

PROXIMAL, DISTAL AND ASYMPTOTIC POINTS IN COMPACT CONE METRIC SPACES

P. RAJA

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Abstract. In this paper, proximal points, distal points, and asymptotic points of a self-mapping of cone metric spaces are introduced and some conditions on contractive self-mappings are obtained which ensure the existence of such points.

1. Introduction

The study of the behavior of contractive mappings is an interesting subject in mathematics, see [5], [17]. Recently Guang and Xian (see [11]) generalized the concept of metric spaces by considering ordered Banach spaces, leading to *cone metric spaces*. Subsequently the study of fixed point theorems for contractive mappings in these cone metric spaces has been considered by some mathematicians, see [1] and [12].

This paper aims to define proximal points, distal points, and asymptotic points in cone metric spaces and then, to obtain some conditions to ensure their existence. The proposed definitions and the related results are presented in Section 2 after representing a few introductory materials in Section 1, as follows.

Let E be a real Banach space. A subset P of E is called a *cone* if and only if the following hold:

- (i) P is closed, non-empty, and $P \neq \{\theta\}$,
- (ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$, imply that $ax + by \in P$,
- (iii) $x \in P$ and $-x \in P$, imply $x = \theta$.

Given a cone $P \subset E$, a *partial ordering* \preceq with respect to P is defined by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int } P$, where $\text{int } P$ denotes the interior of P .

The cone P is called *normal* if there exists a number $K > 0$ such that $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$, for every $x, y \in E$. The least positive number that satisfies this condition is called the *normal constant* of P .

In what follows, it is supposed that E is a real Banach space, P is a cone in E with $\text{int } P \neq \emptyset$, and \preceq is a partial ordering with respect to P .

Let X be a non-empty set. As it has been defined in [11], a function $d : X \times X \rightarrow E$ is called a *cone metric* on X if it satisfies the following conditions:

- (i) $d(x, y) \succeq \theta$, for every $x, y \in X$, and $d(x, y) = \theta$ if and only if $x = y$,
- (ii) $d(x, y) = d(y, x)$, for every $x, y \in X$,
- (iii) $d(x, y) \preceq d(x, z) + d(y, z)$, for every $x, y, z \in X$.

Then, (X, d) is called a *cone metric space*. The sequence $\{x_n\}$ in X is *convergent* to $x \in X$ if for every $c \in E$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \ll c$, for

every $n \geq n_0$, and it is a *Cauchy sequence* if for every $c \in E$ with $\theta \ll c$, there is $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) \ll c$, for every $m, n \geq n_0$. A cone metric space (X, d) is a *compact cone metric space* if every Cauchy sequence in X has a convergent subsequence in X . A self-map T on X is *continuous* if $\lim_{n \rightarrow \infty} x_n = x$ implies that $\lim_{n \rightarrow \infty} T(x_n) = T(x)$, for every sequence $\{x_n\}$ in X .

Lemma 1.1. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K , and $\{x_n\}$ be a sequence in X . Then:*

- (i) [11, Lemma 1] $\{x_n\}$ converges to x if and only if $\lim_{n \rightarrow \infty} d(x_n, x) = \theta$.
- (ii) [11, Lemma 2] $\lim_{n \rightarrow \infty} x_n$ is unique, if $\{x_n\}$ is convergent.
- (iii) [11, Lemma 5] $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$, where $\{y_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} y_n = y$ and $\lim_{n \rightarrow \infty} x_n = x$.

2. Contractive Mappings

Definition 2.1. *Let (X, d) be a cone metric space, where P is a cone and f is a self map on X . For every $x, y \in X$, x is proximal to y under f , if for every $c \gg \theta$, there exists $n \in \mathbb{N}$ such that*

$$d(f^n(x), f^n(y)) \prec c.$$

If x and y are not proximal, they are called distal. If for every $c \gg \theta$, there exists $n \in \mathbb{N}$ such that

$$d(f^m(x), f^m(y)) \prec c,$$

for every $m \geq n$, then x and y are said to be asymptotic under f .

It should be recalled that for a function $f : X \rightarrow X$, $x \in X$ is said to have period k whenever k is the least natural number that $f^k(x) = x$.

