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A NOTE ON THE SOLUTIONS TO A TRANSCENDENTAL EQUATION

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Abstract. We show that the transcendental equation $\cos z + H\dfrac{\sin z}{z} = 0$, with H a real number has only real solutions, which are countably many, simple, and there exists a positive H_0 such that the positive solutions satisfy

$$z_n(H) = (n - \frac{1}{2})\pi + \frac{H}{(n-\frac{1}{2})\pi} - \frac{H^2}{2(n-\frac{1}{2})^3\pi^3} + \mathcal{O}(\frac{1}{n^3}), \text{ as } n \to \infty,$$

for each $H \geq -H_0$, and

$$z_n(H) = (n + \frac{1}{2})\pi + \frac{H}{(n+\frac{1}{2})\pi} - \frac{H^2}{2(n+\frac{1}{2})^3\pi^3} + \mathcal{O}(\frac{1}{n^3}), \text{ as } n \to \infty.$$

if $H < -H_0$.

1. The Zeros of the Transcendental Equation $\cos z + H\dfrac{\sin z}{z} = 0$.

Throughout this paper, H is considered a real number. We investigate the complex solutions of the equation

$$\cos z + H\frac{\sin z}{z} = 0. \tag{1.1}$$

We show first that $z \in \mathbb{C}$ is a solution to (1.1) if and only if $\lambda = z^2$ satisfies the eigenvalue problem

$$\begin{cases} -u''(x) = \lambda u(x), \ x \in (0,1) \\ u(0) = 0 = u'(1) + Hu(1). \end{cases} \tag{1.2}$$

This will help in showing that the solutions to (1.1) are real. One can see easily that if $z \in \mathbb{C}$ satisfies (1.1), then $\lambda = z^2$ is an eigenvalue of (1.2) with the associated eigenfunction $u(x) = \dfrac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}$. Conversely, if λ is an eigenvalue of the boundary value problem (1.2), then there exists a non-identically zero function $u(x)$ that satisfies (1.2). It follows that $u'(0) \neq 0$, as otherwise the ODE and the boundary condition $u(0) = 0$ would imply that $u(x) = 0$, for all $x \in [0,1]$ (the initial value problem $-u''(x) = \lambda u(x)$, $u(0) = 0 = u'(0)$ has only one solution, namely the zero solution). Hence, $v(x) := \dfrac{u(x)}{u'(0)}$ is well-defined. It follows that $v'(0) = 1$, and $v(0) = 0$, $-v''(x) = \lambda v(x)$, for $x \in (0,1)$, due to the properties of $u(x)$. This means that $v(x) = \dfrac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}$, because $\dfrac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}$ is the only solution to this initial value

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problem. Thus, $\cos(\sqrt{\lambda})+H\dfrac{\sin(\sqrt{\lambda})}{\sqrt{\lambda}}=v'(1)+Hv(1)=\dfrac{u'(1)}{u'(0)}+H\dfrac{u(1)}{u'(0)}=0$, since $u(x)$ satisfies (1.2). We showed this way that $z=\sqrt{\lambda}$ is a solution to (1.1).

Next we show that the eigenvalues λ's of (1.2) are real valued. Let $\lambda \in \mathbb{C}$ be an eigenvalue of (1.2) with $u(x)$ an associated eigenfunction. Taking the complex conjugate in (1.2), multiplying the ODE just obtained by $u(x)$, multiplying the ODE of (1.2) by $\bar{u}(x)$ (the complex conjugate of $u(x)$), and subtracting the two new equations one from another we arrive at:

$$\frac{d}{dx}(\bar{u}u'-\bar{u}'u)(x)=(\bar{\lambda}-\lambda)\bar{u}(x)u(x),\ x\in(0,1). \tag{1.3}$$

Integrating (1.3) from $x=0$ to $x=1$, and using the boundary conditions that u and $\bar{u}$ satisfy and the fact that $H\in\mathbb{R}$ we get:

$$0=(\bar{\lambda}-\lambda)\int_0^1|u(x)|^2dx,$$

from which we readily have $\bar{\lambda}=\lambda$, since $u\neq 0$ as an eigenfunction. Thus, $\lambda\in\mathbb{R}$.

Due to the previous two paragraphs, the zeros of $f(z):=\cos z+H\dfrac{\sin z}{z}$ are either real or pure imaginary ($z=\pm\sqrt{\lambda}$ where λ is an eigenvalue of (1.2), so λ is real). We show that f cannot have zeros that are pure imaginary. Suppose that $z=iy$, with $y\in\mathbb{R}-\{0\}$, is a zero of f. Then $f(iy)=0$, which is equivalent to $\left(e^{-y}+e^{y}\right)+H\dfrac{e^{-y}-e^{y}}{-y}=0$, by means of the known formulas $\cos z=\dfrac{e^{iz}+e^{-iz}}{2}$ and $\sin z=\dfrac{e^{iz}-e^{-iz}}{2i}$. So $y\neq\pm H$, since otherwise the second equation for y would give $2e^{\pm y}=0$. Hence, the second equation for y is further equivalent to $e^{2y}=\frac{H-y}{H+y}$. Clearly, the graphs of e^{2y} and $\frac{H-y}{H+y}$ intersect only at $y=0$, but our y was assumed nonzero. This is the needed contradiction.

