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Galileo and Aristotle's Wheel

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Abstract

At the beginning of his last major work, Galileo tackles an old paradox, Aristotle's Wheel, in order to produce a model of the continuum that explains (at least to him) how line segments of different length could be put into a one-to-one correspondence. His argument seems like a playful digression. However, it is precisely this type of a one-to-one correspondence that he needs to support his work on free fall. In this article, we investigate how Galileo's model for the wheel paradox informs his work on free fall. We also examine some of the reasons his results on free fall—results that were grounded in his notion of the continuum—were not readily accepted in his time.

1. Introduction

In 1638, Galileo was infirm, blind, and under house arrest by the Catholic Church for his defense of the heretical Copernican cosmological view. It was at that time that he wrote his magnum opus, *The Discourses on the Two New Sciences*, in which he offered his refined and final thoughts on topics that he had pondered over a lifetime. At the center of that work we find what Galileo finally deemed to be a satisfactory mathematical proof of his law for bodies uniformly accelerated by free fall: the distance x that a body falling from rest travels in time t is given by $x = gt^2/2$ where g is a proportionality constant that depends on the units chosen.

What seems so unremarkable now was ground-breaking at the time. While natural philosophers of the middle ages had studied uniformly accelerated motion per se, their models for bodies in free fall were very different. (We return to this later.) Galileo deduced his model of free fall experimentally and then investigated its consequences mathematically. The rest is physics history.

> [S]ince nature does employ a certain kind of acceleration for descending heavy things we decided to look into their properties so that we might be sure that the definition of accelerated motion that we are about to adduce [i.e., uniform acceleration] agrees with the essence of naturally accelerated motion.[1]

Here is Galileo's own statement of the law of free fall (Proposition II, Theorem II, Day 3):

> If a moveable descends from rest in uniformly accelerated motion, the spaces run through in any times whatever are to each other as the duplicate ratios of their times; that is, as the squares of those times.

The simple phrase, "through in any times whatever," is critical to Galileo's model of free fall. It implies a continuous and scale-free dependence of distance on the continuum of time. But here is how Galileo refers to time as he is describing uniform acceleration in his prelude to Proposition II:

> Thus, taking any equal particles of time whatever from the first instant in which a moveable departs from rest and decent is begun, the degree of swiftness acquired in the first and second little parts of time [together] is double the degree that the moveable acquired in the first little part [of time].

We see a struggle between a model of time in which time is made up of "particles," one following another like the frames of a movie, versus the continuous model implied by unlimited divisibility of any measurable span time. It's a struggle as old as Zeno. A resolution, or at least a certain reconciliation of these models, was important to Galileo as he laid out his "new science" of motion. How Galileo thought about the continuum comes early in the *Two New Sciences* in his analysis of the paradox known as Aristotle's Wheel. As we shall see, his internalization of this model informed both his refutation of the then current model of free fall and his mathematical justification of his own results on uniform acceleration.

[1] All quotes are from [5] unless otherwise noted.

2. The Wheel

The paradox of the wheel is found in *The Mechanical Problems*, a book of unknown authorship but attributed to Aristotle in Galileo's time. (For a history of the paradox see [2].) The paradox is embodied in the Figure 1.[2] The length of AB is equal to the circumference of the outer circle. Imagine that the wheel makes one complete revolution so that the point on the wheel initially in contact with the line at A lands on B, a smooth roll. Now look at the roll of the inner circle. The length CD equals that of AB, but it should also equal the length of the shorter circumference of the inner circle. The paradox: $AB = CD$ and $AB \neq CD$.

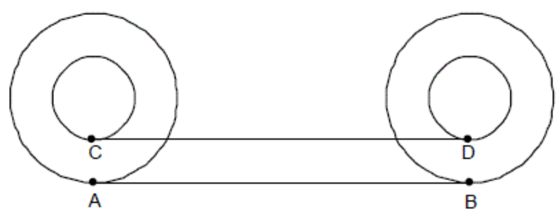

Figure 1: Aristotle's Wheel.

