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FINSLER-METRIZABILITIES OF SPRAY MANIFOLDS

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Abstract. A spray manifold (M, S) is said to be Finsler-metrizable in a broad sense or projectively Finsler, if there exists a Finsler structure $L\colon TM \to \mathbb{R}$ such that the Finsler manifold (M, L) is projectively equivalent to (M, S). If, in particular, the canonical spray of (M, L) coincides with the given spray S, then we say that (M, S) is Finsler-metrizable in a natural sense or that S is a Finsler-variational spray. In his influential paper [3] M. Crampin presented a stimulating intrinsic reformulation of the famous Helmholtz conditions from the classical inverse problem of the calculus of variations through the existence of a 2-form on the tangent manifold. Prescribing some extra condition on this 2-form we derive necessary and sufficient conditions for metrizability of a spray in both senses.

1. Introduction

The investigation of the geometry of spray manifolds, in classical terminology the *general geometry of paths* started in the twenties-thirties of the last century. Several outstanding mathematicians worked on the local study of sprays, e.g. L. Berwald, E. Cartan, J. Douglas, M. S. Knebelmann, T. Y. Thomas, O. Veblen and others. One of the Hungarian geometers, András Rapcsák, enriched the theory to a significant extent in the 1960's.

A renaissance of spray geometry began in the 1970's, by recognizing the fundamental role of sprays in the geometrical background of Lagrangian mechanics [3], [5], and, in particular in the foundation of Finsler geometry [10], [11]. Indeed, in differential–geometric terms, the dynamics of a time–independent Lagrangian dynamical system is determined by a spray acting on the tangent manifold of the configuration space of the system. The 'canonical spray' of a Finsler manifold arises from the energy determined by a suitable Lagrangian. J. Klein, J. Grifone and M. Crampin did pioneering work in this field.

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Simply put, a general Finsler structure on a differentiable manifold M is a function

$$L\colon TM \to \mathbb{R}$$

satisfying appropriate differentiability, homogeneity and regularity conditions. In conformity with the demands of Finsler geometry, the smoothness is not required or assured on the whole tangent manifold TM. With the help of Finsler structure L we can introduce the *canonical spray* of a Finsler manifold and the canonical horizontal endomorphism generated by the spray which is called the *Barthel endomorphism* (see definition in section 2).

One of the main topics of our paper belongs to the territory of *projective geometry of sprays*. Roughly speaking, two sprays over the same manifold are said to be projectively equivalent if they have the same geodesics as point sets, i.e., if they have common *pregeodesics*. (Recall: a curve is called a pregeodesic of a spray if it has a reparametrization as a geodesic.) Since every Finsler manifold is a spray manifold, we can also speak of the projective equivalence of a spray manifold and a Finsler manifold, and of that of two Finsler manifolds (which have, of course, a common carrier manifold). A spray manifold (M, S) is said to be *Finsler-metrizable in a broad sense* or — following Shen's terminology [20] — *projectively Finsler*, if there exists a Finsler structure $L\colon TM \to \mathbb{R}$ such that the Finsler manifold (M, L) is projectively equivalent to (M, S). If, in particular, the canonical spray of (M, L) coincides with the given spray S, then we say that (M, S) is *Finsler-metrizable in a natural sense* or that S is a *Finsler-variational* spray. The latter concept is a faithful analogue of the variationality of a spray (or a semispray) used in the classical inverse problem of the calculus of variations ([7], [12], [13], [14]).

For the problem of Finsler-metrizability in a broad sense, the key ingredients will be *the fundamental equations of projective equivalence*. These provide equivalent (and, in our presentation, intrinsically formulated) second order partial differential equations for the Finsler structure to be determined (see **5.1**). Their coordinate version was discovered by A. Rapcsák in the early sixties ([16], [17], [19], [18] and [20]), hence we call them *Rapcsák equations*.

M. Crampin in [3] presented a intrinsic reformulation of the *Helmholtz conditions* through the existence of a 2-form on the tangent manifold. Prescribing some extra condition on this 2-form concerning the homogeneity and regularity *we derive necessary and sufficient conditions for metrizability of a spray in both senses*. Next we investigate the question of the metrizability from another point of view. We present equivalent conditions for a spray to be Finsler–variational assuming the existence of a symmetric, non–degenerate (0,2) tensor field on the vertical bundle with some further properties. Similarly, our main result (**5.3**) gives a characterization of the projectively Finsler spray manifolds through the existence of a $(0, 2)$ tensor field on the tangent manifold, satisfying certain algebraic conditions, some of them are quite complicated. Nevertheless, the its real significance lies in the fact that it reduces the problem of Finsler-metrizabilty in a broad sense to a *first order* partial differential equation. This theorem summarizes its integrability condition and, supplementing by some conditions on the homogeneity and regularity, we derive through the Rapcsák equation necessary and sufficient conditions for a spray to be projectively Finsler.

The paper is organized as follows. In *section 2* we recall some basic concepts and facts. We apply mainly the calculus of vector–valued differential forms elaborated by A. Frölicher and A. Nijenhuis [9] combining it with (and simplifying at the same time) a systematic use of a moving frame field consisting of vertically and completely (or vertically and horizontally) lifted vector fields. To make this section as short as possible we do not recall the definition of the Frölicher-Nijenhuis bracket and do not even give the evaluation formulas in cases we deal with later. The reader can find them in their original work [9]; for a list of some useful identities we refer to [23]. After their definitions we briefly summarize the relationship between the horizontal endomorphisms and semisprays ([2], [4], [10]). We mention only the definition of the well-known *Berwald connection*, for details we refer [11] or [21]. We recall the intrinsic definition of a projective change and summarize the most important facts concerning a Finsler manifold.

