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RESEARCH ARTICLE

On The Order and Type of Entire Matrix Functions in Complete Reinhardt Domain

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In this paper, we deals with the study of analytic functions of complex matrices and a standard method for evaluating the order and type of the entire function of complex matrices is given independently of the scalar entire function of complex variables associated with it.

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1. Introduction

In 1959, Gol'dberg gave the definitions of the order and type of entire function in several complex variables [1]. For more details about the study of the order and type of entire functions we refer to [2–7]. The main aim of this paper is to study of analytic function of several complex matrices in complete Rinhardt domains, also called polycylinderical regions and define the order and type of entire function of several complex matrices in complete Reinhardt domains.

We will let $\mathbb{C}$ represents the field of complex variables. Let $\mathbf{z} = (z_1, z_2, ..., z_n)$ be an element of $\mathbb{C}^n$; the space of several complex variables, a closed complete Reinhardt domain of radii $(\alpha_s r > 0)$; $s \in \mathbf{I} = 1, 2, 3, ..., n$ is here denoted by $\bar{\Gamma}_{[\alpha r]}$ and is given by

$$\bar{\Gamma}_{[\alpha r]} = \{\mathbf{z} \in \mathbb{C}^n : | z_s | \leq \alpha_s r; s \in \mathbf{I}\},$$

where α_s are positive numbers.

The open Reinhardt domain is here denoted by $\Gamma_{[\alpha r]}$ and is given by

$$\Gamma_{[\alpha r]} = \{\mathbf{z} \in \mathbb{C}^n : | z_s | < \alpha_s r; s \in \mathbf{I}\}.$$

Consider unspecified domain containing the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha r]}$. This domain will be of radii $\alpha_s r_1$; $r_1 > r$, then making a contraction to this domain, we will get the domain $\bar{D}([\alpha \mathbf{r}^+]) = \bar{D}([\alpha_1 r^+, \alpha_2 r^+, ..., \alpha_n r^+])$, where r^+ stands for the right-limit of r_1 at r (see [8]).

The order and type of entire functions of several complex variables in Reinhardt domain are given as follows:

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Definition 1.1 [1, 5, 8] The order ρ of the entire function $f(\mathbf{z})$ for the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha r]}$ is defined as follows:

$$\rho = \limsup_{r \longrightarrow \infty} \frac{\ln \ln M[\alpha r]}{\ln r},$$

where

$$M[\alpha r] = M[\alpha_1 r, \alpha_2 r, ..., \alpha_n r] = \max_{\bar{\Gamma}_{[\alpha r]}} |f(\mathbf{z})|.$$

Definition 1.2 [1, 5, 8] The type τ of the entire function $f(\mathbf{z})$ for the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha r]}$ is defined as follows:

$$\tau = \limsup_{r \longrightarrow \infty} \frac{\ln M[\alpha r]}{r^{\rho}},$$

where $0 < \rho < \infty$.

Theorem 1.3 [1, 5, 8] The necessary and sufficient condition that the entire function $f(\mathbf{z})$ of several complex variables should be of order ρ in the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha r]}$ is that

$$\rho = \limsup_{\langle \mathbf{m} \rangle \longrightarrow \infty} \frac{\langle \mathbf{m} \rangle \ln \langle \mathbf{m} \rangle}{-\ln(|a_{\mathbf{m}}| \prod_{s=1}^{n} \alpha_s^{m_s})},$$

where $\langle \mathbf{m} \rangle = m_1 + m_2 + m_3 + ... + m_n$ and $\mathbf{m} = (m_1, m_2, m_3 ..., m_n)$.

Theorem 1.4 [1, 5, 8] The necessary and sufficient condition that the entire function $f(\mathbf{z})$ of several complex variables should be of type τ in the closed complete Reinhardt domain $\bar{\Gamma}_{[\alpha r]}$ is that

$$\tau = \frac{1}{e\rho} \limsup_{\langle \mathbf{m} \rangle \longrightarrow \infty} \langle \mathbf{m} \rangle \left\{ |a_{\mathbf{m}}| \prod_{s=1}^{n} \alpha_s^{m_s} \right\}^{\frac{\rho}{\langle \mathbf{m} \rangle}}.$$

For simplicity, we consider only two complex matrices, though the results can easily be extended to several complex matrices.

