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MEAN VALUE OF HARDY SUMS OVER SHORT INTERVALS

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Abstract. The main purpose of this paper is to study the mean value properties of certain Hardy sums over short intervals by using the mean value theorems of the Dirichlet L-functions, and to give two interesting asymptotic formulae.

1. Introduction and the main result

For a positive integer q and an arbitrary integer h, the Dedekind sum $S(h,q)$ is defined by

$$S(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer,}\end{cases}$$

$[x]$ is the largest integer less than or equal to x. The various properties of $S(h,q)$ were investigated by many authors. Maybe the most famous property of the Dedekind sums is the reciprocity formula (see references [1, 3, 7])

$$S(h,q)+S(q,h)=\frac{h^2+q^2+1}{12hq}-\frac{1}{4} \tag{1.1}$$

valid for all $(h,q)=1, h>0, q>0$. A three term version of (1.1) was discovered by H. Rademacher [6]. In this paper we study the Hardy sums

$$H(h,k)=\sum_{j=1}^{k-1}(-1)^{j+1+[hj/k]}.$$

Since the sums $H(h,k)$ can be expressed explicitly in terms of Dedekind sums, they are connected with Dedekind sums. Some arithmetical properties of

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$H(h,k)$ can be found in B. C. Berndt [2] and R. Sitaramachandra Rao [8]. In [10] an asymptotic formula for the $2m$-th power mean of these sums was proved, namely

$$\sum_{h=1}^{p-1} |H(h,p)|^{2m} = p^{2m}\frac{\zeta^2(2m)(1-4^{-m})}{\zeta(4m)(1+4^{-m})} + O\left(p^{2m-1}\exp\left(\frac{6\ln p}{\ln\ln p}\right)\right),$$

here p is an odd prime, m is a positive integer and $\zeta(s)$ is the Riemann zeta function.

What about the mean value of $H(h,p)$ when h runs through an interval $[1,\lambda p]$, $0<\lambda<1$? It seems difficult to obtain an asymptotic formula even in case like $\lambda = \frac{1}{3}, \frac{1}{4}$. Here we study the better behaved mean value:

$$\sum_{a<p/3}\sum_{b<p/3} H(2a\bar{b},p), \qquad \sum_{a<p/3}\sum_{b<p/4} H(2a\bar{b},p),$$

where $\bar{b}$ is defined by the equation $b\bar{b}\equiv 1(\mathrm{mod}p)$. The factor 2 is necessary since $H(d,p)=0$ for an odd number d (see Lemma 1).

Theorem 1. *Let $p\geq 5$ be a prime. We have*

(I)

$$\sum_{a\leq p/3}\sum_{b\leq p/3} H(2a\bar{b},p) = \frac{1}{5}p^2 + O(p^{1+\varepsilon}).$$

(II)

$$\sum_{a\leq p/3}\sum_{b\leq p/4} H(2a\bar{b},p) = \frac{27}{320}p^2 + O(p^{1+\varepsilon}).$$

There are also five other types of Hardy sums, all of them closely connected with Dedekind sums [4]. Since their relation to L-functions is quite similar to that used here (see Lemma 1), analogues of our theorem can be obtained for these sums.

2. Some lemmas

To prove the theorem, we need the following lemmas.

Lemma 1. *Let p be a prime and h be a positive integer with $(h,p)=1$. We have*

$$H(h,p) = \begin{cases} -\frac{16p}{\pi^2(p-1)} \sum\limits_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(h)\left|L(1,\chi\chi_2^0)\right|^2, & \text{if } 2|h; \\ 0, & \text{if } 2\nmid h, \end{cases}$$

where χ_2^0 denotes the principal character modulo 2 and the sum is over the odd character $(\mathrm{mod}p)$.