We observe that if $z\neq 0$ is a zero of f, then $\sin z\neq 0$ (otherwise $\cos z=-H\frac{\sin z}{z}=0$, contradicting $\sin^2 z+\cos^2 z=1$). It follows next that $\cot z=-\frac{H}{z}$. Conversely, if $z\neq 0$ is such that $\cot z=-\frac{H}{z}$, then $f(z)=0$.

Hence, by the discussion in the previous two paragraphs, the nonzero zeros of f are the real-valued solutions of $\cot z=-\frac{H}{z}$, i.e. they are the z-coordinate of the intersection points of the curves $w=\cot z$ and $w=-\frac{H}{z}$ of real variable z (for $H\neq 0$). Graphing these two functions, we observe that their intersection points are countably many, they form pairs symmetric with respect to the origin of the coordinate axes, and the n'th point of intersection located in the right half plane has the z-coordinate near $(n+\frac{1}{2})\pi$, if $H<-H_0$, and near $(n-\frac{1}{2})\pi$, if $H\geq -H_0$, for all $n\geq 1$. The number $-H_0$ is the smallest negative H such that

$$\lim_{z\to 0^+}\left(\cos z+H\frac{\sin z}{z}\right)\cdot\lim_{z\to\pi^-}\left(\cos z+H\frac{\sin z}{z}\right)<0. \tag{1.4}$$

In other words, $-H_0$ is the smallest negative H such that the curves $w=\cot z$ and $w=-\frac{H}{z}$ of real variable z have their first intersection point of the right half plane located in the first vertical stripe $0<z<\pi$. An asymptotic formula for z_n can be found in [**2**, Problem 4.2, page 171]. However, we sharpen it in Theorem 1. Thus, we showed that the nonzero zeros of f are countably many.

Now we argue that the nonzero zeros of f are simple. If $H = 0$ then $f(z) = \cos z$, and thus it has the zeros $\pm\left(n - \frac{1}{2}\right)\pi$, which clearly are simple. So we assume further that $H \neq 0$. Let $\tilde{z} \neq 0$ be a zero of f. Then $\tilde{z} \in \mathbb{R} - \{0\}$ (as argued above), $f(\tilde{z}) = 0$ and

$$f'(\tilde{z}) = -\sin\tilde{z} + H\frac{\cos\tilde{z}}{\tilde{z}} - H\frac{\sin\tilde{z}}{\tilde{z}^2} = \cos\tilde{z}\cdot\left(\frac{\tilde{z}}{H} + \frac{H+1}{\tilde{z}}\right),$$

by substituting $-\sin\tilde{z}$ and $-H\frac{\sin\tilde{z}}{\tilde{z}}$ from $f(\tilde{z}) = 0$ into the formula of $f'(\tilde{z})$. Since $H \neq 0$, we have that $\cos\tilde{z} \neq 0$, because otherwise it would imply that $\cos\tilde{z} = 0 = \sin\tilde{z}$ (see $f(\tilde{z}) = 0$), contradicting $\sin^2\tilde{z} + \cos^2\tilde{z} = 1$. If $H \notin (-1, 0]$, then $\frac{\tilde{z}}{H} + \frac{H+1}{\tilde{z}} \neq 0$, again because $\tilde{z} \in \mathbb{R} - \{0\}$. If $H \in (-1, 0)$, then $\frac{\tilde{z}}{H} + \frac{H+1}{\tilde{z}} = 0$ *if only if* $\tilde{z} = \pm\sqrt{-H(H+1)}$, which due to $f(\tilde{z}) = 0$ would imply that $\sqrt{-H(H+1)}\cdot\cos\sqrt{-H(H+1)} = -H\cdot\sin\sqrt{-H(H+1)}$. Graphing the two functions of H on $(-1, 0)$ one sees no intersection points. Actually, the graph of $\sqrt{-H(H+1)}\cdot\cos\sqrt{-H(H+1)}$ stays above the graph of $-H\cdot\sin\sqrt{-H(H+1)}$ for $H \in (-1, 0)$. See Figure 1. Hence, $\frac{\tilde{z}}{H} + \frac{H+1}{\tilde{z}} \neq 0$, if $H \in (-1, 0)$. Thus $f'(\tilde{z}) \neq 0$, which means that $\tilde{z}$ is a simple zero of f.

If $H = -1$, then

$$f(z) = \cos z - \frac{\sin z}{z} = \left(\frac{1}{3!} - \frac{1}{2!}\right)z^2 + \left(\frac{1}{4!} - \frac{1}{5!}\right)z^4 + \ldots,$$

by the power series of $\cos z$ and $\sin z$ (see [**1**, page 38]). Hence $z = 0$ is a double zero of f, in this case.