Any real system of concentric wheels would screech loudly because the inner wheel could not be carried by a smooth roll of the outer wheel without slipping. However, to the mathematician, there seems to be no problem. Let's say the inner circle is of radius 1 and the outer is of radius 2. Any point on either circle can be named uniquely by θ, the angle that the common extended radius makes with the perpendicular as in Figure 2 (see the next page). So the points on the two circles are in one-to-one correspondence. In a smooth roll of the outer circle, a point named θ maps to a point a distance 2θ away from A in Figure 1. The mathematician running Aristotle's wheel simply maps the point on the inner circle named θ to a point a distance 2θ away from C: $f(\theta) = 2\theta$. Each and every point on each of the spans AB and CB is touched exactly once by a point named θ, putting the points of CD in one-to-one correspondence with the points of AB. Easy.

[2] All figures except for Figure 3 were produced by the author.

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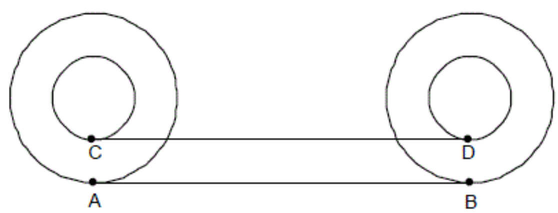

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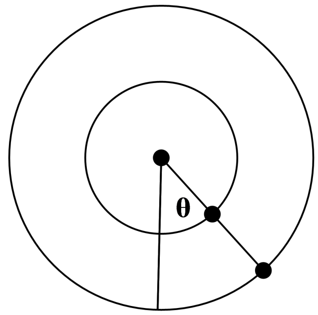

Figure 2: Points named by θ.

But "$f(\theta) = 2\theta$" does not get to the heart of the paradox because it does not explain how the rotation of the wheel can simultaneously lay out the points of each circumference—points that are in one-to-one correspondence via the parameterizing angle—one at a time, one after another, yet have them map out different spaces, one twice the length of the other. Galileo developed a model of the continuum that accommodates, if not explains, the paradox.

To understand the motion of the inner and outer circles, Galileo analyzed the motion of concentric regular polygons. In his diagram (see Figure 3) he uses concentric hexagons.

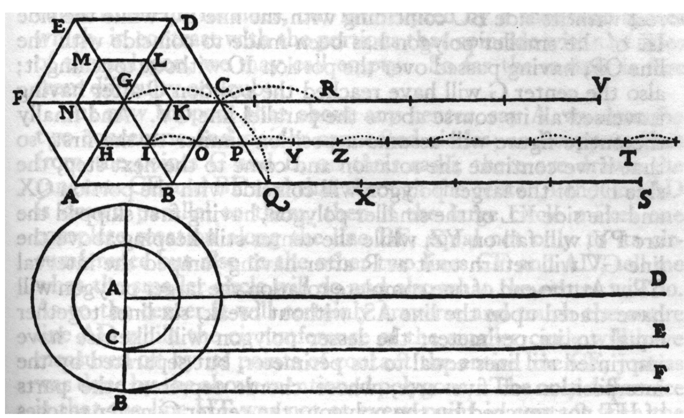

Figure 3: Galileo's diagram. Image from [4]. See `http://galileoandeinstein.physics.virginia.edu/tns_draft/tns_001to061.html` (*pages 1-61: Text and figures, part of the First Day*) for the diagram in context.

Imagine rolling the outer hexagon so that it stamps out the base with line segments that have no gaps or overlaps, the polygonal equivalent of a slide-free roll of the outer circle. The inner polygon does something very different. It stamps out discontinuous flat segments, with gaps or "voids" in between. If a polygon with more sides is used, as the number of sides increases, the number of gaps increases, and their lengths shorten. In a complete rotation, the number of footprints that the n sides of the smaller polygon leave is in one-to-one correspondence with the number of footprints that the n larger sides leave. Also, the total length of smaller sides together with the lengths of the $n - 1$ voids in between sums to the length that the larger polygon stamps out.[3] These two observations form the basis for Galileo's model of the continuum.