Proof. See [10, Lemma 3]. □

Lemma 2. *Let $p \geq 5$ be a prime. For any non-principal character χ modulo p, we have*

$$\sum_{a \leq p/3} \chi(a) = \frac{3\tau(\chi)}{2\pi i} L(1, \overline{\chi}\chi_3^0)$$

and

$$\sum_{a \leq p/4} \chi(a) = \frac{\tau(\chi)}{2\pi i} \left[2 + \overline{\chi}(2) - \overline{\chi}(4)\right] L(1, \overline{\chi}),$$

where χ_3^0 is the principal character modulo 3, and $\tau(\chi) = \sum_{a=1}^{p} \chi(a) e^{2\pi i a/p}$ is the Gauss sum.

Proof. These identities can be easily deduced from the Fourier expansion for primitive character sums(see [5])

$$\sum_{a \leq \lambda p} \chi(a) = \begin{cases} \frac{\tau(\chi)}{\pi} \sum_{n=1}^{+\infty} \frac{\overline{\chi}(n) \sin(2\pi n \lambda)}{n}, & \text{if } \chi(-1) = 1; \\ \frac{\tau(\chi)}{\pi i} \sum_{n=1}^{+\infty} \frac{\overline{\chi}(n)(1 - \cos(2\pi n \lambda))}{n}, & \text{if } \quad \chi(-1) = -1. \end{cases}$$

See also reference [9]. □

Lemma 3. *Let χ_2^0 and χ_3^0 be the principal characters modulo 2 and 3, respectively. If $r_1(n) = \sum_{d|n} \chi_2^0(d)$, $r_2(n) = \sum_{d|n} \chi_2^0(d)\chi_3^0(\frac{n}{d})$, then for any integer $m \geq 0$, we have the identity*

$$\sum_{n=1}^{\infty} \frac{r_1(n) r_2(2^m n)}{n^2} = \frac{\pi^4}{40}.$$

Proof. Noting that $r_1(n)$, $r_2(n)$ are multiplicative functions, we can write

$$\sum_{n=1}^{\infty} \frac{r_1(n) r_2(2^m n)}{n^2} = \left(r_2(2^m) + \sum_{j=1}^{\infty} \frac{r_1(2^j) r_2(2^{m+j})}{4^j} \right) \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} \frac{r_1(n) r_2(n)}{n^2} \tag{2.1}$$

$$= \left(1 + \sum_{j=1}^{\infty} \frac{1}{4^j} \right) \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} \frac{r_1(n) r_2(n)}{n^2}$$

$$= \frac{4}{3} \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} \frac{r_1(n) r_2(n)}{n^2}.$$

For the summation $\displaystyle\sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} \frac{r_1(n) r_2(n)}{n^2}$, we can write

$$\sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} \frac{r_1(n) r_2(n)}{n^2} = \prod_{p \geq 3} \left(1 + \frac{r_1(p) r_2(p)}{p^2} + \frac{r_1(p^2) r_2(p^2)}{p^4} + \dots \right)$$

by using the Euler product formula. Note that

$$r_1(p^\alpha) = 1 + \alpha$$

and

$$r_2(p^\alpha) = \begin{cases} 1, & \text{if } p = 3, \\ \alpha + 1, & \text{if } p \neq 3, \end{cases}$$

for any positive integer α and an odd prime p. Hence

$$\begin{aligned} \sum_{\substack{n=1 \\ 2\nmid n}}^{\infty} \frac{r_1(n) r_2(n)}{n^2} &= \\ &= \left(1 + \frac{2}{3^2} + \frac{3}{3^4} + \dots\right) \prod_{p \geq 5} \left(1 + \frac{2^2}{p^2} + \frac{3^2}{p^4} + \dots + \frac{(\alpha+1)^2}{p^{2\alpha}} + \dots\right) \\ &= \frac{81}{64} \prod_{p\geq 5} \left(1 - \frac{1}{p^2}\right)^{-4} \left(1 - \frac{1}{p^4}\right) = \frac{27}{100} \zeta^4(2)/\zeta(4) = \frac{3\pi^4}{160}, \end{aligned} \tag{2.2}$$

where we used that $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$. Therefore, from (2.1) and (2.2), we have

$$\sum_{n=1}^{\infty} \frac{r_1(n) r_2(2^m n)}{n^2} = \frac{\pi^4}{40}.$$

This proves Lemma 3. □

Lemma 4. *For any integer $m \geq 0$, we have the asymptotic formula*

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^m) \left|L(1, \chi\chi_2^0)\right|^2 L(1, \chi\chi_3^0) L(1, \chi) = \frac{\pi^4}{90 \cdot 2^m}(p-1) + O_m(p^\varepsilon).$$

