1)T_4(z) \\ + D_h^{\{2\}}R(0)T_5(z) + D_h^{\{2\}}R(1)T_6(z), \end{aligned} \tag{4}$$

where

$$\begin{aligned} T_1(z) &= (1-z)^3(1+3z-6z^2) \\ T_2(z) &= z^3(10-15z+6z^2), \\ T_3(z) &= z(1-6z^2+8z^3-3z^4), \\ T_4(z) &= z^3(-4+7z-3z^2), \\ T_5(z) &= z^2(1-z)^3/2, \\ T_6(z) &= z^3(1-z)^2/2. \end{aligned}$$

Now, we are set to answer the Problem 1, in the following:

Theorem 2.1 For any $h > 0$, there exists a unique deficient discrete quintic polynomial $S(x,h) \in B^*(5,1,P,h)$ which satisfies the conditions (2) and (3).

Proof Denoting $x = x_i + pt$, $0 \le t \le 1$, we can expressed (4) in the form of the restriction $S_i(x,h)$ of the deficient discrete quintic spline $S(x,h)$ on $[x_i, x_{i+1}]$ as follows:

$$\begin{aligned} S(x,h) = f(x_i)T(x) + f(x_{i+1})T_2(x) + PD_n^{\{1\}}f(x_i)T_3(x) + PD_n^{\{1\}}f(x_{i+1})T_4(x) \\ + PD_n^{\{2\}}f(x_i)T_5(x) + PD_n^{\{2\}}f(x_{i+1})T_6(x), \end{aligned} \tag{5}$$

where

$$T_1(x) = \frac{1}{P^5}(x_{i+1}-x)^3[P^2+3P(x-x_i)-6(x-x_i)^2]$$

$$T_2(x) = \frac{(x-x_i)^3}{P^5}[10P^2-15(x-x_i)P+6(x-x_i)^2]$$

$$T_3(x) = \frac{(x-x_i)}{P^5}[P^4-6(x-x_i)^2P^2+8P_i(x-x_i)^3-3(x-x_i)^4]$$

$$T_4(x) = \frac{(x-x_i)^3}{P^5}(x_{i+1}-x)(3(x-x_i)-4P)$$

$$T_5(x) = \frac{(x-x_i)^2}{2P^5}(x_{i+1}-x)^3$$

$$T_6(x) = \frac{(x-x_i)^3(x_{i+1}-x)^2}{2P^5}.$$

Observing (4) it may easily be verified that $S_i(x,h)$ is a quintic on $[x_i, x_{i+1}]$ for $i = 0, 1, \ldots, n-1$ satisfying (2) and (3) and writing $H(a,b) = ap_i^2 + bh^2$, for real a, b and $D_h^{\{2\}}S_i(x,h) = m_i(h) = m_i$, (say), we shall apply continuity of the third difference of $S(x,h)$ at x_i in (5) to see that

$$m_{i+1}H(1,10) + 2H(3,10) - m_{i-1}H(1,10) = F_i - F_{i-1} \tag{6}$$

where $F_i = [H(20,120)\Delta f(x_i) - H(12,6)D_h^{\{1\}}f(x_i)P - PD_h^{\{1\}}f(x_{i+1})H(8,60)]$ and $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$.

We can easily see that excess of the absolute value of the coefficient of m_i over the sum of absolute value of the coefficients of m_{i-1} and m_{i+1} in (6) under the condition of Theorem 2.1 is given by

$$l_i(h) = 2H(2,0)$$

which is clearly positive, thus the coefficient matrix of the system of equations (6) is diagonally dominant and hence invertible, therefore the system of equation (6) has "a unique" solution, which completes the proof of the Theorem 2.1. ■

Remark 2.2 The studies concerning discrete splines smaller values of h, have special significance for simple region that "discrete" spline reduce to continuous spline as $h \to 0$.

3. Error Bounds

For a given $h > 0$, we introduce the set

$$R_h = \{x_0 + jh |\ j \textit{ is an integer}\}$$

and define a discrete interval as follows

$$[0,1]_h = [0,1] \bigcap R_h$$

for a function f and discrete points x_1, x_2, x_3 in the domains the first and second divided difference are defined as

$$[x_1, x_2]_f = \frac{[f(x_1) - f(x_2)]}{(x_1 - x_2)}$$

and

$$[x_1, x_2, x_3]_f = \frac{[x_2, x_3]_f - [x_1, x_2]_f}{(x_3 - x_1)}.$$

respectively similarly, we can define the higher order divided difference. Now in this section, we shall obtain the precise estimate of the error bounds for deficient discrete quintic spline interpolation for s of Theorem 2.1, i.e. $e = f - s$ over the discrete interval $[0, 1]_h$.