By thinking of a circle as a polygon with an infinite number of sides, Galileo develops his model:

> And just so, I shall say, in circles (which are polygons with infinitely many sides), the line passed over by the infinitely many sides of the large circle, arranged continuously [in a straight line], is equal in length to the line passed over by the infinitely many sides of the smaller, but in the latter case, with the interposition of as many voids between them. And just as the "sides" [of the circles] are not quantified, but there are infinitely many, so the interposed voids are not quantified, but are infinitely many. But imagining the line resolved into unquantifiable parts—that is, into its infinitely many indivisibles—we can conceive it immensely expanded without any quantified void spaces, though not without infinitely many indivisible voids.

In Galileo's vocabulary, a line segment with nonzero length is "quantified." Such a segment is divisible into two similarly divisible quantified segments. The process of dividing can be iterated any number of times, but at the n^{th} stage, no matter how big n is, we are left with a set of quantified intervals of finite length. Alternatively, an "unquantified part" such as a point has no length and cannot be bisected. It is an "indivisible". To imagine

[3] Roughly. It's missing a bit at the beginning and end (for example HT versus AS in Figure 3) that becomes negligible as the number of sides of the considered polygon increases.

"the line resolved" is to imagine how we pass from a line made up of finite (but divisible) segments to a line made up of an infinite number of indivisible points. The circle, an infinite-sided polygon with unquantified points as its sides, lets us accomplish such a resolution. When concentric polygons roll along, the segments stamped out by the smaller polygon need quantized voids in between in order to be in one-to-one correspondence with the segments stamped out by the larger polygon and to span the same total length when the roll is completed. Galileo asserts that, in the same way, the unquantified points of the continuous segment touched by the smaller circle need unquantified voids among them in order to be in one-to-one correspondence with the continuous segment touched by the larger circle and (most importantly) to span the same length. Galileo imagines a continuous line segment to be an aggregate of an infinite number of indivisible points, with the points of any two segments in one-to-one correspondence. The interspersed unquantified voids accommodate for different lengths.

Certainly, Dedekind is not anticipated here, nor the calculus of infinitesimals. Galileo offers no arithmetic of unquantified points and voids. His is a mental model with which he can justify the one-to-one correspondence between segments of different lengths. The introduction of unquantified voids does this for him. It is the careful use of precisely these one-to-one correspondences that allows him to refute the theory of free fall current in his time and then go on to prove his own results on free fall. Galileo offers us a model, but not necessarily an understanding:

> Let us remember that we are among infinities and indivisibles, the former incomprehensible to our finite understanding by reason of their largeness and the latter by their smallness. Yet we see that human reason does not want to abstain from giddying itself about them.

3. Ideas Uprooted

Before he presented his theory of free fall, Galileo first argued against the then current notion that, in free fall, "[s]peed goes increasing according to the increase of space traversed," which is to say that the speed of an object falling is proportional to distance fallen. We would model this as $dx/dt = kx$. With x measured from the point of release, the initial condition is $x(0) = 0$ and the solution is $x(t) = 0$, leaving our rock suspended in midair.

To disprove this notion, Galileo first reminds us that if one object travels a certain distance at a fixed speed and another object travels double the distance at double that speed, the time it takes for both trips is identical. Comparing the distances traveled by objects moving at non-constant speeds is less straightforward. Galileo thinks in terms of "degrees of speed" and of "total speed." A degree of speed is a quantifiable attribute of a body at a point of time and location. The intensities of different degrees of speed allow for ratios between them. Total speed is the aggregate of all the degrees of speed that an object has acquired over the time passed during its trip. The total speed "consumed" determines how far an object has gone. Galileo's task is to find the ratio between total speeds consumed, a task made difficult because total speed is the aggregate of infinitely many degrees and so the usual operations are not available.