Proof. For convenience, we put

$$A(y, \chi) = \sum_{N < n \leq y} \chi(n) r_1(n), \quad B(y, \chi) = \sum_{N < n \leq y} \overline{\chi}(n) r_2(n),$$

where N is a parameter with $p \leq N \leq p^4$ and r_1, r_2 were defined in Lemma 3. Then from Abel's identity, we have

$$L(1, \chi\chi_2^0) L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n) r_1(n)}{n} = \sum_{1 \leq n \leq N} \frac{\chi(n) r_1(n)}{n} + \int_N^{\infty} \frac{A(y, \chi)}{y^2} dy;$$

$$L(1, \overline{\chi}\chi_2^0) L(1, \overline{\chi}\chi_3^0) = \sum_{n=1}^{\infty} \frac{\overline{\chi}(n) r_2(n)}{n} = \sum_{1 \leq n \leq N} \frac{\overline{\chi}(n) r_1(n)}{n} + \int_N^{\infty} \frac{B(y, \chi)}{y^2} dy.$$

Hence, we can write

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}} \chi(2^m)\left|L(1,\chi\chi_2^0)\right|^2 L(1,\chi\chi_3^0)L(1,\chi) = \tag{2.3}$$

$$\begin{aligned}
&= \sum_{\chi(-1)=-1} \chi(2^m)\left(\sum_{1\le n_1\le N}\frac{\chi(n_1)r_1(n_1)}{n_1}+\int_N^\infty \frac{A(y,\chi)}{y^2}dy\right)\times\\
&\quad\times\left(\sum_{1\le n_2\le N}\frac{\overline{\chi}(n_2)r_1(n_2)}{n_2}+\int_N^\infty \frac{B(y,\chi)}{y^2}dy\right)\\
&= \sum_{\chi(-1)=-1} \chi(2^m)\left(\sum_{1\le n_1\le N}\frac{\chi(n_1)r_1(n_1)}{n_1}\right)\left(\sum_{1\le n_2\le N}\frac{\overline{\chi}(n_2)r_1(n_2)}{n_2}\right)\\
&\quad+\sum_{\chi(-1)=-1} \chi(2^m)\left(\sum_{1\le n_1\le N}\frac{\chi(n_1)r_1(n_1)}{n_1}\right)\left(\int_N^\infty \frac{B(y,\chi)}{y^2}dy\right)\\
&\quad+\sum_{\chi(-1)=-1} \chi(2^m)\left(\sum_{1\le n_2\le N}\frac{\overline{\chi}(n_2)r_1(n_2)}{n_2}\right)\left(\int_N^\infty \frac{A(y,\chi)}{y^2}dy\right)\\
&\quad+\sum_{\chi(-1)=-1} \chi(2^m)\left(\int_N^\infty \frac{A(y,\chi)}{y^2}dy\right)\left(\int_N^\infty \frac{B(y,\chi)}{y^2}dy\right)\\
&:= M_1+M_2+M_3+M_4.
\end{aligned}$$

Now, we calculate each term in the expression (2.3).