Theorem 1. *For a fixed $H \in \mathbb{R}$, let $z_n(H)$ be the n'th positive zero of $f(z) := \cos z + H\dfrac{\sin z}{z}$, for $n \geq 1$. Then for each $H \geq -H_0$, and respectively for each $H < -H_0$:*

$$z_n(H) = (n - \frac{1}{2})\pi + \frac{H}{(n-\frac{1}{2})\pi} - \frac{H^2}{2(n-\frac{1}{2})^3\pi^3} + \mathcal{O}(\frac{1}{n^3}), \text{ as } n \to \infty, \tag{1.5}$$

$$z_n(H) = (n + \frac{1}{2})\pi + \frac{H}{(n+\frac{1}{2})\pi} - \frac{H^2}{2(n+\frac{1}{2})^3\pi^3} + \mathcal{O}(\frac{1}{n^3}), \text{ as } n \to \infty. \tag{1.6}$$

Proof: Let $G(H, z) := \cos z + H\frac{\sin z}{z}$, for $H, z \in \mathbb{R}$. This function is continuously differentiable with respect to both H and z. Fix $n \geq 1$ and let $z_n^* := \left(n - \frac{1}{2}\right)\pi$, and $z_n^{**} := \left((n+1) - \frac{1}{2}\right)\pi = \left(n + \frac{1}{2}\right)\pi$. Then:

$$\begin{cases} G(0, z_n^*) = 0, \\ \frac{\partial G}{\partial z}(0, z_n^*) = -\sin z_n^* = (-1)^n \neq 0. \end{cases} \tag{1.7}$$

It follows by the Implicit Function Theorem that there exists a small neighborhood $[-\delta_1, \delta_1]$ of $H = 0$ and a unique continuously differentiable function $Z_n : [-\delta_1, \delta_1] \to \mathbb{R}$ such that:

$$\begin{cases} Z_n(0) = z_n^*, \\ G(H, Z_n(H)) = 0, \text{ for all } H \in [-\delta_1, \delta_1]. \end{cases} \tag{1.8}$$

We shall prove that the function Z_n above extends to a continuously differentiable function on all of $[-H_0, +\infty)$, has the property $G(H, Z_n(H)) = 0$, for all $H \in [-H_0, +\infty)$, and $Z_n(H)$ equals the right hand side of (1.5), for each $H \in [-H_0, +\infty)$. If these are established, then for an arbitrary but fixed $H \in$

$[-H_0, +\infty)$, $Z_n(H)$ will be a real-valued zero of $f(z) := \cos z + H\frac{\sin z}{z}$ in the neighborhood of z_n^*, for n large. So, $\sin Z_n(H) \approx \sin z_n^* = \pm 1 \neq 0$. This will further mean that $Z_n(H)$ is the real solution of $\cot z = -\frac{H}{z}$, which is closest to z_n^*. Graphing the functions $\cot z$ and $-\frac{H}{z}$ of real variable z (Remember $H \geq -H_0$!), one observes that their n'th intersection point located in the right half plane has the abscissa z closest to z_n^*. Thus, $Z_n(H)$ will be the n'th positive solution of $\cot z = -\frac{H}{z}$, and so the n'th positive zero of $f(z)$ for the chosen H. Therefore $Z_n(H) = z_n(H)$, by our numbering of the zeros of $f(z)$, from which the asymptotics formula (1.5) of $z_n(H)$ follows.

Taking the derivative with respect to H in the second identity of (1.8) and using the definition of $G(H, z)$ we get:

$$Z_n'(H) = \frac{Z_n(H)\sin Z_n(H)}{Z_n(H)^2 \sin Z_n(H) - HZ_n(H)\cos Z_n(H) + H\sin Z_n(H)}, \text{ for } H \in [-\delta_1, \delta_1]. \tag{1.9}$$

Then for any $H \in [-\delta_1, \delta_1]$ the following calculations hold:

$$\begin{aligned} Z_n(H) - z_n^* &= \int_0^H Z_n'(H')dH' \\ &= \int_0^H \frac{Z_n(H')\sin Z_n(H')}{Z_n(H')^2 \sin Z_n(H') - H'Z_n(H')\cos Z_n(H') + H'\sin Z_n(H')} dH' \\ &\approx \int_0^H \frac{z_n^*}{(z_n^*)^2 + H'} dH' = z_n^* \ln\left(1 + \frac{H}{(z_n^*)^2}\right). \end{aligned} \tag{1.10}$$

In (1.10) we used $Z_n(H') \approx Z_n(0) = z_n^*$, possible because Z_n is a continuous function and H' is close to 0, by being between 0 and $H \in [-\delta_1, \delta_1]$. Thus $\cos Z_n(H') \approx 0$, and $\sin Z_n(H') \approx (-1)^{n+1}$. Using the Taylor series expansion of $\ln(1+x)$ about $x = 0$ (note that $\dfrac{H}{(z_n^*)^2} = \mathcal{O}(\dfrac{1}{n^2})$), formula (1.10) yields for each $H \in [-\delta_1, \delta_1]$:

$$Z_n(H) = z_n^* + \frac{H}{z_n^*} - \frac{H^2}{2(z_n^*)^3} + \mathcal{O}(\frac{1}{n^3}), \text{ as } n \to \infty. \tag{1.11}$$

Thus we have the desired asymptotics of $Z_n(H)$, but only for $H \in [-\delta_1, \delta_1]$. We shall use (1.11) to achieve our goal stated above about $Z_n(H)$.