The brief *section 3* is devoted to discussing the *strong convexity* of Finsler manifolds. This condition is usually formulated by prescribing the pointwise symplecticity of the fundamental 2-form dd_JE, where E is the energy of the Finsler manifold (see **2.6**). This condition guarantees the non-degeneracy (and therefore the positive definiteness [15]) of the Riemann-Finsler metric g. In **3.1** we give the condition of strong convexity in terms of the 2-form dd_JL and of a symmetric $(0,2)$ tensor field $\overline{\mu}$ defined analogously as g is derived from dd_JE. In the next proposition we restate (in our terminology) Carthéodory's result [1]: if there exists a strongly convex, 1-homogeneous Lagrangian then there also exists a positive one.

We begin *section 4* with Crampin's theorem. This theorem works with a 2-form living on the tangent bundle. In **4.2** we supplement Crampin's theorem by extra conditions concerning homogeneity and regularity (strong convexity) and we derive necessary and sufficient conditions for a spray to be Finsler-variational. In **4.4** we give equivalent conditions through the existence of a symmetric (0,2) tensor field on the vertical bundle.

The concept of the last section is similar. First we recall the Rapcsák equations of projectively equivalence. In **5.2** we give necessary and sufficient conditions for a spray to be projectively Finsler through the existence of a 2-form on the tangent bundle. Our main result is summarized in the theorem **5.3**. We derive equivalent conditions through the existence of a symmetric (0,2) tensor field on the vertical bundle. Our aim was to make it clear that conditions (2), (3) and (4) guarantee the integrability, condition (1) the homogeneity and condition (5) the regularity (strong convexity).

2. Basic facts

2.1. Throughout this paper, M will denote a connected, smooth (i.e., C^∞) manifold of dimension $n \geq 2$. $C^\infty(M)$ is the ring of real-valued smooth functions on M, $\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of vector fields on M. For $(r,s) \in \mathbb{N} \times \mathbb{N}$, $\mathcal{T}^r_s(M)$ is the $C^\infty(M)$-module of smooth tensor fields (briefly tensors) of type (r,s), contravariant of order r and covariant of order s. $\Omega^k(M)$ $(0 \leq k \leq n)$ is the module of differential forms on M, $\Omega^\circ(M) := C^\infty(M)$. The differential forms constitute the graded algebra $\Omega(M) := \overset{n}{\underset{k=0}{\oplus}} \Omega^k(M)$, with multiplication given by the wedge product. The *operator of the exterior derivative* will be denoted by d while i_X and $\mathcal{L}_X$ stand for the *insertion operator* and the *Lie derivative* respectively. If $K \in \operatorname{End}\mathfrak{X}(TM)$ and $\omega \in \Omega^l(M)$

$$K^*\omega\left(X_1, \ldots, X_\ell\right) := \omega\left(K\left(X_1\right), \ldots, K\left(X_\ell\right)\right) \quad \left(X_i \in \mathfrak{X}(M),\ 1 \leq i \leq \ell\right).$$

A vector k-form on the manifold M is a skew-symmetric $C^\infty(M)$-multilinear map $[\mathfrak{X}(M)]^k \to \mathfrak{X}(M)$ if $k \in \mathbb{N}^+$, and a vector field on M if $k = 0$. The set of all vector k-forms on M is a $C^\infty(M)$-module, denoted by $\Psi^k(M)$. In particular, the elements of $\Psi^1(M)$ are just the $(1,1)$ tensor fields on M. One of the basic tool in our calculations is the Frölicher-Nijenhuis theory of vector forms and derivations. For definitions and identities we refer their original paper [9], or to [23].

2.2. The tangent bundle. The tangent bundle of the manifold M will be denoted by $\pi\colon TM \to M$, while $\pi_0\colon \mathcal{T}M \to M$ stands for the subbundle of the nonzero tangent vectors. The kernel of the tangent map $T\pi\colon TTM \to TM$ is a distinguished subbundle of TTM, the *vertical subbundle*, whose total space will be denoted by T^vTM. The sections of this bundle constitute the $C^\infty(TM)$-module $\mathfrak{X}^v(TM)$ of the *vertical vector fields*. In our calculations we shall frequently use the *vertical lift* X^v and the *complete lift* X^c of a vector field $X \in \mathfrak{X}(M)$. Their usefulness is established by the following simple observation (1st local basis property). *If $(X_i)_{i=1}^n$ is a local basis for the module $\mathfrak{X}(M)$, then $(X_i^v, X_i^c)_{i=1}^n$ is a local basis for $\mathfrak{X}(TM)$.*

Tangent bundle geometry is dominated by two canonical objects: the *Liouville vector field (the canonical vertical vector field) $C \in \mathfrak{X}^v(TM)$* and the *vertical endomorphism (the canonical almost tangent structure) $J \in \mathcal{T}^1_1(TM) \cong \operatorname{End}\mathfrak{X}(TM)$.*

2.3. Horizontal endomorphisms and semisprays. In our approach the role of a "nonlinear connection" is played by the *horizontal endomorphisms*. A vector 1-form $h \in \operatorname{End}\mathfrak{X}(TM)$, *smooth only on $\mathcal{T}M$*, is said to be a horizontal endomorphism on M if it is a *projector* (i.e., $h^2 = h$) and $\operatorname{Ker} h = \mathfrak{X}^v(TM)$. $v := 1_{\mathfrak{X}(TM)} - h$ is the *vertical projector* belonging to h. $\mathfrak{X}^h(TM) := \Im h$ is called the module of *horizontal vector fields.* The mapping $X \in \mathfrak{X}(M) \mapsto X^h := hX^c \in \mathfrak{X}^h(TM)$ is called the *horizontal lifting* by h. We have a *2nd local basis* *property: if* $(X_i)_{i=1}^n$ *is a local basis for the module* $\mathfrak{X}(M)$*, then* $(X_i^v, X_i^h)_{i=1}^n$ *is a local basis for* $\mathfrak{X}(TM)$. Suppose that h is a horizontal endomorphism on the manifold M. The vector forms

$$\begin{aligned} H &:= [h, C] \in \Psi^1(TM), \\ t &:= [J, h] \in \Psi^2(TM), \\ R &:= -N_h := -\frac{1}{2}[h, h] \in \Psi^2(TM) \end{aligned}$$

are called the *tension*, the *torsion* and the *curvature* of h, respectively. A horizontal endomorphism is said to be *homogeneous* if its tension vanishes.