Galileo considers two objects in free fall, the first falling a span of 2 meters from rest and the second a span of 4 meters. As per the model he seeks to refute, he assumes that the speed of the second object is twice that of the first when it has gone twice as far. He must then determine the ratio of their total speeds so that he can determine the ratio of the times taken. To do so, he puts the positions in the two spans into one-to-one correspondence: position x in the two meter span corresponds with position $2x$ in the four meter span. This puts individual degrees of speed into one-to-one correspondence: the speed at position x of the first object corresponds to the degree of speed at position $2x$ of the second object. Since the speed at $2x$ is twice the speed at x, at each and every x, the total speed of the second object is twice that of the first. So they complete their journeys in the same time. But that is impossible!

To see how Galileo's resolution of Aristotle's Wheel lies behind his argument, let's first think about how we might approach the problem of finding total speed. In Figure 4 displayed on the next page, the vertical legs of the triangles represent the 2 and 4 meter spans respectively. At any position x along a span, the degree of speed is represented by the length of the horizontal line from x to the hypotenuse. To find and compare the total speeds—the aggregate of the horizontal lines— we would be inclined to compute and compare the areas of the two triangles. But since the area of the second triangle is four times that of the first, the total speed of the object falling 4 meters would be *four* times the total speed of the first object. It would complete its journey in half the time of the object falling 2 meters. Galileo puts the span of 2 meters and the span of 4 meters in one-to-one correspondence in the same way he corresponded the points on the line traced out by the inner circle of Aristotle's wheel to those traced out by the outer circle, a correspondence made possible to him because of his unquantified voids. The points in each span are in one-to-one correspondence; the speeds at each point are in one-to-one correspondence; corresponding speeds are doubles; the total speed is doubled. The paradoxes are similar: the line traced out by the smaller circle is the same length as that traced out by the larger, just as the time it takes an object to fall 2 meters is the same as the time it would take it to go 4 meters.

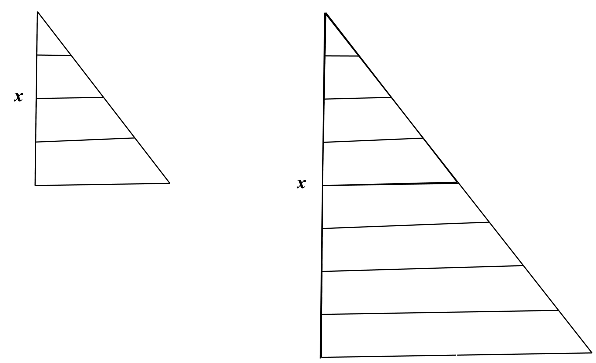

Figure 4: Speed in proportion to distance traveled.

A bit of the spirit of Galileo's model lives with us in our calculus-based approach to the same problem. Suppose that we have two objects that traverse two intervals, $[0, a]$ and $[0, 2a]$, and that their velocities at position x are $v_1(x)$ and $v_2(x)$, respectively. Also suppose that $v_2(x) = 2v_1(x/2)$. Now compute the times T_1 and T_2 that it takes to complete each trip. So $T_1 = \int_0^a 1/v_1(x)\,dx$ and $T_2 = \int_0^{2a} 1/(2v_1(x/2))\,dx$. To compute T_2 we make the u-substitution $u = x/2$. The times are identical; the role of the voids in providing the stretch to the intervals is taken by the u-substitution $2du = dx$.

Another prevalent idea Galileo challenged was the notion that a heavy object falling arrives immediately at great speed. Instead, Galileo argues:

> There will be no degree of speed, however small, ... such that the moveable will not be found to have this [at some time] after its departure from infinite slowness, that is, rest.

In other words, on the way down, before an object falling from rest has reached a certain speed s, it will have passed through all possible speeds from 0 to s, continuously. He argues first from symmetry. An object dropped from rest from a certain height travels with the same speeds as that with which an object falls back to earth after it was thrown up to exactly that height. On the trip down, the speeds gained by the thrown object will mirror (in reverse order) the speeds it lost (or consumed) on the way up. And these he cannot imagine as not passing through every degree of slowness. Anticipating the argument that, if there are infinitely many degrees of slowness to go through, then nothing could come to rest, Galileo counters that, since the object remains at a speed for only an (unquantified) instant, and "[s]ince in any finite time there are infinitely many instants, there are enough to correspond to the infinitely many degrees of diminished speed." Here he places the continuum of possible speeds in one-to-one correspondence with the instants in an interval of time.