Let $\hat{H} := \delta_1$, and let $\hat{z}_n := Z_n(\hat{H})$. It follows from (1.11), after ignoring all terms $\mathcal{O}(\frac{1}{n^4})$ and lower, that:

$$\begin{cases} \hat{z}_n = z_n^* + \frac{\hat{H}}{z_n^*} + \mathcal{O}(\frac{1}{n^3}), \\ \sin \hat{z}_n = \sin z_n^* \cos\left(\frac{\hat{H}}{z_n^*} + \mathcal{O}(\frac{1}{n^3})\right) + \cos z_n^* \sin\left(\frac{\hat{H}}{z_n^*} + \mathcal{O}(\frac{1}{n^3})\right) = (-1)^{n+1}\left(1 - \frac{\hat{H}^2}{2(z_n^*)^2}\right), \\ \cos \hat{z}_n = \cos z_n^* \cos\left(\frac{\hat{H}}{z_n^*} + \mathcal{O}(\frac{1}{n^3})\right) - \sin z_n^* \sin\left(\frac{\hat{H}}{z_n^*} + \mathcal{O}(\frac{1}{n^3})\right) = (-1)^n\left(\frac{\hat{H}}{z_n^*} + \mathcal{O}(\frac{1}{n^3})\right), \\ \frac{1}{\hat{z}_n} = \frac{1}{z_n^* + \left(\frac{\hat{H}}{z_n^*} + \mathcal{O}(\frac{1}{n^3})\right)} = \frac{1}{z_n^*}\frac{1}{1 + \left(\frac{\hat{H}}{(z_n^*)^2} + \mathcal{O}(\frac{1}{n^4})\right)} = \frac{1}{z_n^*}\left(1 - \frac{\hat{H}}{(z_n^*)^2} + \mathcal{O}(\frac{1}{n^4})\right) = \frac{1}{z_n^*} + \mathcal{O}(\frac{1}{n^3}). \end{cases} \tag{1.12}$$

We used in (1.12) the Taylor series expansions about $x = 0$

$$\begin{cases} \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots, \\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots, \\ \frac{1}{1+x} = 1 - x + x^2 - \dots. \end{cases}$$

It follows from (1.8), and the definition of $G(H, z)$ together with (1.12) that:

$$G(\hat{H}, \hat{z}_n) = G(\hat{H}, Z_n(\hat{H})) = 0, \tag{1.13}$$

$$\begin{aligned} \frac{\partial G}{\partial z}(\hat{H}, \hat{z}_n) &= (-1)^n \left(1 - \frac{\hat{H}^2}{2(z_n^*)^2}\right) + \hat{H}(-1)^n \left(\frac{\hat{H}}{z_n^*} + \mathcal{O}(\frac{1}{n^3})\right) \left(\frac{1}{z_n^*} + \mathcal{O}(\frac{1}{n^3})\right) \\ &\quad + \hat{H}(-1)^n \left(1 - \frac{\hat{H}^2}{2(z_n^*)^2}\right) \left(\frac{1}{(z_n^*)^2} + \mathcal{O}(\frac{1}{n^4})\right) \\ &= (-1)^n \left(1 + \frac{\hat{H}^2 + 2\hat{H}}{2(z_n^*)^2}\right) + \mathcal{O}(\frac{1}{n^4}) \\ &\neq 0, \text{ since } \tfrac{\hat{H}^2+2\hat{H}}{2(z_n^*)^2} = \mathcal{O}\left(\tfrac{1}{n^2}\right). \end{aligned} \tag{1.14}$$

The Implicit Function Theorem, (1.13), (1.14) imply the existence of a neighborhood $[\hat{H} - \delta_2, \hat{H} + \delta_2]$ of $H = \hat{H}$ and of a unique continuously differentiable function $\hat{Z}_n : [\hat{H} - \delta_2, \hat{H} + \delta_2] \to \mathbb{R}$ such that:

$$\begin{cases} \hat{Z}_n(\hat{H}) = \hat{z}_n, \\ G(H, \hat{Z}_n(H)) = 0, \text{ for all } H \in [\hat{H} - \delta_2, \hat{H} + \delta_2] \subset (0, \infty). \end{cases} \tag{1.15}$$

Note that we can take $\delta_2 < \hat{H} = \delta_1$, by shrinking the interval around $\hat{H}$, because clearly $G(H, \hat{Z}_n(H)) = 0$ would hold on any subinterval of the interval asserted by the implicit function theorem. If we show that

$$\hat{Z}_n(H) = z_n^* + \frac{H}{z_n^*} - \frac{H^2}{2(z_n^*)^3} + \mathcal{O}(\frac{1}{n^3}), \text{ for each } H \in [\hat{H} - \delta_2, \hat{H} + \delta_2], \tag{1.16}$$

then (1.16), (1.11), and the second identity in each of (1.15) and (1.8) will tell that for $H \in [-\delta_1, \delta_1] \cap [\hat{H} - \delta_2, \hat{H} + \delta_2]$, both $\hat{Z}_n(H)$ and $Z_n(H)$ are real-valued zeros of $f(z)$ and they are the closest to z_n^* such zeros. It follows by the same discussion as in the third paragraph of the proof of Theorem 1 that both $\hat{Z}_n(H)$ and $Z_n(H)$ are the n'th positive zero $z_n(H)$ of $f(z)$, and thus

$$\hat{Z}_n(H) = Z_n(H), \text{ for all } H \in [-\delta_1, \delta_1] \cap [\hat{H} - \delta_2, \hat{H} + \delta_2].$$