A *semispray* on the manifold M is a mapping $S\colon TM \to TTM$, $v \mapsto S_v \in T_vTM$ satisfying the following conditions: S is smooth on $\mathcal{T}M$ and $JS = C$. A semispray S is called a *spray* if S is of class C^1 on TM and $[C, S] = S$ (i.e., S is positive homogeneous of degree 2). A manifold M endowed with a spray S will be mentioned as a *spray manifold.* A spray S is said to be *affine* (or *quadratic*) if it is C^2 on TM.

We recall (see [10]) that any horizontal endomorphism $h \in \operatorname{End}\mathfrak{X}(TM)$ gives rise to a unique *almost complex structure* $F \in \operatorname{End}\mathfrak{X}(TM)$, smooth over $\mathcal{T}M$, defined by $F := h[S, h] - J$ where $S := h(S')$ (S' is an arbitrary semispray on M.)

The fundamental relation between the horizontal endomorphisms and the semisprays was discovered, independently, by M. Crampin and J. Grifone [2], [4], [10]. Their main result can be summarized as follows.

(i) If $h \in \operatorname{End}\mathfrak{X}(TM)$ is a horizontal endomorphism and S' is an arbitrary semispray on M, then $S := hS'$ is also a semispray on M. This semispray does not depend on the choice of S', it is horizontal with respect to h and satisfies the relation $h[C, S] = S$. S is called the *semispray associated to h.*

(ii) Any semispray $S\colon TM \to TTM$ generates in a canonical way a horizontal endomorphism which can be given by the formula

$$(*) \qquad h := \frac{1}{2}\left(1_{\mathfrak{X}(TM)} + [J, S]\right).$$

Then h is torsion free (i.e., $t = 0$) and the semispray associated to h is $\frac{1}{2}(S + [C, S])$. If, in addition, S is a spray, then h is homogeneous and its associated semispray is just the starting spray S.

(iii) A horizontal endomorphism is generated by a semispray according to $(*)$ if and only if it is torsion free.

2.4. Berwald connection. Let us suppose that a horizontal endomorphism h is given. We define the mapping

$$\overset{\circ}{D}\colon \mathfrak{X}(TM) \times \mathfrak{X}(TM) \to \mathfrak{X}(TM), \quad (X, Y) \mapsto \overset{\circ}{D}_X Y$$

by the following rules:

$$\overset{\circ}{D}_{JX} JY := J[JX, Y],$$
$$\overset{\circ}{D}_{hX} JY := v[hX, JY],$$
$$\overset{\circ}{D}_{vX} hY := h[vX, Y],$$
$$\overset{\circ}{D}_{hX} hY := hF[hX, JY]$$

and

$$\overset{\circ}{D}_X Y := \overset{\circ}{D}_{vX} vY + \overset{\circ}{D}_{hX} vY + \overset{\circ}{D}_{vX} hY + \overset{\circ}{D}_{hX} hY.$$

$\overset{\circ}{D}$ is said to be the *Berwald connection induced by* h.

Suppose $\overset{\circ}{D}$ is a Berwald connection on the manifold $\mathcal{T}M$. We introduce the operators

$$\overset{\circ}{D}_J, \overset{\circ}{D}_h \colon \mathcal{T}^r_s(TM) \to \mathcal{T}^r_{s+1}(TM)$$

by the rules

$$i_X \overset{\circ}{D}_J A := \overset{\circ}{D}_{JX} A, \quad i_X \overset{\circ}{D}_h A := \overset{\circ}{D}_{hX} \quad (X \in \mathfrak{X}(TM)).$$

2.5. Projective change. Two sprays S and $\overline{S}$ over a manifold M are said to be *projectively equivalent* if there is a function $\lambda\colon TM \to \mathbb{R}$ satisfying the conditions

(i) λ is smooth on $\mathcal{T}M$, and C^1 on TM;
(ii) $\overline{S} = S + \lambda C$.

Then λ is automatically 1-homogeneous (i.e., $C\lambda = \lambda$). Conversely, if a spray S and a 1-homogeneous function λ, satisfying (i), are given, then $\overline{S} = S + \lambda C$ is also a spray. In this case we speak of a *projective change* of the spray, and we say that the spray manifolds (M, S) and $(M, \overline{S})$ are *projectively equivalent*.

2.6. Finsler manifolds. By a *Lagrange function*, briefly *Lagrangian*, we mean a continuous function $L\colon TM \to \mathbb{R}$ which is smooth on $\mathcal{T}M$ and satisfies the condition $L(0) = 0$.

A Lagrangian $L\colon TM \to \mathbb{R}$ is said to be a *Finsler structure* on M, if L is 1-homogeneous (positive homogeneous of degree 1, that is $L(tv) = tL(v)$, where $v \in TM$, $\ t > 0$ or, equivalently, $CL = L$), and the *fundamental 2-form* $\omega := dd_J E := \frac{1}{2} dd_J L^2$ is symplectic. $E := \frac{1}{2}L^2$ is mentioned as the *energy function* (briefly the *energy*) of the Finsler structure L. A Finsler structure L is called a *positive Finsler structure* if $L(v) > 0$ for all $v \in \mathcal{T}M$.

A manifold endowed with a Finsler structure is called a *Finsler manifold*. A horizontal endomorphism h on a Finsler manifold with energy E is *conservative* if $d_h E = 0$.

Nondegeneracy of the fundamental 2-form guarantees that for any 1-form α there exists a unique vector field $\alpha^{\#}$ (read: α *sharp*) on $\mathcal{T}M$ such that $i_{\alpha^{\#}}\omega = \alpha$.