(i) From the orthogonality of Dirichlet characters we can write

(2.4)

$$\begin{aligned}
M_1 &= \\
&\sum_{\chi(-1)=-1} \chi(2^m)\left(\sum_{1\le n_1\le N}\frac{\chi(n_1)r_1(n_1)}{n_1}\right)\left(\sum_{1\le n_2\le N}\frac{\overline{\chi}(n_2)r_1(n_2)}{n_2}\right)\\
&= \frac{1}{2}\sum_{1\le n_1\le N}\sum_{1\le n_2\le N}\frac{r_1(n_1)r_2(n_2)}{n_1n_2}\sum_{\chi \bmod p}(1-\chi(-1))\chi(2^m n_1\overline{n}_2)\\
&= \frac{p-1}{2}\left(\sideset{}{'}\sum_{\substack{1\le n_1\le N\\ 2^m n_1\equiv n_2 (\bmod p)}}\sideset{}{'}\sum_{1\le n_2\le N}\frac{r_1(n_1)r_2(n_2)}{n_1n_2}-\sideset{}{'}\sum_{\substack{1\le n_1\le N\\ 2^m n_1\equiv -n_2 (\bmod p)}}\sideset{}{'}\sum_{1\le n_2\le N}\frac{r_1(n_1)r_2(n_2)}{n_1n_2}\right),
\end{aligned}$$

where $\sum_{1\le n_1\le N}{}'$ denotes the summation over all $1\le n_1\le N$ such that $(n_1,p)=1$.

For convenience, we split the sum over n_1 or n_2 into following cases:

a.) $p/2^m \le n_1 \le N$, $p \le n_2 \le N$;
b.) $1 \le n_1 \le p/2^m - 1$, $p \le n_2 \le N$;
c.) $p/2^m \le n_1 \le N$, $1 \le n_2 \le p-1$;
d.) $1 \le n_1 \le p/2^m - 1$, $1 \le n_2 \le p-1$.

Then we have

$$\frac{p-1}{2} \sideset{}{'}\sum_{\substack{p/2^m \le n_1 \le N \\ 2^m n_1 \equiv n_2 (\bmod p)}} \sideset{}{'}\sum_{p \le n_2 \le N} \frac{r_1(n_1) r_2(n_2)}{n_1 n_2} \ll$$

$$\ll p \sum_{1 \le s_1 \le 2^m N/p} \sum_{1 \le s_2 \le N/p} \sum_{\substack{\ell_1=1 \\ \ell_1 \equiv \ell_2 (\bmod p)}}^{p-1} \sum_{\ell_2=1}^{p-1} \frac{r_1(s_1 p + \ell_1) r_2(s_2 p + \ell_2)}{(s_1 p + \ell_1)(s_2 p + \ell_2)}$$

$$\ll p \sum_{1 \le s_1 \le 2^m N/p} \sum_{1 \le s_2 \le N/p} \sum_{\ell_1=1}^{p-1} \frac{[(s_1 p + \ell_1)(s_2 p + \ell_1)]^{\varepsilon}}{(s_1 p + \ell_1)(s_2 p + \ell_1)}$$

$$\ll p^{\varepsilon} \sum_{1 \le s_1 \le 2^m N/p} \sum_{1 \le s_2 \le N/p} \frac{1}{s_1 s_2} \ll_m p^{\varepsilon}$$

and

$$\frac{p-1}{2} \sideset{}{'}\sum_{\substack{1 \le n_1 \le p/2^m - 1 \\ 2^m n_1 \equiv n_2 (\bmod p)}} \sideset{}{'}\sum_{p \le n_2 \le N} \frac{r_1(n_1) r_2(n_2)}{n_1 n_2} \ll p \sum_{1 \le n_1 \le p/2^m - 1} \sum_{1 \le r \le N/p} (rpn_1)^{\varepsilon - 1} \ll p^{\varepsilon}.$$

Moreover,

$$\frac{p-1}{2} \sideset{}{'}\sum_{\substack{p/2^m \le n_1 \le N \\ 2^m n_1 \equiv n_2 (\bmod p)}} \sideset{}{'}\sum_{1 \le n_2 \le p} \frac{r_1(n_1) r_2(n_2)}{n_1 n_2} \ll p^{\varepsilon},$$

where we have used the estimate $r_1(n) \ll n^{\varepsilon}$, and $r_2(n) \ll n^{\varepsilon}$.