We shall need the following Lemma due to Lyche [10].

Lemma 3.1 Let $\{a_i\}_{i=1}^m$ and $\{b_j\}_{j=1}^n$ be given sequence of non-negative real numbers such that $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$ then for any real valued function f, defines on discrete interval $[0, 1]_h$, we have

$$\begin{aligned} |\sum_{i=1}^{m} a_j[x_{j_0}, x_{j_1}, ..., x_{j_k}]_f - \sum_{j=1}^{n} b_j[y_{j_0}, y_{j_1}, ..., y_{j_k}]_f| \\ \leq w(D_n^{(K)} f, |1 - kh|) \frac{\sum a_j}{k!} \end{aligned}$$

for relevant values of j, k. It may be observed that system of equation (6) may be written as

$$A(h)M(h) = F \tag{7}$$

where $A(h)$ is the coefficients matrix and $M(h) = m_i(h)$. However, already shown in the proof of Theorem 2.1 $A(h)$ is invertible. Denoting the inverse of $A(h)$ by $A^{-1}(h)$, we note that row max norm $||A^{-1}(h)||$ satisfies the following inequality

$$||A^{-1}(h)|| \leq y(h) \tag{8}$$

where $y(h) = \max\{l_i(h)\}^{-1}$.

For convenience, we write $f^{\{1\}}$ for $D_h^{\{1\}} f, f_i^{\{2\}}$ for $D_h^{\{2\}} f(x_i)$ and $w(f, p)$ for modules of continuity of "f" the discrete norm of a function f over the interval $[0, 1]_h$ is defined by

$$||f|| = \max |f(x)|. \tag{9}$$

Theorem 3.2 Suppose $S(x, h)$ is the deficient discrete quintic splines interpolate of Theorem 2.1 then

$$||e(x)|| \leq k_1(p, h) w(f^{\{1\}}, p), \tag{10}$$

$$||e^{\{1\}}(x)|| \leq k_2(p, h) w(f^{\{1\}}, p), \tag{11}$$

and

$$||e^{\{2\}}(x)|| \leq y(h) k^*(p, h) w(f^{\{1\}}, p) \tag{12}$$

$$||e^{\{2\}}(x)|| \leq k(p,h)w(f^{\{1\}},p)$$

where $k_1(p,h), k_2(p,h), k(p,h)$ and $k^*(p,h)$ are some positive functions of p and h.

Proof Writing $f(x_i) = f_i$. Equation (7) may be written as

$$A(h) \cdot e^{\{2\}}(x_i) = F_i(h) - A(h)f_i^{\{2\}} = L_i \qquad \text{(say)}. \tag{13}$$

We shall estimate $L_i(f)$ by using Lemma 3.1, due to Lyche [10]. It may observe that the

$$|L_i(f)| = |\sum_{i=1}^{6} a_i[x_{i0}, x_{i1}]_f - \sum_{j=1}^{8} b_j[y_{j0}, y_{j1}]_f| \tag{14}$$

where

$$\begin{aligned}
a_1 &= H(20,120), \quad b_1 = H(12,60)\\
a_2 &= H(20,120), b_2 = H(8,60)\\
a_3 &= \frac{1}{h}H(1,10), \quad b_3 = H(12,60)\\
a_4 &= \frac{1}{h}H(3,10), \quad b_4 = H(8,60)\\
a_5 &= \frac{1}{h}H(1,10), \quad b_5 = \frac{1}{h}H(1,10)\\
a_6 &= \frac{1}{h}H(3,10), \quad b_6 = \frac{1}{h}H(3,10)\\
b_7 &= \frac{1}{h}H(1,10)\\
b_8 &= \frac{1}{h}H(3,10)
\end{aligned}$$

and

$$\begin{aligned}
&x_{10} = x_{21} = x_{41} = y_{60} = x_{61} = y_{80} = x_i,\\
&x_{11} = x_{50} = y_{71} = x_{i+1},\\
&y_{10} = y_{40} = x_{40} = x_{60} = x_i - h,\\
&y_{11} = x_i + h = y_{41} = y_{61} = y_{81},\\
&y_{20} = x_{i+1} - h = y_{70},\\
&y_{21} = x_{i+1} + h = x_{51},\\
&x_{20} = x_{30} = y_{51} = x_{i-1},\\
&x_{31} = x_{i-1} + h = y_{31},\\
&y_{30} = y_{50} = x_{i-1} - h.
\end{aligned}$$