4. Free Fall

As early as the 14th century, medieval scholastic philosophers such as Nicole Oresme and the Merton School at Oxford, had deduced the Mean Speed Rule [1, pages 199–219 and 340–344]. This rule asserts that a body uniformly accelerated from rest over an interval of time goes as far as a body traveling at a uniform speed that is half the terminal speed of the accelerated body. Galileo's proof of the same law occurs in Proposition I, Theorem I on Day Three of the *Two Sciences*. It is not known if Galileo knew the older results. What he did that his medieval predecessors did *not* do is to link free fall and uniformly accelerated motion:

> Proposition I. Theorem I. The time in which a certain space is traversed by a moveable in uniformly accelerated movement from rest is equal to the time in which the same space would be traversed by the same moveable carried in uniform motion whose degree of speed is one-half the maximum and final degree of speed of the previous, uniformly accelerated motion.

The proof of his Theorem I is carried out with an argument the resembles his disproof that speed in free fall is proportional to distance fallen. At its center is a one-to-one correspondence. The essence of his proof is captured by the following excerpt and is accompanied by Galileo's diagram (Figure 5). In that diagram, the "line AB represent(s) the time in which the space CD is traversed by a moveable in uniformly accelerated movement from C."

> Since each instant and all instants of time AB correspond to each point and all points of the line AB, from which points the parallels drawn and included in the triangle AEB represent increasing degrees of the increased speed, while the parallel contained within the parallelogram represent in the same way just as many degrees of speed not increased but equable, it appears that there are just as many momenta of speed consumed in the accelerated motion according to the increasing parallels of triangle AEB, as in the equable motion according to the parallels of the parallelogram GB. For the deficit of the first half of the motion (the momenta represented by the parallels in triangle AGI falling short) is made up by the momenta represented by the parallels of triangle IEF.

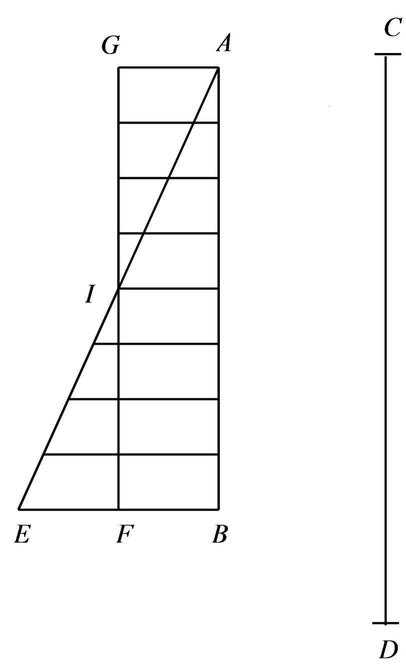

Figure 5: Galileo's Diagram, as found in *The Two Sciences* [5], redrawn by the author.

To compare the total speed consumed by an object traveling with constant velocity to that of the object that is constantly accelerated, Galileo invokes the one-to-one correspondence of the parallels in AIG with the parallels in IEF. The excesses are in one-to-one correspondence with the deficits. In short, the total of the speeds in the rectangle AGFB is the same as the total in the triangle AEB.

If we extend our triangle as in Figure 6, marking the tick of each time unit by a horizontal parallel, we can see that the distances (like the boxes) traversed increase as 1, 3, 5,.... After t units of time the space traversed adds up to $1+3+5+\cdots+(2t-1)=t^2$. Galileo called this the Odd Number Rule. The rule holds no matter what unit of time is chosen; it is independent of scale. In so far as time is a continuum, distance and velocity also vary continuously with time.