2. Analytic Functions of Complex Matrices

Consider the space $\mathbb{C}^{N \times N}$ of all matrices $X = [x_{ij}]$ and $Y = [y_{ij}]$, where x_{ij} and y_{ij} are complex numbers; $i, j = 1, 2, 3, ..., N$. Let $F(X, Y)$ be a matrix function such that

$$F = [f_{ij}]; \quad f_{ij} = f(x_{ij}, y_{ij}) \quad \forall \ i, j = 1, 2, 3, ..., N.$$

Suppose that the matrix function $F(X, Y)$ of two square complex matrices is given by a power series in the form

$$F(X, Y) = \sum_{m,n} a_{m,n} X^m Y^n; \qquad m, n \geq 0 \tag{1}$$

where

$$X^m = \sum_{k_1, k_2, ... k_{m-1}} x_{ik_1} x_{k_1 k_2} .... x_{k_{m-1} j}$$

and

$$Y^n = \sum_{k_1,k_2,...k_{n-1}} x_{ik_1}x_{k_1k_2}....x_{k_{n-1}\,j},$$

in the assumption that $X^0 = Y^0 = I$, I is unit matrix of order N and X^mY^n is equal to a square complex matrix $Z = [z_{ij}]$, where $z_{ij} = \sum_{k=1}^{N}\{x^m\}_{ik}\{y^n\}_{kj}$.
Hence

$$f_{ij} = \sum_{m,n} a_{m,n}z_{ij}; \qquad m,n \geq 0. \tag{2}$$

Thus, we can say that the function $F(X,Y)$ is convergent if the elements f_{ij} given in (2) are convergent series for all $i,j = 1,2,...,N$. Consider the domain which is a subset of the space determined by the two inequalities

$$\begin{aligned} |X| &< \|\alpha_1 R\|, \\ |Y| &< \|\alpha_2 R\|. \end{aligned} \tag{3}$$

The symbol $|X|$ denotes the matrix $(|x_{ij}|)$ whose elements are the modulii of the elements x_{ij} of the matrix X, and the symbol $\|a\|$ denotes a matrix each of its elements is equal to the positive number. Thus (3) implies that

$$|x_{ij}| < \alpha_1 R \quad and \quad |y_{ij}| < \alpha_2 R; \quad i,j = 1,2,3,...,N.$$

Therefore, there is a number r where $0 < r < R$ such that

$$|x_{ij}| \leq \alpha_1 r \quad and \quad |y_{ij}| \leq \alpha_2 r; \quad i,j = 1,2,3,...,N,$$

where $(x_{ij}, y_{ij}) \in \bar{\Gamma}_{[\alpha_s R]}$; $\alpha_s R(> 0)$, α_s are positive numbers, $s = 1,2$.

Now, Let $F(z,w) = \sum_{m,n} a_{m,n}z^m w^n$ be the scalar function of two variables z and w associated with the matrix function in (1), that $F(z,w)$ is analytic function in the complete Reinhardt domain $\bar{\Gamma}_{[\alpha_s NR]}$.

Since

$$F(z,w) = \sum_{m,n} a_{m,n}z^m w^n, \quad M[\alpha_s(NR)] = \max_{\bar{\Gamma}_{[\alpha_s NR]}} |F(z,w)| \tag{4}$$

and

$$|a_{m,n}| \leq \frac{M}{\alpha_1^m \alpha_2^n (NR)^{m+n}}; \quad m,n \geq 0. \tag{5}$$

We have

$$\begin{aligned}\Big|f_{ij}\Big| =&\Big|\sum_{m,n} a_{m,n} z_{ij}\Big| \leq \sum_{m,n}\Big|a_{m,n}\Big|\Big|\sum_{k=1}^{N}\{x^m\}_{ik}\{y^n\}_{kj}\Big| \\ \leq&\sum_{m,n}\Big|a_{m,n}\Big|\sum_{k=1}^{N} N^{m-1}(\alpha_1 r)^m N^{n-1}(\alpha_2 r)^n = \frac{M}{N}\sum_{m,n}(\frac{r}{R})^{m+n} \\ =&\frac{M}{N}\sum_{\nu=0}^{\infty}(\frac{r}{R})^{\nu} = \frac{M}{N(1-\frac{r}{R})^2}; \quad i,j=1,2,...,N; \quad (x_{ij},y_{ij})\in\bar{\Gamma}_{[\alpha_s R]},\end{aligned} \tag{6}$$

i. e., the matrix function $F(X,Y)$ as given in (1) is absolute convergence. Since r can be chose arbitrary near to R, then the following theorem follows:

Theorem 2.1 If the function $F(z,w)$ as given in (4) is analytic in $\bar{\Gamma}_{[\alpha_s NR]}$, then the function $F(X,Y)$ as given in (1) will be analytic in $\bar{\Gamma}_{[\alpha_s R]}$ and bounded on $\bar{\Gamma}_{[\alpha_s R]}$, where N is the common order of the matrices X and Y.