Thus we obtain an isomorphism from $\Omega^1(\mathcal{T}M)$ onto $\mathfrak{X}(\mathcal{T}M)$, called the *sharp operator* with respect to ω.

Any Finsler manifold is a spray manifold in a natural manner. — Indeed, if $S := -(dE)^\#$ over $\mathcal{T}M$ and $S(0) := 0$, then S is a spray on M, called the *canonical spray* of (M, L).

On any Finsler manifold there exists a unique conservative horizontal endomorphism generated by the canonical spray. — This is the *fundamental lemma of Finsler geometry* due to J. Grifone [10]. The horizontal endomorphism in question we call the *Barthel endomorphism* of the Finsler manifold.

Two *Finsler manifolds* (with common base manifold) are said to be *projectively equivalent* if they are projectively equivalent as spray manifolds.

A spray manifold (M, S) is projectively equivalent to a Finsler manifold $(M, \overline{L})$ if S is projectively equivalent to the canonical spray of $(M, \overline{L})$. In this case (M, S) is called *Finsler-metrizable in a broad sense* or *projectively Finsler*.

A spray manifold (M, S) is said to be *Finsler-metrizable in a natural sense* if there exists a Finsler structure on M, whose canonical spray is the given spray S. Then we also say that S is *Finsler-variational*.

3. Strong convexity

Proposition 3.1. *Suppose $L\colon TM \to \mathbb{R}$ is a 1-homogeneous Lagrangian. L is a Finsler structure if and only if L is positive and* $\operatorname{rank} dd_J L = 2(n-1)$.

Proof. Let us define the tensor fields g and $\overline{\mu}$ by

$$g(JX, JY) := dd_J E(JX, Y), \quad \overline{\mu}(JX, JY) := dd_J L(JX, Y), \tag{3.1a, b}$$

for all $X, Y \in \mathfrak{X}(TM)$. First we show that $dd_J E$ is non-degenerate if and only if $\operatorname{rank} g = n$. Take a local basis $(X_i)_{i=1}^n$ of $\mathfrak{X}(M)$ and consider the local basis $(X_i^v,\ X_i^c)_{i=1}^n$ of $\mathfrak{X}(TM)$. Using the following notations $g_{ij} := g(X_i^v, X_j^v) = dd_J E(X_i^v, X_j^c)$ and $\alpha_{ij} := dd_J E(X_i^c, X_j^c)$ the matrix of $dd_J E$ has the form

$$\begin{pmatrix} 0 & \dots & 0 & & \\ \vdots & \ddots & \vdots & & (g_{ij}) \\ 0 & \dots & 0 & & \\ & & & 0 & \\ (-g_{ij}) & & (-\alpha_{kl}) & \ddots & (\alpha_{kl}) \\ & & & & 0 \end{pmatrix}.$$

From this the implication is immediate. Next we prove that $\operatorname{rank} dd_J L = 2(n-1)$ if and only if $\operatorname{rank} \overline{\mu} = n-1$. Take the Liouville vector field C and a semispray $S_\circ$ and extend them to a basis of $\mathfrak{X}(TM)$. Note first that $i_C dd_J L = 0$ and for any $X \in \mathfrak{X}(TM)$ we have $i_{S_\circ} dd_J L(JX) = -JX(L) - J[S_\circ, JX]L = 0$. Suppose $(C, JX_1, \dots, JX_{n-1}, S_\circ, X_1, \dots, X_{n-1})$ is our considered basis. Let us introduce $\overline{\mu}_{ij} := \overline{\mu}(JX_i, JX_j) = dd_J L(JX_i, X_j)$ and $\beta_{ij} := dd_J L(X_i, X_j)$ $(1 \le i, j \le n-1)$. The matrix of $dd_J L$ is written in the following form:

$$\begin{pmatrix} 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & (\overline{\mu}_{ij}) & \\ 0 & \dots & 0 & 0 & & \\ 0 & \dots & 0 & 0 & & \\ \vdots & (-\overline{\mu}_{ij}) & & (-\beta_{kl}) & \ddots & (\beta_{kl}) \\ 0 & & & & & 0 \end{pmatrix}.$$

From this one can easily deduce our statement. Finally we show that g is of rank n if and only if $\operatorname{rank} \overline{\mu} = n-1$ and L is positive. Since

$$dd_J E = dd_J(\frac{1}{2}L^2) = d(L d_J L) = dL \wedge d_J L + L dd_J L,$$

so for any vector field X, Y on (TM)

$$dd_J E(JX, Y) = dL(JX) \otimes dL(JY) + L dd_J L(JX, Y).$$

By the definition of g and $\overline{\mu}$ we obtain that

$$(*) \qquad g(JX, JY) = dL(JX) \otimes dL(JY) + L\overline{\mu}(JX, JY).$$

dL has a $n-1$-dimensional kernel on the vertical bundle which does not contain the Liouville vector field C. Now choose $n-1$ vector fields from the kernel and extend with C to a basis of the vertical bundle. In this case the equation $(*)$ has the form

$$\begin{pmatrix} & & \\ & (g_{ij}) & \\ & & \end{pmatrix} = \begin{pmatrix} L^2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} + L \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & (\overline{\mu}_{ij}) & \\ 0 & & & \end{pmatrix},$$

which proves 3.1 □ □

Definition ([6])**.** A Finsler structure L is said to be *positive semidefinite* if the tensor $\overline{\mu}$ defined above has the property: for any $v \in \mathcal{T}M$ and $X \neq \lambda C \in \mathfrak{X}(TM)$ $(\lambda \in C^\infty(TM))$ $\overline{\mu}(X, X)(v) > 0$. In this case we also call the tensor $\overline{\mu}$ positive semidefinite.