This means that the function Z_n could be extended from $[-\delta_1, \delta_1]$ to $[-\delta_1, \delta_1] \cup [\hat{H} - \delta_2, \hat{H} + \delta_2] = [-\delta_1, \delta_1 + \delta_2]$ by preserving the analyticity, the asymptotics, and the property $G(H, Z(H)) = 0$. The extension is performed by patching together the two functions Z_n and $\hat{Z}_n$.

Hence, we can repeat the above steps with $[-\delta_1, \delta_1 + \delta_2]$ in place of $[-\delta_1, \delta_1]$, and $\hat{H}$ being the right end-point of this new interval. Note that doing this, $\hat{H}$ would increase. Nevertheless, (1.14) remains true, because when $|\hat{H}|$ is large (i.e comparable to z_n^*), $\hat{H}^2$ dominates in $\hat{H}^2 + 2\hat{H}$, so $\dfrac{\hat{H}^2 + 2\hat{H}}{2(z_n^*)^2}$ stays positive, keeping

$(-1)^n \left(1 + \dfrac{\hat{H}^2 + 2\hat{H}}{2(z_n^*)^2}\right) + \mathcal{O}(\dfrac{1}{n^4})$ away from zero. Having (1.14) true guarantees the existence of a neighborhood of the right end-point $\hat{H}$ of the current interval where z can be explicitly found in terms of H, via the Implicit Function Theorem. That means that we can keep enlarging the interval. So, the function Z_n can be extended to $[-\delta_1, +\infty)$ by preserving the three properties above.

Now we proceed in proving (1.16). Differentiating with respect to H in the second identity of (1.15) and using the definition of $G(H,z)$ we obtain $\hat{Z}_n'(H)$ for all $H \in [\hat{H} - \delta_2, \hat{H} + \delta_2]$, which further gives:

$$\begin{aligned}
\hat{Z}_n(H) - \hat{z}_n &= \int_{\hat{H}}^{H} \hat{Z}_n'(H')dH' \\
&= \int_{\hat{H}}^{H} \frac{\sin \hat{Z}_n(H')}{\hat{Z}_n(H')\sin \hat{Z}_n(H') - H'\cos \hat{Z}_n(H') + H'\frac{\sin \hat{Z}_n(H')}{\hat{Z}_n(H')}} dH' \\
&= \int_{\hat{H}}^{H} \frac{\sin \hat{Z}_n(H')}{\hat{Z}_n(H')\sin \hat{Z}_n(H') - H'\cos \hat{Z}_n(H') - \cos \hat{Z}_n(H')} dH'. \quad (1.17)
\end{aligned}$$

To obtain the last equality in (1.17) we used the second identity of (1.15).

Due to the continuity of $\hat{Z}_n$, and because H' is between $\hat{H}$ and $H \in [\hat{H} - \delta_2, \hat{H} + \delta_2]$ we can approximate $\hat{Z}_n(H')$ by $\hat{Z}_n(\hat{H}) = \hat{z}_n$ in (1.17), and we can also use $\frac{\sin \hat{z}_n}{\hat{z}_n} = -\frac{\cos \hat{z}_n}{\hat{H}}$, which is due to the definition of $\hat{z}_n$ and the second identity of (1.8). Thus, (1.17) implies that for $H \in [\hat{H} - \delta_2, \hat{H} + \delta_2]$:

$$\begin{aligned}
\hat{Z}_n(H) - \hat{z}_n &\approx \int_{\hat{H}}^{H} \frac{\sin \hat{z}_n}{\hat{z}_n \sin \hat{z}_n - H'\cos \hat{z}_n - \cos \hat{z}_n} dH' \\
&= \int_{\hat{H}}^{H} \frac{\sin \hat{z}_n}{\hat{z}_n \sin \hat{z}_n + H'\left(\hat{H}\frac{\sin \hat{z}_n}{\hat{z}_n}\right) + \hat{H}\frac{\sin \hat{z}_n}{\hat{z}_n}} dH', \\
&\qquad (\sin \hat{z}_n \neq 0, \text{ as otherwise } \cos \hat{z}_n = -\hat{H}\tfrac{\sin \hat{z}_n}{\hat{z}_n} = 0) \\
&= \int_{\hat{H}}^{H} \frac{1}{\left(\hat{z}_n + \frac{\hat{H}}{\hat{z}_n}\right) + H'\left(\frac{\hat{H}}{\hat{z}_n}\right)} dH', \\
&\qquad (\text{dividing by } \sin \hat{z}_n \neq 0. \text{ Recall } \sin^2 \hat{z}_n + \cos^2 \hat{z}_n = 1.) \\
&= \frac{\hat{z}_n}{\hat{H}} \ln \left(\frac{\left(\hat{z}_n + \frac{\hat{H}}{\hat{z}_n}\right) + H\left(\frac{\hat{H}}{\hat{z}_n}\right)}{\left(\hat{z}_n + \frac{\hat{H}}{\hat{z}_n}\right) + \hat{H}\left(\frac{\hat{H}}{\hat{z}_n}\right)} \right) = \frac{\hat{z}_n}{\hat{H}} \ln \left(1 + \frac{(H - \hat{H})\left(\frac{\hat{H}}{\hat{z}_n}\right)}{\left(\hat{z}_n + \frac{\hat{H}}{\hat{z}_n}\right) + \hat{H}\left(\frac{\hat{H}}{\hat{z}_n}\right)} \right) \\
&= \frac{\hat{z}_n}{\hat{H}} \ln \left(1 + \frac{\hat{H}(H - \hat{H})}{\hat{z}_n} \cdot \frac{1}{\hat{z}_n + \frac{\hat{H} + \hat{H}^2}{\hat{z}_n}} \right) = \frac{\hat{z}_n}{\hat{H}} \ln \left(1 + \frac{\hat{H}(H - \hat{H})}{\hat{z}_n^2} \cdot \frac{1}{1 + \frac{\hat{H} + \hat{H}^2}{\hat{z}_n^2}} \right) \\
&= \frac{\hat{z}_n}{\hat{H}} \left[\frac{\hat{H}(H - \hat{H})}{\hat{z}_n^2} \frac{1}{1 + \frac{\hat{H} + \hat{H}^2}{\hat{z}_n^2}} - \frac{\hat{H}^2(H - \hat{H})^2}{2\hat{z}_n^4} \left(\frac{1}{1 + \frac{\hat{H} + \hat{H}^2}{\hat{z}_n^2}} \right)^2 + \dots \right]. \quad (1.18)
\end{aligned}$$

The last equality in (1.18) is due to the Taylor's series expansion $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ about $x = 0$, which is permitted because $\frac{\hat{H}(H-\hat{H})}{\hat{z}_n^2}\frac{1}{1+\frac{\hat{H}+\hat{H}^2}{\hat{z}_n^2}} = \mathcal{O}(\frac{1}{n^2})$, as we shall see next.

Using (1.11) we write

$$\hat{z}_n := Z_n(\hat{H}) = z_n^* + \frac{\hat{H}}{z_n^*} - \frac{\hat{H}^2}{2(z_n^*)^3} + \mathcal{O}(\frac{1}{n^3}), \tag{1.19}$$

from which we obtain:

$$\begin{aligned}
\frac{1}{\hat{z}_n} &= \frac{1}{z_n^*}\cdot\frac{1}{1+\left(\frac{\hat{H}}{(z_n^*)^2} - \frac{\hat{H}^2}{2(z_n^*)^4} + \mathcal{O}(\frac{1}{n^4})\right)} \\
&= \frac{1}{z_n^*}\left[1 - \left(\frac{\hat{H}}{(z_n^*)^2} - \frac{\hat{H}^2}{2(z_n^*)^4} + \mathcal{O}(\frac{1}{n^4})\right) + \left(\frac{\hat{H}}{(z_n^*)^2} - \frac{\hat{H}^2}{2(z_n^*)^4} + \mathcal{O}(\frac{1}{n^4})\right)^2 - \ldots\right] \\
&= \frac{1}{z_n^*} - \frac{\hat{H}}{(z_n^*)^3} + \mathcal{O}(\frac{1}{n^5}).
\end{aligned} \tag{1.20}$$

In the second identity of (1.20) we used the Taylor series expansion $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$ about $x = 0$. From (1.20) we get further:

$$\frac{1}{(\hat{z}_n)^2} = \frac{1}{(z_n^*)^2} - \frac{2\hat{H}}{(z_n^*)^4} + \mathcal{O}(\frac{1}{n^6}) \tag{1.21}$$

$$\frac{1}{(\hat{z}_n)^3} = \frac{1}{(z_n^*)^3} + \mathcal{O}(\frac{1}{n^5}), \text{ by multiplying (1.20) with (1.21).} \tag{1.22}$$

Using again the Taylor's series expansion of $\frac{1}{1+x}$ about $x = 0$ and (1.21) we have:

$$\frac{1}{1+\frac{\hat{H}+\hat{H}^2}{\hat{z}_n^2}} = 1 - \frac{\hat{H}+\hat{H}^2}{\hat{z}_n^2} + \left(\frac{\hat{H}+\hat{H}^2}{\hat{z}_n^2}\right)^2 - \ldots = 1 - \frac{\hat{H}+\hat{H}^2}{(z_n^*)^2} + \mathcal{O}(\frac{1}{n^4}), \tag{1.23}$$

and

$$\left(\frac{1}{1+\frac{\hat{H}+\hat{H}^2}{\hat{z}_n^2}}\right)^2 = 1 - \frac{2(\hat{H}+\hat{H}^2)}{(z_n^*)^2} + \mathcal{O}(\frac{1}{n^4}). \tag{1.24}$$