Since $a_1 = b_1 + b_2$, $a_2 = b_3 + b_4$, $a_3 = b_5$, $a_4 = b_6$, $a_5 = b_7$, $a_6 = b_8$. Observing that

$$\sum_{i=1}^{6} a_i = \sum_{j=1}^{8} b_j = [2H(20,120) + \frac{2}{h}H(4,20) = K^*(P,h) \quad \text{(say)}. \tag{15}$$

Thus apply Lemma 3.1 for $m = 6$, $n = 8$ and $K = 1$

$$|L_i(f)| \leq K^*(P,h)w(f^{\{1\}},P). \tag{16}$$

Now using the equations (8) and (16) in (13), we get

$$||e^{\{2\}}(x_i)|| \leq y(h)K^*(P,h)w(f^{\{1\}},P). \tag{17}$$

This complete the proof of Theorem 3.2.

To obtain the bound of $e(x)$, we replace m_i by $e(x_i)$ in equation (5) to get

$$e(x) = P^2[e_i^{\{2\}}T_5(t) + e_{i+1}^{\{2\}}T_6(t) + L_i^*(f)] \tag{18}$$

where

$$L_i^*(f) = P^2[\{f_i^{\{2\}}T_5(t) + f_{i+1}^{\{2\}}T_6(t)\} + f_iT_1(t) + f_{i+1}T_2(t) + P\{f_i^{\{1\}}T_3(t) + f_{i+1}^{\{1\}}T_4(t)\} - f(x)].$$

Now $L_i^*(f)$ in (18) may be rewritten as in the form of divided difference as follows

$$|L_i(f)| = |\sum_{i=1}^{5} a_i[x_{i0}, x_{i1}]f - \sum_{j=1}^{3} b_j[y_{j0}, y_{j1}]f|$$

where

$$\begin{aligned}
a_1 &= P[10t^3 - 15t^4 + 6t^5] \\
a_2 &= P[t - 6t^3 + 8t^4 - 3t^5] \\
a_3 &= P[-4t^3 + 7t^4 - 3t^5] \\
a_4 &= P^2(t^2 - 3t^3 + 3t^4 - t^5)/2 \\
a_5 &= P^2(t^2 - 2t^4 + t^5)/2 \\
b_1 &= P^2(t^2 - 3t^3 + 3t^4 - t^5)/2 \\
b_2 &= P^2(t^2 - 2t^4 + t^5)/2 = a_5, \\
b_1 &= a_4, \\
b_3 &= a_1 + a_2 + a_3, \\
b_3 &= tP
\end{aligned}$$

and

$$\begin{aligned}
&x_{10} = x_i = x_{40} = y_{11} = y_{30}, \\
&x_{30} = x_{i+1} - h = y_{20}, \\
&x_{11} = x_{i+1} = x_{50} = y_{21}, \\
&x_{31} = x_{i+1} + h = x_{51}, \\
&x_{20} = x_i - h = y_{10}, \quad y_{31} = x, \\
&x_{21} = x_i + h = x_{11}.
\end{aligned}$$

Since $a_5 = b_2$, $a_4 = b_1$, $a_1 + a_2 + a_3 = b_3$. Therefore

$$\sum_{i=1}^{5} a_i = \sum_{j=1}^{3} b_j = \frac{P^2}{2}(t + t^2 - 3t^3 + t^4) + Pt = K(P, h) \quad \text{(say)}.$$

Therefore, applying Lemma 3.1 for $m = 5$, $n = 3$ and $K = 1$ in (18) we get

$$|L_i^*(f)| \leq K(P, h)\, w(f^{\{1\}}, P) \tag{19}$$

and finally applying bounds of (17) and (19) in (18), we get inequality (10) of Theorem 3.2.