For the case $1 \le n_1 \le p/2^m - 1$, $1 \le n_2 \le p-1$, the solution of the congruence $2^m n_1 \equiv n_2 (\bmod p)$ is $2^m n_1 = n_2$. Hence,

$$\frac{p-1}{2} \sideset{}{'}\sum_{\substack{1 \le n_1 \le p/2^m - 1 \\ 2^m n_1 \equiv n_2 (\bmod p)}} \sideset{}{'}\sum_{1 \le n_2 \le p-1} \frac{r_1(n_1) r_2(n_2)}{n_1 n_2} = \frac{p-1}{2^{m+1}} \sum_{1 \le n_1 \le p/2^m - 1} \frac{r_1(n_1) r_2(2^m n_1)}{n_1^2}$$

$$= \frac{p-1}{2^{m+1}} \sum_{n=1}^{\infty} \frac{r_1(n) r_2(2^m n)}{n^2} + O_m(p^{\varepsilon}).$$

Now, from Lemma 3, we can immediately get

$$\frac{p-1}{2} \sideset{}{'}\sum_{\substack{1 \le n_1 \le N \\ 2^m n_1 \equiv n_2 (\bmod p)}} \sideset{}{'}\sum_{1 \le n_2 \le N} \frac{r_1(n_1) r_2(n_2)}{n_1 n_2} = \frac{\pi^4}{90 \cdot 2^m}(p-1) + O_m(p^{\varepsilon}). \tag{2.5}$$

Similarly, we can also get

$$\frac{p-1}{2}\sideset{}{'}\sum_{\substack{1\le n_1\le N\\ 2^m n_1\equiv -n_2(\bmod p)}}\sideset{}{'}\sum_{1\le n_2\le N}\frac{r_1(n_1)r_2(n_2)}{n_1n_2}= \tag{2.6}$$

$$=\frac{p-1}{2}\sideset{}{'}\sum_{\substack{1\le n_1\le N\\ 2^m n_1+n_2=p}}\sideset{}{'}\sum_{1\le n_2\le N}\frac{r_1(n_1)r_2(n_2)}{n_1n_2}+\frac{\phi(p)}{2}\sideset{}{'}\sum_{\substack{1\le n_1\le N\\ 2^m n_1+n_2=\ell p,\ell\ge 2}}\sideset{}{'}\sum_{1\le n_2\le N}\frac{r_1(n_1)r_2(n_2)}{n_1n_2}$$

$$\ll p\sum_{1\le n\le p-1}\frac{2^m r_1(\frac{p-n}{2^m})r_2(m)}{n(p-n)}+p\sideset{}{'}\sum_{1\le n_2\le N}\sum_{2\le n_2\le\frac{N+n_2}{p}}\frac{2^m r_1(\frac{\ell p-n_2}{2^m})r_2(n_2)}{(\ell p-n_2)n_2}$$

$$\ll_m \sum_{1\le n\le p-1}\frac{(n(p-n))^\varepsilon}{n}+\sum_{1\le n_2\le N}\sum_{2\le n_2\le\frac{N+n_2}{p}}\frac{n_2^\varepsilon(\ell p-n_2)^\varepsilon}{\ell n_2-n_2^2/p}$$

$$\ll_m p^\varepsilon+p^\varepsilon\sum_{n_2=1}^{N}\sum_{\ell=1}^{N}\frac{n_2^\varepsilon\ell^\varepsilon}{\ell n_2}\ll_m p^\varepsilon.$$