Lemma 2.2. *Let (X, d) be a compact cone metric space, where P is a normal cone with normal constant K and f is a self-map on X . If $x \in X$ is proximal to $f^k(x)$ for some $k \in \mathbb{N}$ and f is continuous, then there exists $z \in X$ of a period not more than k .*

Proof. For every $n \in \mathbb{N}$, there exists $c_n \gg \theta$ such that $K||c_n|| < \frac{1}{n}$. By assumption, there exists $m_n \in \mathbb{N}$ such that

$$d(f^{m_n}(x), f^{m_n+k}(x)) \prec c_n, \text{ for every } n \in \mathbb{N}.$$

Hence

$$||d(f^{m_n}(x), f^{m_n+k}(x))|| \leq K||c_n|| \leq \frac{1}{n}, \text{ for every } n \in \mathbb{N}.$$

Since X is compact, there are subsequences $\{f^{m_{n_r}}(x)\}$ and $\{f^{m_{n_r}+k}(x)\}$ and $z, w \in X$ such that $\lim_{r \rightarrow \infty} f^{m_{n_r}}(x) = z$, $\lim_{r \rightarrow \infty} f^{m_{n_r}+k}(x) = w$. Clearly, $w = z$ and the continuity of f implies that $f^k(z) = z$. So z is periodic of a period less than or equal to k , and the proof is complete. \square

Theorem 2.3. *Let (X, d) be a compact cone metric space, where P is a normal cone with normal constant K , f is a continuous self-map on X , and for $x, y \in X$ with $\theta \prec d(x, y)$, there exists $n \in \mathbb{N}$ such that*

$$d(f^n(x), f^n(y)) \prec d(x, y).$$

Then every pair of points in X is proximal under f .

Proof. Fix $x, y \in X$. If $f^n(x) = f^n(y)$, for some $n \in \mathbb{N}$, then x and y are proximal. So we may assume that $f^n(x) \neq f^n(y)$, for every $n \in \mathbb{N}$. Let $\{n_i\} \subseteq \mathbb{N}$ be chosen such that

$$d(x, y) \succ d(f^{n_1}(x), f^{n_1}(y)) \succ \cdots \succ d(f^{n_k}(x), f^{n_k}(y)) \succ \cdots.$$

Suppose that for every $i \in \mathbb{N}$, n_i is chosen as small as possible in order to satisfy this condition. Now, by contradiction, assume that x and y are distal. So there is $c \gg \theta$ such that $d(f^m(x), f^m(y)) \succeq c$, for every $m \in \mathbb{N}$. So

$$\|d(f^m(x), f^m(y))\| \geq \frac{1}{K} \|c\|,$$

for every $m \in \mathbb{N}$. By compactness of X , there are subsequences $\{f^{n_i^r}(x)\}$ and $\{f^{n_i^s}(y)\}$ and $z, w \in X$ such that $\lim_{r \rightarrow \infty} f^{n_i^r}(x) = z$ and $\lim_{s \rightarrow \infty} f^{n_i^s}(y) = w$. By (i) of Lemma 1.1, $z \neq w$, and by (ii) of Lemma 1.1, we have

$$\begin{aligned} d(f^k(w), f^k(z)) &= \lim_{r, s \rightarrow \infty} d(f^{n_i^r+k}(x), f^{n_i^s+k}(y)) \\ &\succeq \lim_{r, s \rightarrow \infty} d(f^{n_i^r+k}(x), f^{n_i^s+k}(y)) = d(w, z), \end{aligned}$$

for every $k \in \mathbb{N}$, which is a contradiction. This completes the proof. \square

Corollary 2.4. Let (X, d) be a compact cone metric space, where P is a normal cone with normal constant K , and f is a self-map on X such that

$$d(f(x), f(y)) \prec d(x, y),$$

for every $x, y \in X$ with $d(x, y) \succ \theta$. Then every pair of points is asymptotic under f .

Corollary 2.5. Let (X, d) be a compact cone metric space, where P is a normal cone with normal constant K , and f is a continuous self-map on X such that for $x, y \in X$ with $\theta \prec d(x, y)$, there exists $n \in \mathbb{N}$ such that

$$d(f^n(x), f^n(y)) \prec d(x, y).$$

Then f has a unique fixed point in X .