If the matrix function

$$\begin{aligned}F(X,Y) =&f_1(X)f_2(Y) = (\sum_{m=0}^{\infty} a_m^1 X^m)(\sum_{n=0}^{\infty} a_n^2 Y^n) \\ =&\sum_{m,n=0}^{\infty} a_{m,n}X^mY^n; \quad a_{m,n}=a_m^1 a_n^2\end{aligned} \tag{7}$$

associated with the scalar function

$$\begin{aligned}F(z,w) =&f_1(z)f_2(w) = (\sum_{m=0}^{\infty} a_m^1 z^m)(\sum_{n=0}^{\infty} a_n^2 w^n) \\ =&\sum_{m,n=0}^{\infty} a_{m,n}z^mw^n; \quad a_{m,n}=a_m^1 a_n^2.\end{aligned} \tag{8}$$

We get the following corollary:

Corollary 2.1 If the functions f_1 and f_2 of the single variables z and w are analytic in $|z| < \alpha_1 NR$ and $|w| < \alpha_2 NR$, then the matrix function $F(X,Y)$ of square complex matrices X and Y each of them of order N, as given in (7) will be analytic in $\bar{\Gamma}_{[\alpha_s R]}$.

Now, suppose that the scalar functions

$$F(z,w) = \sum_{m,n=0}^{\infty} a_{m,n}z^mw^n, \quad G(z,w) = \sum_{m,n=0}^{\infty} b_{m,n}z^mw^n \tag{9}$$

are analytic in $\bar{\Gamma}_{[\alpha_s NR]}$, then according to (5), we get

$$|a_{m,n}| \leq \frac{M_1}{\alpha_1^m\alpha_2^n(NR)^{m+n}}; \quad m,n\geq 0, \quad M_1\geq 1, \tag{10}$$

and

$$|b_{m,n}| \leq \frac{M_2}{\alpha_1^m \alpha_2^n (NR)^{m+n}}; \quad m,n \geq 0, \quad M_2 \geq 1. \tag{11}$$

Let $F(X,Y)$ and $G(X,Y)$ be the matrix functions associated with the scalar functions (9) in the form

$$F(X,Y) = \sum_{m,n=0}^{\infty} a_{m,n} X^m Y^n, \quad G(X,Y) = \sum_{m,n=0}^{\infty} b_{m,n} X^m Y^n. \tag{12}$$

Write the product matrix function $P(X,Y)$ as follows

$$P(X,Y) = F(X,Y).G(X,Y) = \sum_{m,n=0} C_{m,n} X^m Y^n, \tag{13}$$

where

$$C_{m,n} = \sum_{h=0}^{m} \sum_{k=0}^{n} a_{h,k} b_{m-h,n-k}.$$

From (10) and (11), we have

$$\begin{aligned} |C_{m,n}| &\leq \sum_{h=0}^{m} \sum_{k=0}^{n} \mid a_{h,k} b_{m-h,n-k} \mid \\ &\leq \sum_{h=0}^{m} \sum_{k=0}^{n} \frac{M_1 M_2}{\alpha_1^m \alpha_2^n (NR)^{m+n}} = (m+1)(n+1) \frac{M_1 M_2}{\alpha_1^m \alpha_2^n (NR)^{m+n}}. \end{aligned} \tag{14}$$

Thus

$$\begin{aligned} &\max_{\bar{\Gamma}_{[\alpha_s N r]}} \| \sum_{m,n=0} C_{m,n} X^m Y^n \| \\ &\leq \sum_{m,n=0} \mid C_{m,n} \mid \max_{\bar{\Gamma}_{[\alpha_s N r]}} \| X^m Y^n \| \\ &\leq \sum_{m,n=0} (m+1)(n+1) \frac{M_1 M_2 (r)^{m+n}}{(R)^{m+n}} < \infty, \end{aligned} \tag{15}$$

i. e., the product matrix function $P(X,Y)$ given in (13) is analytic function in the complete Reinhardt domain $\bar{\Gamma}_{[\alpha_s NR]}$. Since r can be chose arbitrary near to R, then the following theorem follows

Theorem 2.2 The matrix function $P(X,Y)$ as given in (13) is absolute convergence in $\bar{\Gamma}_{[\alpha_s NR]}$ and analytic in some region if the functions $F(z,w)$ and $G(z,w)$ as given in (9) are analytic in $\bar{\Gamma}_{[\alpha_s NR]}$.