Proposition 3.2. *Let $\overline{L}$ be a 1-homogeneous Lagrangian on the manifold M and $\overline{\mu}$ as before. Suppose that $\overline{L}$ is positive semidefinite. Then there exists – locally – a positive Finsler structure L such that the tensor μ arising from L according to 3.1 coincides with $\overline{\mu}$.*

Proof. (Cf. [1], page 243.) Suppose $\overline{L}$ is a 1-homogeneous Lagrangian on M. Fix a vector $v \in \mathcal{T}M$ and choose a $w \in \mathcal{T}M$ such that $\pi(v) = \pi(w)$ and $w \neq \xi v$ $(\xi \geq 0)$. Taking a chart $(U, (u)_{i=1}^n)$ on M and introducing the induced chart $\left(\pi^{-1}(U), x^i, y^i\right)_{i=1}^n$ $(x^i := u^i \circ \pi,\ y^i := du^i)$, consider the Taylor expansion of $\overline{L}$

$$\overline{L}(w) = \overline{L}(v) + (w^i - v^i)\frac{\partial \overline{L}}{\partial y^i}(v) + \frac{1}{2}(w^i - v^i)(w^j - v^j)\frac{\partial^2 \overline{L}}{\partial y^i \partial y^j}(z),$$

where $z = v + \theta(w - v)$ $(0 < \theta < 1)$, $w = w^i\left(\frac{\partial}{\partial u^i}\right)_{\pi(w)}$, $v = v^i\left(\frac{\partial}{\partial u^i}\right)_{\pi(v)}$. This expression can be written in the form

$$\overline{L}(w) = \overline{L}(v) - C(v)\overline{L} + w_v^{\uparrow}\overline{L} + \frac{1}{2}(w-v)_z^{\uparrow}\Big((w-v)_z^{\uparrow}\overline{L}\Big),$$

where $w_v^{\uparrow}$ is the vertical lift of w to v, that is $w_v^{\uparrow} = w^i\left(\frac{\partial}{\partial y^i}\right)_v$. By the definition $\overline{\mu}$ this yields

$$\overline{L}(w) = \overline{L}(v) - C(v)\overline{L} + w_v^{\uparrow}\overline{L} + \frac{1}{2}\overline{\mu}_z\big((w-v)_z^{\uparrow}, (w-v)_z^{\uparrow}\big).$$

Since $\overline{L}$ is 1-homogeneous we obtain that

$$\overline{L}(w) - w_v^{\uparrow}\overline{L} = \frac{1}{2}\overline{\mu}_z\big((w-v)_z^{\uparrow}, (w-v)_z^{\uparrow}\big).$$

By our assumption v and w have not the same direction, therefore $(w-v)_z^{\uparrow} \neq \lambda(v)C(v)$ $(\lambda \in C^\infty(TM))$, but then $\overline{\mu}_z$ is strictly positive, whence

$$(*) \qquad \overline{L}(w) - w_v^{\uparrow}\overline{L} = \overline{L}(w) - w^i\frac{\partial \overline{L}}{\partial y^i}(v) > 0.$$

Identifying $T_{\pi(v)}(M)$ with $\mathbb{R}^n$ equipped with the canonical Euclidean inner product we take the half of the Euclidean unit ball of $T_{\pi(v)}M$ which is in opposite direction to v, that is the set B defined by

$$(y^1)^2 + (y^2)^2 + \cdots (y^n)^2 = 1, \quad \sum_{i=1}^n y^i v^i \leq 0.$$

This is a compact set. Let k denote the minimum of the function $\overline{L}(w) - w_v^{\uparrow}\overline{L}$ on B and define c_i by

$$c_i = \frac{k}{2}\frac{v^i}{\sqrt{(v^1)^2 + (v^2)^2 + \cdots + (v^n)^2}} - \frac{\partial \overline{L}}{\partial y^i}(v).$$

Set $L(w) := \overline{L}(w) + c_i y^i(w)$, $\big(w \in T_{\pi(v)}M \setminus \{0\}\big)$. We show that $L(w) > 0$ on the Euclidean unit ball of $T_{\pi(v)}M$. If $\sum_{i=1}^n v^i w^i > 0$, then $L(w) > 0$ by $(*)$. Suppose that $\sum_{i=1}^n v^i w^i \leq 0$. In this case $\overline{L}(w) - w^i\frac{\partial \overline{L}}{\partial y^i}(v)$ is decreased by

$$\frac{k}{2}\frac{|\sum_{i=1}^n v^i w^i|}{\sqrt{(v^1)^2 + \cdots + (v^n)^2}} = \frac{k}{2}\frac{|\sum_{i=1}^n v^i w^i|}{\sqrt{(v^1)^2 + \cdots + (v^n)^2}\sqrt{(w^1)^2 + \cdots + (w^n)^2}}$$
$$= \frac{k}{2}\frac{|< v, w >|}{\|v\|\|w\|} \leq \frac{k}{2},$$

where $< \ , \ >$ denotes the Euclidean inner product and $\| \ \|$ is the corresponding Euclidean norm. Therefore we obtain that $L(w) \geq \frac{k}{2} > 0$. By the homogeneity of L this yields $L(w) > 0, \ w \in T_{\pi(v)}M \setminus \{0\}$. Since $c_i y^i = (c_i u^i)^c$, so L and $\overline{L}$ differs in only a complete lift of a function. We obtain by a routine calculation that for all $X, Y \in \mathfrak{X}(TM) : dd_J f^c(JX, Y) = 0$, whence $\overline{\mu} = \mu$. □

4. Metrizability in a natural sense (Finsler-variationality)

Theorem 4.1 (A theorem of M. Crampin)**.** *Let S be a semispray over the manifold M, and let h be the horizontal endomorphism generated by S. If a 2-form ω on TM satisfies the conditions*

$$\mathcal{L}_S\omega = 0, \tag{4.1a}$$

$$\omega(JX, JY) = 0 \quad (X, Y \in \mathfrak{X}(TM)), \tag{4.1b}$$

$$d\omega(hX, JY, JZ) = 0 \quad (X, Y, Z \in \mathfrak{X}(TM)), \tag{4.1c}$$

then there is a smooth function K defined on an open subset of TM such that $\omega = dd_J K$.