Proof. By Theorem 2.3, x and $f(x)$ are proximal under f , for every $x \in X$. Hence by Lemma 2.2, f has a fixed point. By assumption, the uniqueness is clear. \square

Theorem 2.6. Let (X, d) be a compact cone metric space, where P is a normal cone with normal constant K , and f is a continuous self-map on X which satisfies the following condition:

$$\exists c \gg 0 \ni \forall x, y \in X [\theta \prec d(x, y) \ll c \Rightarrow d(f^n(x), f^n(y)) \prec d(x, y)],$$

for some $n \in \mathbb{N}$. Then, x and y are proximal under f , for every $x, y \in X$ with $d(x, y) \ll c$.

Proof. Fix $x, y \in X$ with $d(x, y) \ll c$ and the rest of the proof is similar to the proof of Theorem 2.3. \square

Corollary 2.7. Let (X, d) be a compact cone metric space, where P is a normal cone with normal constant K , and f is a continuous self-map on X . If there exists $c \gg \theta$ such that

$$d(f(x), f(y)) \prec d(x, y),$$

for every $x, y \in X$ with $\theta \prec d(x, y) \ll c$, then, x and y are asymptotic under f , for every $x, y \in X$ with $d(x, y) \ll c$.

Corollary 2.8. *Let (X, d) be a compact cone metric space, where P is a normal cone with normal constant K , and f is a continuous self-map on X which satisfies the following condition:*

$$\exists c \gg \theta \ni \forall x, y \in X [\theta \prec d(x, y) \ll c \Rightarrow d(f^n(x), f^n(y)) \prec d(x, y)],$$

for some $n \in \mathbb{N}$. Then X has a periodic point under f .

Proof. Fix $x \in X$, and suppose that x is not periodic, then $\{f^n(x)\}$ is an infinite set and by compactness of X , there exists $m > n$ such that

$$d(f^m(x), f^n(x)) \prec c.$$

Hence by Theorem 2.6, $f^m(x)$ is proximal to $f^n(x)$. Thus by Theorem 2.3, there is $z \in X$ of period at most $m - n$. This completes the proof. \square

The following example shows that we may have a self-map that have some non-asymptotic proximal points.

Example 2.9. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\} \subseteq \mathbb{R}^2$, $X \subseteq \mathbb{R}^2$, and $d : X \times X \rightarrow E$ defined by $d((x, y), (z, w)) = (|x - z|, \alpha|y - w|)$, where $\alpha \geq 0$ is a constant. Then it is easily check that (X, d) is a cone metric space. Let $A_0 = \{(\frac{1}{i}, 0) : i \in \mathbb{N} \text{ and } i \neq 0\} \cup \{(0, 0)\}$, and $A_n = \{(\frac{1}{i}, \frac{1}{n}) : 0 < |i| \leq n\} \cup \{(0, \frac{1}{n})\}$, for every $n \in \mathbb{N}$, and $X = \bigcup_{n \in \mathbb{N}} A_n \cup A_0$ with the above cone metric. Define f on X as follows. For every $n \in \mathbb{N}$, $(a, b) \in A_n$ and $(a, b) \neq (0, 0), (1, 0), (1, \frac{1}{n})$ imply that there is $(d, e) \in A_n$ immediately to the right of (a, b) . For each n and each such (a, b) , define $f(a, b) = (d, e)$. Define $f(0, 0) = (0, 0)$, $f(1, 0) = (-1, 0)$ and for $n \in \mathbb{N}$ define $f(1, \frac{1}{n}) = (-1, \frac{1}{n} + 1)$. Now, for x and y in X with $x \neq y$ and x not in A_0 , x and y are proximal but not asymptotic under f .

Theorem 2.10. *Let (X, d) be a compact cone metric space, where P is a normal cone with normal constant K , and f is a continuous self-map on X which satisfies the following condition:*

$$\exists c \gg \theta \ni \forall x, y \in X [\theta \prec d(x, y) \ll c \Rightarrow d(f^n(x), f^n(y)) \prec d(x, y)],$$

for some $n \in \mathbb{N}$. Also, let $\theta \prec e \prec c$. If $\theta \prec p \prec e$, then there exists $n_0 \in \mathbb{N}$ such that for every $x, y \in X$, $p \prec d(x, y) \prec e$, and $d(f^k(x), f^k(y)) \prec d(x, y)$ imply that

$$d(f^{k+j}(x), f^{k+j}(y)) \prec d(x, y),$$

for some $0 \leq j \leq n_0$.