3. On The Order and Type of Entire Matrix Functions

Let

$$F(X,Y) = \sum_{m,n} a_{m,n} X^m Y^n; \qquad m,n \geq 0, \tag{16}$$

be an entire function of two square complex matrices X and Y each of them is of order N, it follows that

$$M[\alpha_s r] = M[\alpha_1 r, \alpha_2 r] = \max_{i,j} \max_{\bar{\Gamma}_{[\alpha_s r]}} |F(X,Y)|. \tag{17}$$

Hence

$$|a_{m,n}|\alpha_1^m \alpha_2^n \leq \frac{NM[\alpha_s r]}{(rN)^{m+n}}; \quad m,n \geq 0, \tag{18}$$

therefore, the radius of regularity of the matrix function $F(X,Y)$ is infinity, i. e.,

$$\limsup_{m+n\to\infty} \{N^{m+n}|a_{m,n}|\alpha_1^m \alpha_2^n\}^{\frac{1}{m+n}} = 0. \tag{19}$$

The order ρ of the entire matrix function $F(X,Y)$ is given by

$$\rho = \limsup_{r\to\infty} \frac{\ln\ln M[\alpha_s r]}{\ln r}. \tag{20}$$

If the entire matrix function $F(X,Y)$ is given by a power series in (16), then we have the following two lemmas concerning the of the function of two square complex matrices. We begin by proving the following lemma.

Lemma 3.1 If

$$\limsup_{r\to\infty} \frac{\ln\ln M[\alpha_s r]}{\ln r} \leq \gamma, \tag{21}$$

then

$$\limsup_{m+n\to\infty} \frac{(m+n)\ln(m+n)}{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m \alpha_2^n)} \leq \gamma. \tag{22}$$

Proof If $\gamma = \infty$ then there is nothing to prove. If $\gamma_1 > \gamma$, then for a suitable number r_0 (21), yields

$$M[\alpha_s r] < e^{r^{\gamma_1}}; \quad r_0 < r, \tag{23}$$

hence by Cauchy's inequality in (18) gives

$$N^{m+n}|a_{m,n}|\alpha_1^m \alpha_2^n \leq \min_{r_0<r} N \frac{e^{r^{\gamma_1}}}{(r)^{m+n}}; \quad r_0 < r, \tag{24}$$

choose the integer μ such that

$$(\frac{m+n}{\gamma_1})^{\frac{1}{\gamma_1}} > r_0, for \quad m+n > \mu.$$

So that

$$N^{m+n}|a_{m,n}|\alpha_1^m \alpha_2^n \leq \min_{r>r_0} \frac{e^{r^{\gamma_1}}}{(r)^{m+n}} = N(\frac{e\gamma_1}{m+n})^{\frac{m+n}{\gamma_1}}; m+n > \mu.$$

Thus

$$\begin{aligned} &\frac{(m+n)\ln(m+n)}{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)} \\ \leq &\frac{\gamma_1(m+n)\ln(m+n)}{(m+n)\{\ln(m+n)-\gamma_1\ln e\}+\gamma_1\ln N} \leq \gamma_1. \end{aligned} \tag{25}$$

Making $m+n$ tend to infinity, we obtain

$$\limsup_{m+n\to\infty}\frac{(m+n)\ln(m+n)}{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)} \leq \gamma_1.$$

Since γ_1 can be chosen arbitrary near to γ, we thus have (22) and the lemma is proved. The converse result is given in the following lemma. ■

Lemma 3.2 If

$$\limsup_{m+n\to\infty}\frac{(m+n)\ln(m+n)}{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)} \leq \gamma, \tag{26}$$

then

$$\limsup_{r\to\infty}\frac{\ln\ln M[\alpha_s r]}{\ln r} \leq \gamma. \tag{27}$$

Proof Again if $\gamma=\infty$ then there is nothing to prove. If $\gamma_1>\gamma$, then there is an integer μ such that

$$N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n \leq (m+n)^{-\frac{m+n}{\gamma_1}}; \quad m+n>\mu, \tag{28}$$