For a proof we refer to Crampin's paper [3] or the book of de León and Rodrigues [8].

Proposition 4.2. *Let (M, S) be a spray manifold and suppose that a 2-form ω satisfies conditions (4.1a-c). If, in addition, ω is 1-homogeneous and has maximal rank then there exists – locally – a Finsler energy E such that S is the canonical spray of the Finsler manifold (M, E).*

Proof. According to 4.1 there exists a function K such that $\omega = dd_J K$. First we show that the condition $\mathcal{L}_C\omega = \omega$ guarantees the existence of a 2-homogeneous function E such that $\omega = dd_J E$. Since

$$\begin{aligned} dd_J K &= \mathcal{L}_C dd_J K = i_C ddd_J K + di_C dd_J K = di_C dd_J K \\ &= d\mathcal{L}_C d_J K - ddi_C d_J K = d\mathcal{L}_C d_J K = dd_J \mathcal{L}_C K - dd_J K, \end{aligned}$$

it follows that $d(d_J 2K - d_J \mathcal{L}_C K) = 0$. Thus by the Poincaré lemma, there exists a smooth function F such that

$$d_J(2K - \mathcal{L}_C K) = dF.$$

The 1-form on the left-hand side of this relation vanishes on the vertical vector fields, so the function F has to be a vertical lift. Suppose that $F = f^v, f \in C^\infty(M)$. Then

$$d_J(2K - \mathcal{L}_C K) = df^v$$

and, consequently,

$$CK - 2K = f^c + h^v,$$

where h is an another smooth function on M. Now let

$$E := K + \frac{1}{2}h^v + f^c.$$

Then E is 2-homogeneous since

$$CE = C(K + \frac{1}{2}h^v + f^c) = CK + f^c = 2K + h^v + f^c + f^c = 2E.$$

Obviously, we also have $dd_J K = dd_J E$. Since the 2-form $dd_J E$ is of maximal rank, E is a Finsler energy on the manifold M. Consider the canonical spray S_E of (M, E). Now, on the one hand

$$(*) \qquad i_{S_E} dd_J E = -dE,$$

on the other hand

$$0 \overset{(4.1a)}{=} \mathcal{L}_S dd_J E = i_S ddd_J E + di_S dd_J E = di_S dd_J E,$$

so there exists a function G defined on an open subset of TM such that

$$(**) \qquad i_S dd_J E = -dG.$$

Taking the difference of $(*)$ and $(**)$ we obtain

$$i_{S_E - S} dd_J E = d(G - E).$$

$S_E - S$ is a vertical vector field, therefore the 1-form on the left-hand side vanishes on the vertical vector fields by (4.1b). This implies as before, that $G - E$ is a vertical lift. If $G - E = k^v$, then $i_{S_E - S} dd_J E = dk^v := i_{(dk^v)^\#} dd_J E$, therefore

$$S_E = S + (dk^v)^\#.$$

(The sharp operator $\#$ is taken with respect to ω.) The sprays S and S_E are 2-homogeneous. To conclude the proof, we show that $(dh^v)^\#$ is 0-homogeneous and hence it has to vanish.

$$\begin{aligned} i_{[C,(dk^v)^\#]}\omega &= \mathcal{L}_C i_{(dk^v)^\#}\omega - i_{(dk^v)^\#}\mathcal{L}_C\omega = \mathcal{L}_C dk^v - i_{(dk^v)^\#}\omega \\ &= d\mathcal{L}_C k^v - i_{(dk^v)^\#}\omega = -i_{(dk^v)^\#}\omega. \end{aligned}$$

□

Lemma 4.3. *Suppose that g is a non-degenerate, symmetric $(0,2)$-tensor on the vertical subbundle and let $\mathcal{C}_\flat := \overset{\circ}{D}_J J^* g$, where $\overset{\circ}{D}$ is the Berwald connection determined an arbitrarily chosen horizontal endomorphism h and S is a semispray over the manifold M. If*

(i) *$\mathcal{C}_\flat$ is totally symmetric,*
(ii) *$i_S \mathcal{C}_\flat = 0$,*

then the $(0,2)$ tensor g_E defined by $g_E(JX, JY) := dd_J E(JX, Y)$, where $E := \frac{1}{2}g(C, C)$; $X, Y \in \mathfrak{X}(TM)$, equals to the given tensor g.

Proof. For any vector field X on M and semispray S over M we get

$$\begin{aligned} 0 &\overset{(ii)}{=} \mathcal{C}_\flat(X^h, S, S) = \overset{\circ}{D}_J J^* g(X^h, S, S) \overset{(i)}{=} X^v g(C, C) - 2g(J\overset{\circ}{D}_{X^v} S, C) \\ &= X^v g(C, C) - 2g(X^v, C), \end{aligned}$$

whence

$$(*) \qquad X^v g(C,C) = 2g(X^v, C).$$

Similarly, for any vector fields X, Y on M we have

$$0 = \mathcal{C}_\flat(X^h, Y^h, S) = \overset{\circ}{D}_J J^* g(X^h, Y^h, S) = X^v g(Y^v, C) - g(Y^v, J\overset{\circ}{D}_{X^v} S)$$
$$= X^v g(Y^v, C) - g(X^v, Y^v),$$

therefore

$$(**) \qquad X^v g(Y^v, C) = g(X^v, Y^v).$$

Now, by relations $(*)$ and $(**)$ it follows that

$$X^v(Y^v g(C,C)) \overset{(*)}{=} X^v(2g(Y^v, C)) \overset{(**)}{=} 2g(X^v, Y^v).$$

On the other hand,

$$\begin{aligned} 2g_E(X^v, Y^v) &= 2g_E(JX^h, JY^h) = 2dd_J E(X^v, Y^h) \\ &= 2X^v d_J E(Y^h) - 2Y^h d_J E(X^v) - 2d_J E([X^v, Y^h]) \\ &= 2X^v Y^v E = X^v Y^v g(C,C), \end{aligned}$$

which concludes the proof. □

Proposition 4.4. *Let (M, S) be a spray manifold and let $(\overset{\circ}{D}, h)$ denote the Berwald connection determined by the horizontal endomorphism h arising from S. Suppose that g is a non-degenerate, symmetric $(0,2)$-tensor in the vertical bundle, and let $\mathcal{C}_\flat := \overset{\circ}{D}_J J^* g$. If*

(4.4a) $\qquad \mathcal{C}_\flat$ *is totally symmetric* ,

(4.4b) $\qquad i_S \mathcal{C}_\flat = 0,$

(4.4c) $\qquad \overset{\circ}{D}_S J^* g = 0,$

then S is the canonical spray of the Finsler energy $E := \frac{1}{2} g(C, C)$.