Proof. By contradiction, let $\{n_i\}$, $\{x_i\}$ and $\{y_i\}$ be such that

$$p \prec d(x_i, y_i) \prec e,$$

$$d(f^{n_i}(x_i), f^{n_i}(y_i)) \prec d(x_i, y_i),$$

and

$$d(f^{n_i+j}(x_i), f^{n_i+j}(y_i)) \succeq d(x_i, y_i),$$

for every $0 < j \leq i$. We may assume that

$$\lim_{i \rightarrow \infty} f^{n_i}(x_i) = w \quad \text{and} \quad \lim_{i \rightarrow \infty} f^{n_i}(y_i) = z,$$

for some $z, w \in X$. On the other hand, since P is a closed set, then

$$d(f^{n_i+k}(x_i), f^{n_i+k}(y_i)) \succeq d(x_i, y_i)$$

and

$$d(x_i, y_i) \succeq p$$

imply that

$$\lim_{i \rightarrow \infty} d(f^{n_i+k}(x_i), f^{n_i+k}(y_i)) \succeq \lim_{i \rightarrow \infty} d(x_i, y_i),$$

and $\lim_{i \rightarrow \infty} d(x_i, y_i) \succeq p$, for every $k \in \mathbb{N}$. So by (ii) of Lemma 1.1,

$$f^k(w), f^k(z) = \lim_{i \rightarrow \infty} d(f^{n_i+k}(x_i), f^{n_i+k}(y_i)) \succeq \lim_{i \rightarrow \infty} d(x_i, y_i) \succeq p,$$

for every $k \in \mathbb{N}$. Hence w and z are distal under f . But

$$d(w, z) = \lim_{i \rightarrow \infty} d(f^{n_i}(x_i), f^{n_i}(y_i)) \preceq e \prec c.$$

This contradicts Theorem 2.6 and the proof is complete. \square

Corollary 2.11. *Let (X, d) be a compact cone metric space, where P is a normal cone with normal constant K , and f is a continuous self-map on X such that there exists $n \in \mathbb{N}$ such that*

$$d(f^n(x), f^n(y)) \prec d(x, y),$$

for every $x, y \in X$ with $\theta \prec d(x, y)$. Let $c \gg 0$. Then, there exists $n_0 \in \mathbb{N}$ such that $d(x, y) \succ c$ and $d(f^k(x), f^k(y)) \prec d(x, y)$ imply $d(f^{k+j}(x), f^{k+j}(y)) \prec d(x, y)$, for some $0 < j \leq n_0$.

Corollary 2.12. *Let (X, d) be a compact cone metric space, where P is a normal cone with normal constant K , and f is a continuous self-map on X which satisfies the following condition:*

$$\exists c \gg \theta \ni \forall x, y \in X [\theta \prec d(x, y) \ll c \Rightarrow d(f^n(x), f^n(y)) \prec d(x, y)],$$

for some $n \in \mathbb{N}$. Then, for every $x, y, z \in X$ with $\theta \prec d(x, y) \prec c$ and $\theta \prec d(x, z) \prec c$, there exists $n_0 \in \mathbb{N}$ such that

$$d(f^{n_0}(x), f^{n_0}(y)) \prec d(x, y) \text{ and } d(f^{n_0}(x), f^{n_0}(z)) \prec d(x, z).$$

Proof. Suppose that $\{n_i\} \subseteq \mathbb{N}$ is such that $d(f^{n_i}(x), f^{n_i}(y)) \prec d(x, y)$, for every $i \in \mathbb{N}$. By Theorem 2.10, the set $\{n_{i+1} - n_i\}$ is bounded-above. By contradiction, suppose that $d(f^{n_i}(x), f^{n_i}(z)) \succeq d(x, z)$, for every $i \in \mathbb{N}$. Since $\{n_{i+1} - n_i\}$ is bounded, this implies that x and z are distal which contradicts Theorem 2.6. So the result follows and the proof is complete. \square

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P. Raja
 Department of Mathematics,
 Shahid Beheshti University,
 Evin, P. O. Box 1983963113,
 Tehran, Iran.
 p.raja@sbu.ac.ir