Then from (2.4), (2.5) and (2.6), we have

$$M_1=\frac{\pi^4}{5\cdot 2^{m+4}}p+O_m(p^\varepsilon). \tag{2.7}$$

(ii) Note the partition identity

$$B(y,\chi)=\sum_{n\le\sqrt{y}}\chi(n)\chi_2^0(n)\sum_{m\le y/n}\chi(m)\chi_3^0(m)$$
$$+\sum_{m\le\sqrt{y}}\chi(m)\chi_3^0(m)\sum_{n\le y/m}\chi(n)\chi_2^0(n)$$
$$-\sum_{n\le\sqrt{N}}\chi(n)\chi_2^0(n)\sum_{m\le N/n}\chi(m)\chi_3^0(m)-\sum_{m\le\sqrt{N}}\chi(m)\chi_3^0(m)\sum_{n\le N/m}\chi(n)\chi_2^0(n)$$
$$-\left(\sum_{n\le\sqrt{y}}\chi(n)\chi_2^0(n)\right)\left(\sum_{n\le\sqrt{y}}\chi(n)\chi_3^0(n)\right)$$
$$+\left(\sum_{n\le\sqrt{N}}\chi(n)\chi_2^0(n)\right)\left(\sum_{n\le\sqrt{N}}\chi(n)\chi_3^0(n)\right).$$

Applying Cauchy inequality, the estimate

$$\sum_{\chi\neq\chi^0}\left|\sum_{N<n\leq M}\chi(n)\right|^2=\sum_{\chi\neq\chi^0}\left|\sum_{N<n\leq M\leq N+p}\chi(n)\right|^2$$
$$=(p-1)\sum_{N<n\leq M\leq N+p}\chi^0(n)-\left|\sum_{N<n\leq M\leq N+p}\chi^0(n)\right|^2\leq\frac{(p-1)^2}{4}$$

and noting that the identity

$$\sum_{N<n\leq M}\chi(n)\chi_2^0(n)=\sum_{d|2}\mu(d)\chi(d)\sum_{N/d<n\leq M/d}\chi(n),$$

we have

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|B(y,\chi)|^2\ll\sqrt{y}\sum_{n\leq\sqrt{y}}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\sum_{m\leq y/n}\chi(m)\chi_2^0(m)\right|^2$$
$$+\sqrt{y}\sum_{m\leq\sqrt{y}}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\sum_{n\leq y/m}\chi(n)\chi_2^0(n)\right|^2$$
$$+\sum_{m\leq\sqrt{y}}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\sum_{n\leq\sqrt{y}}\chi(n)\chi_2^0(n)\right|^2\times\left|\sum_{n\leq\sqrt{y}}\chi(n)\chi_3^0(n)\right|^2\ll yp^{2+\varepsilon}. \tag{2.8}$$

Similarly,

$$\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}|A(y,\chi)|^2\ll yp^{2+\varepsilon}.$$

Then from Cauchy inequality and (2.8) we can write

$$M_2=\sum_{\chi(-1)=-1}\chi(2^m)\left(\sum_{1\leq n_1\leq N}\frac{\chi(n_1)r_1(n_1)}{n_1}\right)\left(\int_N^\infty\frac{B(y,\chi)}{y^2}dy\right)$$
$$\ll\sum_{1\leq n_1\leq N}n_1^{\varepsilon-1}\int_N^\infty\frac{1}{y^2}\left(\sum_{\chi(-1)=-1}|B(y,\chi)|\right)dy$$
$$\ll N^\varepsilon\int_N^\infty\frac{p^{\frac{3}{2}+\varepsilon}\sqrt{y}}{y^2}dy\ll\frac{p^{\frac{3}{2}+\varepsilon}}{N^{\frac{1}{2}-\varepsilon}}. \tag{2.9}$$

(iii) Similarly to (ii), we can also get

$$M_3\ll\frac{p^{\frac{3}{2}+\varepsilon}}{N^{\frac{1}{2}-\varepsilon}}. \tag{2.10}$$