Inserting (1.20), (1.22), (1.23), (1.24) into (1.18) we obtain for each $H \in [\hat{H} - \delta_2, \hat{H} + \delta_2]$:

$$\begin{aligned}
\hat{Z}_n(H) - \hat{z}_n &= \frac{(H-\hat{H})}{\hat{z}_n} \frac{1}{1 + \frac{\hat{H}+\hat{H}^2}{\hat{z}_n^2}} - \frac{\hat{H}(H-\hat{H})^2}{2\hat{z}_n^3} \left(\frac{1}{1 + \frac{\hat{H}+\hat{H}^2}{\hat{z}_n^2}} \right)^2 + \mathcal{O}(\frac{1}{n^5}) \\
&= (H-\hat{H}) \left(\frac{1}{z_n^*} - \frac{\hat{H}}{(z_n^*)^3} + \mathcal{O}(\frac{1}{n^5}) \right) \left(1 - \frac{\hat{H}+\hat{H}^2}{(z_n^*)^2} + \mathcal{O}(\frac{1}{n^4}) \right) \\
&\quad - \frac{\hat{H}(H-\hat{H})^2}{2} \left(\frac{1}{(z_n^*)^3} + \mathcal{O}(\frac{1}{n^5}) \right) \left(1 - \frac{2(\hat{H}+\hat{H}^2)}{(z_n^*)^2} + \mathcal{O}(\frac{1}{n^4}) \right) + \mathcal{O}(\frac{1}{n^5}) \\
&= \frac{H-\hat{H}}{z_n^*} - \frac{\hat{H}(H-\hat{H})}{(z_n^*)^3} - \frac{(H-\hat{H})(\hat{H}+\hat{H}^2)}{(z_n^*)^3} - \frac{\hat{H}(H-\hat{H})^2}{2(z_n^*)^3} + \mathcal{O}(\frac{1}{n^5}) \\
&= \frac{H-\hat{H}}{z_n^*} - \frac{\hat{H}(H-\hat{H})(H+\hat{H}+4)}{2(z_n^*)^3} + \mathcal{O}(\frac{1}{n^5}). \qquad (1.25)
\end{aligned}$$

Finally, (1.25) and (1.19) yield for each $H \in [\hat{H} - \delta_2, \hat{H} + \delta_2]$:

$$\begin{aligned}
\hat{Z}_n(H) &= z_n^* + \frac{H}{z_n^*} - \frac{H^2}{2(z_n^*)^3} + \frac{(H^2 - \hat{H}^2) - \hat{H}(H-\hat{H})(H+\hat{H}+4)}{2(z_n^*)^3} + \mathcal{O}(\frac{1}{n^3}) \\
&= z_n^* + \frac{H}{z_n^*} - \frac{H^2}{2(z_n^*)^3} + \frac{(H-\hat{H})(H - H\hat{H} - \hat{H}^2 - 3\hat{H})}{2(z_n^*)^3} + \mathcal{O}(\frac{1}{n^3}), \qquad (1.26)
\end{aligned}$$

which is (1.16), since $H \in [\hat{H} - \delta_2, \hat{H} + \delta_2]$ can be approximated by $\hat{H}$ and so

$$\frac{(H-\hat{H})(H - H\hat{H} - \hat{H}^2 - 3\hat{H})}{2(z_n^*)^3} \approx -\frac{(H-\hat{H})(\hat{H}^2+\hat{H})}{(z_n^*)^3} \approx -\frac{\delta_2 \cdot \hat{H}^2}{n^3} = \mathcal{O}(\frac{1}{n^3}),$$

because we are allowed to take $\delta_2 \leq \dfrac{1}{\hat{H}^2}$ by shrinking the interval $[\hat{H} - \delta_2, \hat{H} + \delta_2]$ around $\hat{H}$, if needed. Note that $\dfrac{1}{\hat{H}^2}$ decreases (thus giving a smaller quantity δ_2), because, as mentioned in the paragraph preceding the paragraph of (1.17), $|\hat{H}|$ increases.

By the same reasoning we showed that Z_n could be extended from $[-\delta_1, \delta_1]$ to $[-\delta_1, \infty)$ we can show that Z_n can be extended to the left of $-\delta_1$. Note that if we take $\hat{H} := -\delta_1$, then $\hat{H}^2 + 2\hat{H} < 0$, because $-\delta_1$ is a small negative number so it falls in $(-2, 0)$. Nevertheless, $(-1)^n \left(1 + \frac{\hat{H}^2+2\hat{H}}{2(z_n^*)^2}\right) + \mathcal{O}(\frac{1}{n^4}) \neq 0$, because $\frac{\hat{H}^2+2\hat{H}}{2(z_n^*)^2} = \mathcal{O}\left(\frac{1}{n^2}\right)$, even if it is negative. So (1.14) holds, making possible the applicability of the Implicit Function Theorem.