Proof. First we show that the function $E := \frac{1}{2} g(C,C)$ is 2-homogeneous. According to (4.4b)

$$0 = \mathcal{C}_\flat(S,S,S) := \overset{\circ}{D}_J J^* g(S,S,S) := (\overset{\circ}{D}_C J^* g)(S,S)$$
$$= Cg(C,C) - 2g(J\overset{\circ}{D}_C C, C) = Cg(C,C) - 2g(C,C) = 2CE - 4E,$$

therefore $CE = 2E$, as we claimed. Since g is non-degenerate, therefore the 2-form $dd_J E$ is symplectic. All these mean that E is a Finsler energy on M. It remains only to check that E is conservative. Applying our conditions,

$$0 \overset{(4.4c)}{=} \overset{\circ}{D}_S J^* g(S,S) = Sg(C,C) - 2g(J\overset{\circ}{D}_S S, C) = Sg(C,C) = 2SE,$$

therefor $SE = 0$. Choose a vector field X on M. Using this observation and (4.4c) again we get

$$\begin{aligned}
0 &\overset{(4.4c)}{=} \overset{\circ}{D}_S J^* g(X^h, S) = Sg(X^v, C) - g(J\overset{\circ}{D}_S X^h, C) - g(X^v, J\overset{\circ}{D}_S S) \\
&= Sg(X^v, C) - g(vX^c, C) \overset{4.3}{=} SX^v E - vX^c E \\
&= [S, X^v]E + X^v SE - vX^c E = X^c E - 2X^h E - vX^c E \\
&= -X^h E = -d_h E(X^c),
\end{aligned}$$

whence h is indeed a conservative horizontal endomorphism on M and therefore it has to coincide with the Barthel endomorphism of (M, E) or, equivalently, S has to be the canonical spray from which h is arising. □

5. Metrizability in a broad sense (Projectively Finsler spray manifolds)

5.1. Rapcsák equations. Let (M, S) be a spray manifold endowed with the Berwald connection $(\overset{\circ}{D}, h)$ induced by S. If $\overline{L}$ is a Finsler structure on M, then the following conditions are equivalent:

(1) (M, S) is projectively equivalent to $(M, \overline{L})$.

(2) $i_S dd_J \overline{L} = 0$.

(3) $d_h d_J \overline{L} = 0$.

(4) $\overset{\circ}{D}_h d_J \overline{L}(X, Y) = \overset{\circ}{D}_h d_J \overline{L}(Y, X) \quad (X, Y \in \mathfrak{X}(\mathcal{T}M))$.

For an index-free proof we refer to [22].
1

Proposition 5.2. *Let S be a spray over the manifold M and ω be a 2-form satisfying conditions (4.1a-c). Define the tensor μ by the relation $\mu(JX, JY) := \omega(JX, Y)$ $(X, Y \in \mathfrak{X}(TM))$. If, in addition, $i_C\omega = 0$ and μ is positive semi-definite, then there exists – locally – a positive Finsler structure L such that its canonical spray is projectively equivalent to the given one.*

Proof. First we show that there exists a 1-homogeneous Lagrangian $\overline{L}$ such that $\omega = dd_J\overline{L}$. By Crampin's theorem $\omega = dd_J K$, where K is a function on a region of TM. Since

$$0 = i_C dd_J K = \mathcal{L}_C d_J K + di_C d_J K = \mathcal{L}_C d_J K = d_J \mathcal{L}_C K - d_J K,$$

therefore

$$d_J(\mathcal{L}_C K - K) = 0.$$

Obviously, the function $\mathcal{L}_C K - K$ vanishes on the vertical vector fields, so it has to be a vertical lift of a function on M. Suppose $\mathcal{L}_C K - K = f^v$. Define $\overline{L}$ by

$$\overline{L} := \mathcal{L}_C K.$$

Since $\mathcal{L}_C\overline{L} := \mathcal{L}_C\mathcal{L}_C K = \mathcal{L}_C K = \overline{L}$, $\overline{L}$ is 1-homogeneous and, obviously, $\omega = dd_J\overline{L}$. According to 3.2 there exists a positive Finsler structure L such that $dd_J L = \omega$. We show that $i_S dd_J L = 0$. Since

$$0 \overset{(4.1a)}{=} \mathcal{L}_S dd_J L = i_S ddd_J L + di_S dd_J L = di_S dd_J L,$$

it follows that $i_S dd_J L = dF$. The 1-form $i_S dd_J L$ vanishes on the vertical vector fields, this implies again that F is a vertical lift of a function on M. Suppose that $F = h^v$. We claim that $i_S dd_J L$ is a 1-homogeneous semibasic 1-form. Indeed,

$$\begin{aligned}\mathcal{L}_C i_S dd_J L &= \mathcal{L}_C(\mathcal{L}_S d_J L - di_S d_J L)\\ &= \mathcal{L}_{[C,S]} d_J L + \mathcal{L}_S\mathcal{L}_C d_J L - \mathcal{L}_C di_S d_J L\\ &= \mathcal{L}_S d_J L - \mathcal{L}_C di_S d_J L\\ &= i_S dd_J L + \mathcal{L}_C di_S d_J L - \mathcal{L}_C di_S d_J L = i_S dd_J L.\end{aligned}$$