By using (16) and (17), we obtain

$$\begin{aligned} M[\alpha_s r] \leq &\max_{i,j}\max_{\bar{\Gamma}_{[\alpha_s r]}}|\sum_{m,n}a_{m,n}X^mY^n| \\ \leq &\frac{1}{N}\sum_{m,n=0}^{\infty}(Nr)^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n. \end{aligned}$$

For a number $r_0>1$ such that $(2r)^{\gamma_1}>\mu$ and $r>r_0$, we can fix the integer n_1 such that

$$n_1 \leq (2r)^{\gamma_1} < n_1+1; \quad r>r_0,$$

then,

$$\begin{aligned} M[\alpha_s r] \leq &\frac{1}{N}\{\sum_{m,n=0}^{\mu}+\sum_{m,n=\mu+1}^{\infty}\}(Nr)^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n \\ = &\frac{1}{N}\{A+\sum_{m,n=\mu+1}^{\infty}(\frac{r^{\gamma_1}}{m+n})^{\frac{m+n}{\gamma_1}}\} \\ = &\frac{1}{N}\{A+\sum_{m,n=\mu+1}^{n_1}(\frac{r^{\gamma_1}}{m+n})^{\frac{m+n}{\gamma_1}}+\sum_{m,n=n_1+1}^{\infty}(\frac{r^{\gamma_1}}{m+n})^{\frac{m+n}{\gamma_1}}\}. \end{aligned} \tag{29}$$

Hence

$$\sum_{m,n=\mu+1}^{n_1}(\frac{r^{\gamma_1}}{m+n})^{\frac{m+n}{\gamma_1}} < r^{n_1}\sum_{m,n=\mu}^{n_1}\frac{1}{(\mu+1)^{\frac{m+n}{\gamma_1}}}$$
$$< r^{n_1}\sum_{m,n=0}^{\infty}\frac{1}{(\mu+1)^{\frac{m+n}{\gamma_1}}} < r^{(2r)^{\gamma_1}}\{1-\frac{1}{(\mu+1)^{\frac{1}{\gamma_1}}}\}^{-2} = Br^{(2r)^{\gamma_1}}, \tag{30}$$

and

$$\sum_{m,n=n_1+1}^{\infty}(\frac{r^{\gamma_1}}{m+n})^{\frac{m+n}{\gamma_1}}\} \leq \sum_{m,n=n_1+1}^{\infty}(\frac{r^{\gamma_1}}{n_1+1})^{\frac{m+n}{\gamma_1}}\}$$
$$< \sum_{m,n=n_1+1}^{\infty}(\frac{1}{2})^{m+n} < \sum_{m,n=0}^{\infty}(\frac{1}{2})^{m+n} = C, \tag{31}$$

a combination of (29), (30) and (31) leads to the relation

$$M[\alpha_s r] \leq K\exp\{(2r)^{\gamma_1}\ln r\}, \quad r > r_0. \tag{32}$$

Making r tend to infinity, we infer from (32) such that

$$\limsup_{r\to\infty}\frac{\ln\ln M[\alpha_s r]}{\ln r} \leq \gamma_1,$$

and the required inequality (27) follows if we note that γ_1 is arbitrary chosen near to γ and thus, we have the following theorem: ■

Theorem 3.3 A necessary and sufficient condition that the entire matrix function $F(X,Y) = \sum_{m,n} a_{m,n}X^mY^n$ should be of order ρ is that

$$\rho = \limsup_{m+n\to\infty}\frac{(m+n)\ln(m+n)}{-\ln(N^{m+n}|a_{m,n}|\alpha_1^m\alpha_2^n)}. \tag{33}$$

Also if $0 < \rho < \infty$ we define the type τ of matrix function of two square complex matrices in form

$$\tau = \limsup_{r\to\infty}\frac{\ln M[\alpha_s r]}{r^\rho}. \tag{34}$$

Now, we introduce the following two lemmas concerning the type τ of the entire function of two square complex matrices.