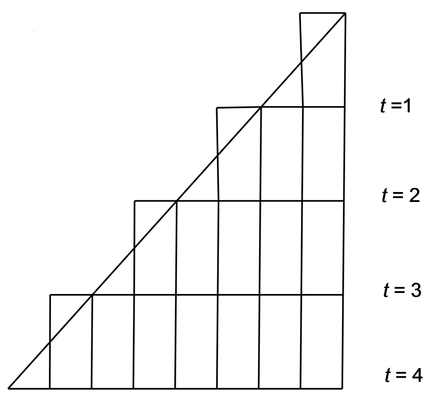

Figure 6: The Odd Number Rule.

5. The Struggle

The idea that velocity increased continuously during free fall was a major impediment to the acceptance of Galileo's model. Those who objected included familiar names in the history of mathematics, such as Mersenne and Descartes. The questions were deep. What is the nature of an instant of time? Is it quantized, with duration? Is physical time different from mathematical time as modeled by the lines of Euclid's geometry? And what causes fall? A common answer to the last question was "intrinsic heaviness," which suggests that, with the removal of what holds an object up, a nonzero degree of velocity is immediately available, without the object passing through every smaller possible degree. Further, for that velocity to occur, a change of time is necessary, which in turn suggests that a moving object stays at uniform speed for a nonzero measure of time. Velocity in free fall would then be modeled by a step function—a discrete model rather than a continuous one.

Let's look at an alternative discrete model for free fall as it was posed by the Jesuit Honorè Fabri and others to refute Galileo's assertions.[4] Grounded in the medieval theory of impetus, it harked back to a 14^{th} century model formulated by Buridan [3, pages 19-22]. In this model, there is a smallest atom of time—an instant—that has nonzero duration, however small. In free fall, speed is accumulated in discrete quanta, at discrete and sequential instants of time, and it increases as the natural numbers. In the first instant, the body has a certain velocity, say v. In the second instant, it has velocity $2v$, then $3v$, and so on. So in the first instant of time, the body travels v units of space; in the second instant, $2v$ units of space, etc. In t units of time, the total distance traveled is $1v + 2v + \cdots + tv = vt(t+1)/2$. Notice that this discrete model requires that the initial velocity be nonzero. If not, the falling body just hangs there! And it could not pass continuously through "every degree of speed" as Galileo asserts it must.

How does this discrete model compare to the Odd Number Rule model? At the scale of Fabri's atomic instant, they are very different. In Galileo's model of free fall from rest, the ratio of the distance traveled in the n^{th} instant to the distance traveled in the first would be $2n-1$, the Odd Number Rule. In Fabri's model, that ratio is simply n. Galileo's model is scale free. For any length of time t, the ratio of the distance traveled in the n^{th} interval of length t, namely $[(n-1)t, nt]$, to the distance traveled in the first interval $[0, t]$ is $\frac{n^2t^2-(n-1)^2t^2}{t^2} = 2n - 1$, the Odd Number Rule again. But Fabri's model is not scale free. For example, suppose that an instant is 10^{-9} sec., a nanosecond, and let $f(t) = \frac{t(t+1)}{2}$. Then in the first, second, and third seconds, an object would travel $f(10^9) = 500000000500000000v$ units, $f(2\cdot 10^9) - f(10^9) = 1500000000500000000v$ units, and $f(3\cdot 10^9) - f(2\cdot 10^9) = 2500000000500000000v$ units, respectively. The ratios of these distances to

[4] For a lovely account of the history of free fall, see [3]. For an account of counter-arguments to Galileo's model, see [6].

the distance traveled in the first, second, and third seconds are not 1, 2, and 3, but 1, 2.999999998, and 4.999999996, very close to the Odd Number Rule. In fact, $\lim_{t\to\infty} \frac{f(nt) - f((n-1)t)}{f(t)} = 2n - 1$, exactly the Odd Number Rule. At the macro level, the models converge.

Limits on precision in any 17$^{\text{th}}$ century experiment, would have made it impossible to distinguish between Fabri's discrete model and Galileo's continuous model, even though their fundamental assumptions are markedly different. Mersenne said as much: without causation established, we could not tell the difference.

> You see therefore that of these descents of bodies, which are commonly called heavy bodies, nothing deeper can be demonstrated as long as the principle, or true an immediate cause is unknown (Mersenne as translated and quoted in [6, page 271]).