(iv) By the same argument as in (ii), we can write

(2.11)

$$
\begin{aligned}
M_4 &= \sum_{\chi(-1)=-1} \chi(2^m) \left(\int_N^\infty \frac{A(y,\chi)}{y^2} dy \right) \left(\int_N^\infty \frac{B(y,\chi)}{y^2} dy \right) \\
&\leq \int_N^\infty \int_N^\infty \frac{1}{y^2 z^2} \sum_{\chi(-1)=-1} |A(y,\chi)||B(z,\chi)| dydz \\
&\ll \int_N^\infty \frac{1}{y^2} \int_N^\infty \frac{1}{z^2} \left(\sum_{\chi(-1)=-1} |A(y,\chi)|^2 \right)^{\frac{1}{2}} \left(\sum_{\chi(-1)=-1} |B(y,\chi)|^2 \right)^{\frac{1}{2}} dydz \\
&\ll \left(\int_N^\infty \frac{p^{1+\varepsilon}}{y^{\frac{3}{2}}} dy \right)^2 \ll \frac{p^{2+\varepsilon}}{N}.
\end{aligned}
$$

Now, taking $N = p^3$, combining (2.3)-(2.11), we obtain the asymptotic formula

$$
\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2^m) \left| L(1, \chi\chi_2^0) \right|^2 L(1, \chi\chi_3^0) L(1, \chi) = \frac{\pi^4}{5 \cdot 2^{m+4}} p + O_m(p^\varepsilon). \quad \square
$$

Lemma 5. *Let χ_2^0 and χ_3^0 be the principal characters modulo 2 and 3, respectively. Then we have*

$$
\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left| L(1, \chi\chi_2^0) \right|^2 \left| L(1, \chi\chi_3^0) \right|^2 = \frac{\pi^4}{180}(p-1) + O(p^\varepsilon).
$$

Proof. From the method of proving Lemma 3, we can easily get this Lemma. $\square$

3. Proof of the Theorem

In this section we complete the proof of our theorem.

Proof. i) From Lemma 1 and Lemma 2 we have

$$
\begin{aligned}
&\sum_{a<\frac{p}{3}} \sum_{b<\frac{p}{3}} H(2a\bar{b}, p) \\
&\quad = -\frac{16p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left| L(1, \chi\chi_2^0) \right|^2 \sum_{a \leq \frac{p}{3}} \chi(a) \sum_{b \leq \frac{p}{3}} \overline{\chi}(b) \\
&\quad = \frac{36p}{\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left| \tau(\chi) \right|^2 \left| L(1, \chi\chi_2^0) \right|^2 \left| L(1, \chi\chi_3^0) \right|^2,
\end{aligned}
$$

using $\tau(\chi)\tau(\overline{\chi}) = p$ if $\chi(-1) = -1$. Thus, applying Lemma 4, we obtain

$$\sum_{a<\frac{p}{3}} \sum_{b<\frac{p}{3}} H(2a\bar{b}, p) = \frac{1}{5}p^2 + O(p^{1+\varepsilon}).$$

This complete the proof of i) in the theorem.

ii) Lemma 1 and Lemma 2 imply

$$\begin{aligned}
&\sum_{a<\frac{p}{3}} \sum_{b<\frac{p}{4}} H(2a\bar{b}, p) \\
&= -\frac{16p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left|L(1, \chi\chi_2^0)\right|^2 \sum_{a\le\frac{p}{3}} \chi(a) \sum_{b\le\frac{p}{4}} \overline{\chi}(b) \\
&= \frac{24p}{\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left[1 + \frac{\chi(2)}{2} - \frac{\chi(4)}{2}\right] |\tau(\chi)|^2 \left|L(1, \chi\chi_2^0)\right|^2 L(1, \chi\chi_3^0) L(1, \overline{\chi}) \\
&= -\frac{12p^2}{\pi^4(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left[2\chi(2) + \chi(4) - \chi(8)\right] \left|L(1, \chi\chi_2^0)\right|^2 L(1, \chi\chi_3^0) L(1, \overline{\chi})
\end{aligned}$$

where using $\tau(\chi)\tau(\overline{\chi}) = p$ if $\chi(-1) = -1$. Thus, applying Lemma 4, we obtain

$$\sum_{a<\frac{p}{3}} \sum_{b<\frac{p}{4}} H(2a\bar{b}, p) = \frac{27}{320}p^2 + O(p^{1+\varepsilon}).$$

This complete the proof of ii) in the theorem. □

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