Lemma 3.4 If

$$\limsup_{r\to\infty}\frac{\ln M[\alpha_s r]}{r^\rho} \leq \gamma, \tag{35}$$

then

$$\frac{N^\rho}{e\rho}\limsup_{m+n\longrightarrow\infty}(m+n)\{|a_{m,n}|\alpha_1^m\alpha_2^n\}^{\frac{\rho}{m+n}} \leq \gamma. \tag{36}$$

Proof If $\gamma = \infty$ then there is nothing to prove. If $\gamma_1 > \gamma$, then for a suitable number r_0, (3.19) yields

$$M[\alpha_2 r] \leq e^{\gamma_1 r^\rho} \quad for \quad r > r_0,$$

hence

$$N^{m+n}|a_{m,n}|\alpha_1^m \alpha_2^n \leq \min_{r_0<r} \frac{Ne^{r^\rho \rho}}{(r)^{m+n}}; \quad r_0 < r,$$

we can choose an integer μ such that.

$$(\frac{m+n}{\rho\gamma_1})^{\frac{1}{\rho}} > r_0, \ for \quad m+n > \mu.$$

So that

$$N^{m+n}|a_{m,n}|\alpha_1^m \alpha_2^n \leq \min_{r_0<r} \frac{Ne^{r^\rho \gamma_1}}{(r)^{m+n}} \leq N(\frac{e\gamma_1\rho}{m+n})^{\frac{m+n}{\rho}}; \quad m+n > \mu,$$

thus

$$\frac{N^\rho}{e\rho} \limsup_{m+n\longrightarrow\infty} (m+n)\{|a_{m,n}|\alpha_1^m \alpha_2^n\}^{\frac{\rho}{m+n}} \leq \gamma_1.$$

As γ_1 can be taken arbitrary near to γ the required inequality of the lemma is established. We now establish the second lemma in the form ■

Lemma 3.5 If

$$\frac{N^\rho}{e\rho} \limsup_{m+n\longrightarrow\infty} (m+n)\{|a_{m,n}|\alpha_1^m \alpha_2^n\}^{\frac{\rho}{m+n}} \leq \gamma, \tag{37}$$

then

$$\limsup_{r\to\infty} \frac{\ln M[\alpha_s r]}{r^\rho} \leq \gamma. \tag{38}$$

Proof Let $\gamma \leq \gamma_1$, choose an integer $\mu > 1$ such that we can have from (21) that

$$|a_{m,n}|\alpha_1^m \alpha_2^n \leq (\frac{e\rho\gamma_1}{N^\rho m+n})^{\frac{m+n}{\rho}}; \quad m+n > \mu,$$

$$\begin{aligned} M[\alpha_s r] \leq & \frac{1}{N} \sum_{m,n=0}^{\infty} (Nr)^{m+n}|a_{m,n}|\alpha_1^m \alpha_2^n \\ \leq & \frac{1}{N} \sum_{m,n=0}^{\infty} (\frac{e\rho\gamma_1 r^\rho}{m+n})^{\frac{m+n}{\rho}}, \end{aligned}$$

for a number $r_0 > 1$ such that $2\gamma_1 e\rho r^\rho > \mu$ and we can fix the integer n_1 such that

$$n_1 \leq 2r^\rho \gamma_1 e\rho < n_1 + 1; \quad r > r_0,$$

then, we have

$$
\begin{aligned}
M[\alpha_s r] \leq & \frac{1}{N}\{\sum_{m,n=0}^{\mu} + \sum_{m,n=\mu+1}^{\infty}\}(\frac{e\rho\gamma_1 r^\rho}{m+n})^{\frac{m+n}{\rho}} \\
\leq & \frac{1}{N}\{A + \sum_{m,n=0}^{n_1}(\frac{e\rho\gamma_1 r^\rho}{m+n})^{\frac{m+n}{\rho}} + \sum_{m,n=\mu+1}^{\infty}(\frac{e\rho\gamma_1 r^\rho}{m+n})^{\frac{m+n}{\rho}}\} \\
\leq & \{A + Be^{\gamma_1 r^\rho} + C\} = Ke^{\gamma_1 r^\rho}.
\end{aligned} \tag{39}
$$

Making r tend to infinity (38) yields

$$\limsup_{r\to\infty} \frac{\ln M[\alpha_s r]}{r^\rho} \leq \gamma_1.$$

As γ_1 can be taken arbitrary near to γ the required inequality of the lemma is established. Again the fundamental result about the order and type of an entire matrix function can be derived Lemmas 2.3 and 2.4 in the following theorem: ■

Theorem 3.6 If the entire matrix function $F(X,Y) = \sum_{m,n} a_{m,n} X^m Y^n$ is of finite order ρ, then the necessary and sufficient condition should be of type τ is that

$$\tau = \frac{N^\rho}{e\rho} \limsup_{m+n\longrightarrow\infty} (m+n)\{|a_{m,n}|\alpha_1^m \alpha_2^n\}^{\frac{\rho}{m+n}}.$$

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