We now proceed to find $e_i^{\{1\}}(x)$. Now

$$\begin{aligned} D_h^{\{1\}} s_i(x) &= f_i T_1^{\{1\}}(t) + f_{i+1} T_2^{\{1\}}(t) + P\{f_i^{\{1\}} T_3^{\{1\}}(t) + f_{i+1}^{\{1\}} T_4^{\{1\}}(t)\} \\ &\quad + P^2\{s_i^{\{2\}} T_5^{\{1\}}(t) + s_{i+1}^{\{2\}} T_6^{\{1\}}(t)\}. \end{aligned}$$

Therefore

$$P D_h^{\{1\}} e_i(x) = P^2[e_i^{\{2\}} T_5^{\{1\}}(t) + e_{i+1}^{\{2\}} T_6^{\{1\}}(t)] + U_i(f) \tag{20}$$

where

$$\begin{aligned} U_i(f) &= P\{f_i T_1^{\{1\}}(t) + f_{i+1} T_2^{\{1\}}(t)\} + P^2\{f_i^{\{1\}} T_3^{\{1\}}(t) + f_{i+1}^{\{1\}} T_4^{\{1\}}(t)\} \\ &\quad + P^3\{f_i^{\{2\}} T_5^{\{1\}}(t) + f_{i+1}^{\{2\}} T_6^{\{1\}}(t)\} - P f_i^{\{1\}} \end{aligned}$$

and

$$\begin{aligned} Q_1^{\{1\}}(t) &= Q_1^{\{1\}}(t) = [-30(t^2 + h^2) + 60t(t^2 + h^2) - 30t^2(t^2 + h^2)] \\ T_2^{\{1\}}(t) &= [30(t^2 + h^2) - 60t(t^2 + h^2) - 30t^2(t^2 + h^2)] \\ T_3^{\{1\}}(t) &= [1 - 18(t^2 + h^2) + 32t(t^2 + h^2) - 15t^2(t^2 + h^2)] \\ T_4^{\{1\}}(t) &= [-12(t^2 + h^2) + 28t(t^2 + h^2) - 15t^2(t^2 + h^2)] \\ T_5^{\{1\}}(t) &= \frac{1}{2}[2t - 9(t^2 + h^2) + 12t(t^2 + h^2) - 5t^2(t^2 + h^2)] \\ T_6^{\{1\}}(t) &= \frac{1}{2}[3(t^2 + h^2) - 8t(t^2 + h^2) + 5t^2(t^2 + h^2)]. \end{aligned}$$

Now, rewriting $U_i(f)$ in terms of Divided difference we have

$$|U_i(f)| = \left| \sum_{i=1}^{5} a_i [x_{i0}, x_{i1}]_f - \sum_{j=1}^{3} b_j [y_{j0}, y_{j1}]_f \right|$$

where

$$
\begin{aligned}
a_1 &= P^2(30 - 60t - 30t^2)(t^2 + h^2)\\
a_2 &= P^2\{1 - (18 + 32t - 15t^2)(t^2 + h^2)\}\\
a_3 &= P^2(-12 + 28t - 15t^2)(t^2 + h^2)\\
a_4 &= \frac{P^3}{h}\frac{1}{2}(2t - (9 + 12t - 5t^2)(t^2 + h^2))\\
a_5 &= \frac{P^3}{h}\frac{1}{2}(3 - 8t + 5t^2)(t^2 + h^2)\\
b_1 &= \frac{P^3}{2h}(2t - (9 + 12t - 5t^2)(t^2 + h^2))\\
b_2 &= \frac{P^3}{2h}(3 - 8t + 5t^2)(t^2 + h^2))\\
b_3 &= P^2
\end{aligned}
$$

and

$$
\begin{aligned}
&x_{10} = x_i = x_{40} = y_{11}, \quad x_{30} = x_{i+1} - h = y_{20},\\
&x_{11} = x_{i+1} = x_{50} = y_{21}, \quad x_{31} = x_{i+1} + h = x_{51},\\
&x_{20} = x_i - h = y_{10} = y_{30},\\
&x_{21} = x_i + h = x_{41} = y_{31}.
\end{aligned}
$$

Since $b_3 = a_1 + a_2 + a_3$, $b_1 = a_4$, $b_2 = a_5$. Now, it can be easily see that

$$
\sum_{i=1}^{5} a_i = \sum_{j=1}^{3} b_j = [P^2 + \frac{P^3}{h}(t - (2t - 3)(t^2 + h^2))] = M^*(t, h, P) \quad (say).
$$

By using Lemma 3.1, we get

$$
|U_i(f)| \leq M(t, h, P)w(f^{\{1\}}, P). \tag{21}
$$

Hence we get the bounds of $e^{\{1\}}(x)$ finally from (20), when we appeal to (21) and (17), thus

$$
||e^{\{1\}}(x)|| \leq k_2(P, h)w(f^{\{1\}}, P). \tag{22}
$$

Thus we get inequality (11) of Theorem 3.2. ∎

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