Galileo did not address the cause of acceleration, neither with prime causes such as "things move towards their source," nor with mechanistic causes such as "air pushes down." The lack of such philosophical underpinnings undermined Galileo's theories for those who sought to account for free fall philosophically, with more than a mathematical formula that predicts position. But Galileo was deliberate. After rehearsing what was offered as cause for acceleration in his day, Galileo said:

> Such fantasies, and other like them, would have to be resolved, with little gain. [It] suffices to demonstrate some attributes of motion so accelerated (whatever be the cause of its acceleration).

Galileo felt that the reason for the position-time relation in the law of free fall could not be deduced or explained from philosophical postulates. It was a repeatable phenomenon, established and credentialed by experimentation.

6. Conclusion

Galileo's conception of the continuum was not a significant step (or even a step at all) in the evolution of mathematics. Rather, it was a step in his own evolution from natural philosopher to physicist. It helped him refine his thoughts about how the continuum worked. His genius was then to model motion in terms of the continuum of time, to observe that, indeed, his model worked, and then to use its mathematics to deduce the ramifications of his model. His theory of motion led to, and was in turn subsumed under, the continuum-based physics of Isaac Newton, physics that would explain free fall near the earth and the motions of the heavens with the same theory—the universal law of gravitation. The change of scope was enormous. It prompted a deep re-imagination of how the universe is configured.

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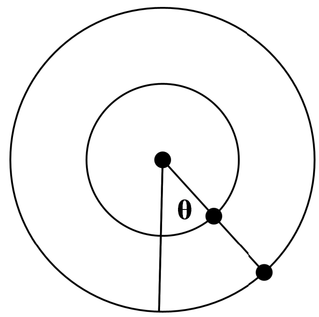

Figure 2: Points named by θ.

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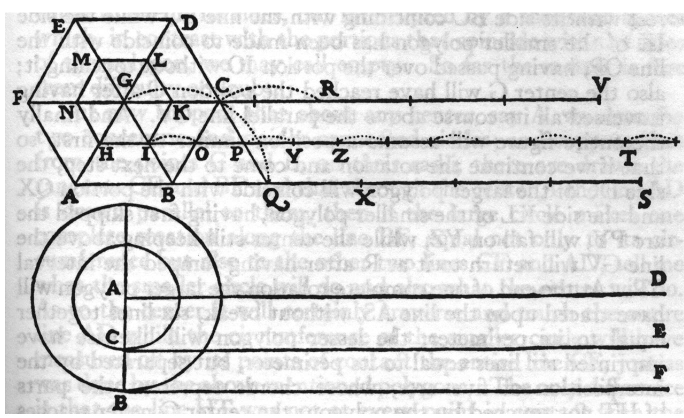

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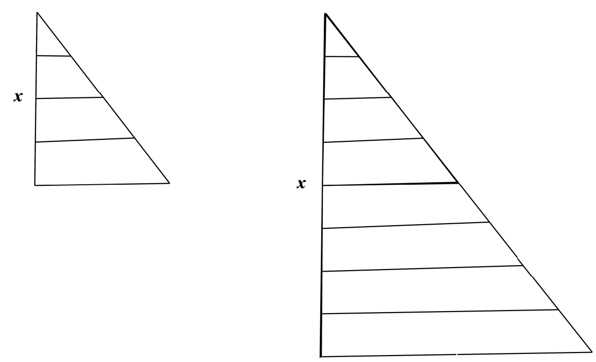

Figure 4: Speed in proportion to distance traveled.

A bit of the spirit of Galileo's model lives with us in our calculus-based approach to the same problem. Suppose that we have two objects that traverse two intervals, $[0, a]$ and $[0, 2a]$, and that their velocities at position x are $v_1(x)$ and $v_2(x)$, respectively. Also suppose that $v_2(x) = 2v_1(x/2)$. Now compute the times T_1 and T_2 that it takes to complete each trip. So $T_1 = \int_0^a 1/v_1(x)\,dx$ and $T_2 = \int_0^{2a} 1/(2v_1(x/2))\,dx$. To compute T_2 we make the u-substitution $u = x/2$. The times are identical; the role of the voids in providing the stretch to the intervals is taken by the u-substitution $2du = dx$.