Since $\mathcal{L}_C dh^v(X^c) = C(dh^v(X^c)) = C(X^c(h^v)) = C(Xh)^v = 0$ it follows that dh^v is 0-homogeneous and therefore it has to vanish. According to 5.1(2) this means that S is projective equivalent to the canonical spray of (M, L). □

Theorem 5.3. *Let (M, S) be a spray manifold and suppose that there exists a symmetric $(0,2)$-tensor μ on the vertical subbundle satisfying the following conditions:*

(1) $i_C\mu = 0$,
(2) $\overset{\circ}{D}_S J^*\mu = 0$,
(3) $\underset{X,Y,Z\in\mathfrak{X}(TM)}{\mathfrak{S}} \mu(JX, R(Y,Z)) = 0$ ($\mathfrak{S}$ *means the cyclic sum*),
(4) $\overset{\circ}{D}_J J^*\mu$ *is totally symmetric* ,
(5) μ *is positive semidefinite.*

Then there exists a positive Finsler structure L such that its canonical spray is projectively equivalent to the given spray S.

Proof. Let h, v and F be the horizontal endomorphism, the vertical projector and the almost complex structure induced by S, respectively. Consider the $(0,2)$ tensor $\overline{\mu}$ given by

$$\overline{\mu}(X,Y) := \mu(vX, vY) + \mu(JX, JY), \quad \text{for all } X, Y \in \mathfrak{X}(TM).$$

Let us also introduce a 2-form ω by the rule

$$\omega(X, FY) := \overline{\mu}(X,Y), \quad \text{for all } X, Y \in \mathfrak{X}(TM).$$

One can easily deduce that

$$(*) \qquad \omega(JX, JY) = \omega(hX, hY) = 0,$$

whence (4.1b) is satisfied. Next we show that (4.1a) is also valid. We check that $\mathcal{L}_S\omega$ vanishes on the following pairs of vector fields:

$$(X^h, Y^h),\ (X^v, Y^h),\ (X^v, Y^v).$$

Case 1.

$$\begin{aligned}\mathcal{L}_S\omega(X^h,Y^h) &= S\omega(X^h,Y^h) - \omega([S,X^h],Y^h) - \omega(X^h,[S,Y^h])\\ &\overset{(*)}{=} \omega([S,X^h],Y^h) + \omega(X^h,[S,Y^h])\\ &= \omega([S,X^h],FY^v) - \omega([S,Y^h],FY^v)\\ &= \mu([S,X^h],Y^v) - \mu([S,Y^h],X^v)\\ &= \mu(v[S,X^h],Y^v) - \mu(v[S,Y^h],X^v)\\ &= \mu(R(S,X^h),Y^v) - \mu(R(S,Y^h),X^v)\\ &\overset{(1)}{=} \mu(R(S,X^h),Y^v) + \mu(R(Y^h,S),X^v) + \mu(R(X^h,Y^h),C)\\ &\overset{(3)}{=} 0.\end{aligned}$$

Case 2.

$$\begin{aligned}\mathcal{L}_S\omega(X^v,Y^h) &= S\omega(X^v,Y^h) - \omega([S,X^v],Y^h) - \omega(X^v,[S,Y^h])\\ &\overset{(*)}{=} S\omega(X^v,FY^v) - \omega([S,X^v],FY^v) - \omega(X^v,h[S,Y^h])\\ &= S\mu(X^v,Y^v) - \mu([S,X^v],Y^v) - \mu(X^v,J[S,Y^h])\\ &= S\mu(X^v,Y^v) - \mu([S,X^v],Y^v) - \mu(X^v,vY^c)\\ &= S\mu(X^v,Y^v) - \mu(vX^c,Y^v) - \mu(X^v,vY^c)\\ &= \mathring{D}_S J^*\mu(X^h,Y^h) \overset{(2)}{=} 0.\end{aligned}$$

Case 3.

$$\begin{aligned}\mathcal{L}_S\omega(X^v,Y^v) &= S\omega(X^v,Y^v) - \omega([S,X^v],Y^v) - \omega(X^v,[S,Y^v])\\ &\overset{(*)}{=} \omega(Y^v,h[S,X^v]) - \omega(X^v,h[S,Y^v])\\ &= -\mu(Y^v,J[S,X^v]) + \mu(X^v,J[S,Y^v])\\ &= -\mu(Y^v,X^v) + \mu(X^v,Y^v) = 0,\end{aligned}$$

due to the symmetry of μ.

Finally, we show that (4) implies (4.1c).

$$\begin{aligned}d\omega(X^h,Y^v,Z^v) &= X^h\omega(Y^v,Z^v) - Y^v\omega(X^h,Z^v) + Z^v\omega(X^h,Y^v)\\ &\quad - \omega([X^h,Y^v],Z^v) + \omega([X^h,Z^v],Y^v) - \omega([Y^v,Z^v],X^v)\\ &= -Y^v\omega(X^h,Z^v) + Z^v\omega(X^h,Y^v),\end{aligned}$$

since ω vanishes on pairs of vertical vector fields, the bracket of vertical lifts is zero, and the bracket of a vertically and a horizontally lifted vector field is vertical. Using the definition μ we obtain

$$d\omega(X^h,Y^v,Z^v) = -Y^v\omega(X^h,Z^v) + Z^v\omega(X^h,Y^v)$$

$$
\begin{aligned}
&= Y^v\omega(Z^v, X^h) - Z^v\omega(Y^v, X^h)\\
&= -Y^v\mu(Z^v, X^v) + Z^v\mu(Y^v, X^v)\\
&= -\mathring{D}_J J^*(X^h, Y^h, Z^h) + \mathring{D}_J J^*(Z^h, Y^h, X^h) \overset{(4)}{=} 0.
\end{aligned}
$$

Now our statement is an immediate consequence of Proposition 5.2. □

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