Another prevalent idea Galileo challenged was the notion that a heavy object falling arrives immediately at great speed. Instead, Galileo argues:

> There will be no degree of speed, however small, ... such that the moveable will not be found to have this [at some time] after its departure from infinite slowness, that is, rest.

by the following excerpt and is accompanied by Galileo's diagram (Figure 5). In that diagram, the "line AB represent(s) the time in which the space CD is traversed by a moveable in uniformly accelerated movement from C."

> Since each instant and all instants of time AB correspond to each point and all points of the line AB, from which points the parallels drawn and included in the triangle AEB represent increasing degrees of the increased speed, while the parallel contained within the parallelogram represent in the same way just as many degrees of speed not increased but equable, it appears that there are just as many momenta of speed consumed in the accelerated motion according to the increasing parallels of triangle AEB, as in the equable motion according to the parallels of the parallelogram GB. For the deficit of the first half of the motion (the momenta represented by the parallels in triangle AGI falling short) is made up by the momenta represented by the parallels of triangle IEF.

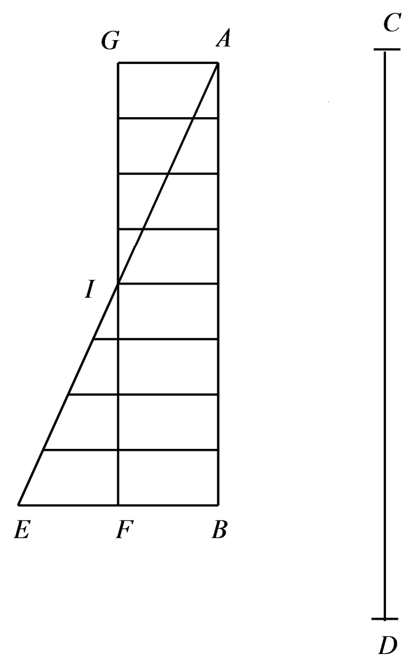

Figure 5: Galileo's Diagram, as found in *The Two Sciences* [5], redrawn by the author.

To compare the total speed consumed by an object traveling with constant velocity to that of the object that is constantly accelerated, Galileo invokes the one-to-one correspondence of the parallels in AIG with the parallels in IEF. The excesses are in one-to-one correspondence with the deficits. In short, the total of the speeds in the rectangle AGFB is the same as the total in the triangle AEB.

If we extend our triangle as in Figure 6, marking the tick of each time unit by a horizontal parallel, we can see that the distances (like the boxes) traversed increase as 1, 3, 5,.... After t units of time the space traversed adds up to $1+3+5+\cdots+(2t-1)=t^2$. Galileo called this the Odd Number Rule. The rule holds no matter what unit of time is chosen; it is independent of scale. In so far as time is a continuum, distance and velocity also vary continuously with time.

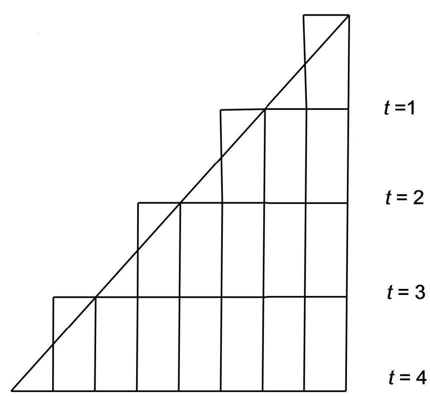

Figure 6: The Odd Number Rule.

5. The Struggle

The idea that velocity increased continuously during free fall was a major impediment to the acceptance of Galileo's model. Those who objected included familiar names in the history of mathematics, such as Mersenne and Descartes. The questions were deep. What is the nature of an instant of time? Is it quantized, with duration? Is physical time different from mathematical time as modeled by the lines of Euclid's geometry? And what causes fall? A common answer to the last question was "intrinsic heaviness," which