Let $-H_0$ be the furtherest margin to the left the function Z_n could be extended to. Note that $-H_0 \neq -\infty$, as otherwise $G(H, Z_n(H)) = 0$, for $H \in (-\infty, +\infty)$ will hold, which together with the asymptotics (1.5) of $Z_n(H)$ would imply that the curves $w = \cot z$ and $w = -\frac{H}{z}$ of real variable z will intersect in the right half plane at (z, w) with $z = Z_n(H) \approx z_n^*$. That would mean that for each $H \in (-\infty, +\infty)$, the two curves will intersect in the right half plane in each vertical stripe $0 < z < \pi$, $\pi < z < 2\pi$, $2\pi < z < 3\pi$, etc, as z_1^*, z_2^*, z_3^*, etc are the midpoints of these intervals. But this is not true because the graphical illustration of the curves $w = \cot z$ and $w = -\frac{H}{z}$ reveals that, for H sufficiently negative, the first intersection point of the two curves in the right half plane is in the vertical stripe $\pi < z < 2\pi$.

Next, take $\hat{H}$ slightly smaller than $-H_0$, and let $\hat{z}_n$ be the abscissa z of the n'th intersection point in the right half plane of the curves $w = \cot z$ and $w = -\frac{\hat{H}}{z}$.

Hence, $\hat{z}_n \approx z_n^{**}$ and $G(\hat{H}, \hat{z}_n) = 0$, which further give:

$$\begin{aligned}\frac{\partial G}{\partial z}(\hat{H}, \hat{z}_n) &= -\sin \hat{z}_n + \hat{H}\frac{\cos \hat{z}_n}{\hat{z}_n} - \hat{H}\frac{\sin \hat{z}_n}{\hat{z}_n^2} = -\sin \hat{z}_n + \hat{H}\frac{\cos \hat{z}_n}{\hat{z}_n} + \frac{\cos \hat{z}_n}{\hat{z}_n} \\ &\approx -\sin z_n^{**} + (\hat{H}+1)\frac{\cos z_n^{**}}{z_n^{**}} = -(-1)^n \neq 0. \qquad (1.27)\end{aligned}$$

Therefore, the Implicit Function Theorem can be applied at this point $(\hat{H}, \hat{z}_n)$: there exists a small interval $[\hat{H} - \tilde{\delta}_1, \hat{H} + \tilde{\delta}_1]$, and a continuously differentiable function $\tilde{Z}_n : [\hat{H} - \tilde{\delta}_1, \hat{H} + \tilde{\delta}_1] \to \mathbb{R}$, such that

$$\begin{cases} \tilde{Z}_n(\hat{H}) = \hat{z}_n \\ G(H, \tilde{Z}_n(H)) = 0, \text{ for all } H \in [\hat{H} - \tilde{\delta}_1, \hat{H} + \tilde{\delta}_1]. \end{cases}$$

By shrinking the interval around $\hat{H}$, we can assume that $[\hat{H} - \tilde{\delta}_1, \hat{H} + \tilde{\delta}_1] \subset (-\infty, -H_0)$. From here we can continue with similar arguments as we presented for the function Z_n, and (1.6) will follow. $\square$

We illustrate the asymptotic formulas (1.5), (1.6) when $H = -10; -0.8; \sqrt{3}; 13$. For a fixed $H \in \mathbb{R}$, the calculations of the positive zeros $z_n(H)$ of $f(z)$, which are the positives $Z_n(H)$ such that $G(H, Z_n(H)) = 0$, were performed using the MATLAB built-in function *fzero* with the specified searching intervals being $[0+\varepsilon, \pi-\varepsilon]$, $[\pi+\varepsilon, 2\pi-\varepsilon]$, etc if H satisfies (1.4), and $[\pi+\varepsilon, 2\pi-\varepsilon]$, $[2\pi+\varepsilon, 3\pi-\varepsilon]$, etc if H is such that the inequality in (1.4) is reversed. We took $\varepsilon = 0.01$ for the numerical experiments. The panels of Figure 2 confirm the order $\mathcal{O}\left(\frac{1}{n^3}\right)$ of the asymptotic formulas (1.5), (1.6). Through numerical experiments, we got $-H_0 \approx -1$.

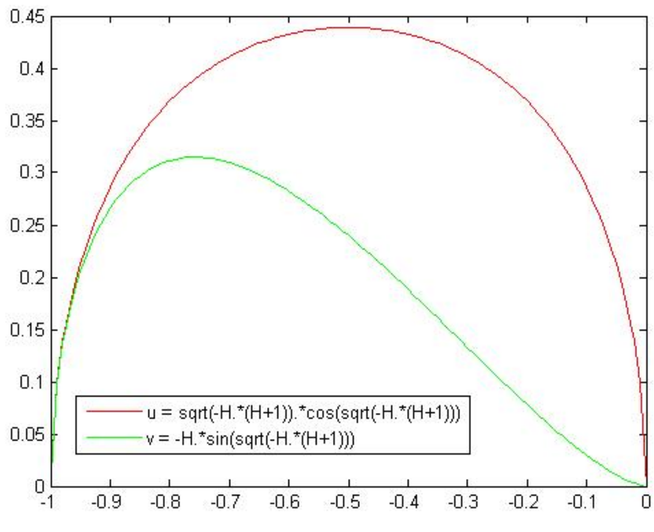

FIGURE 1. The graphs of $u = \sqrt{-H(H+1)} \cdot \cos\sqrt{-H(H+1)}$ and of $v = -H \cdot \sin\sqrt{-H(H+1)}$.

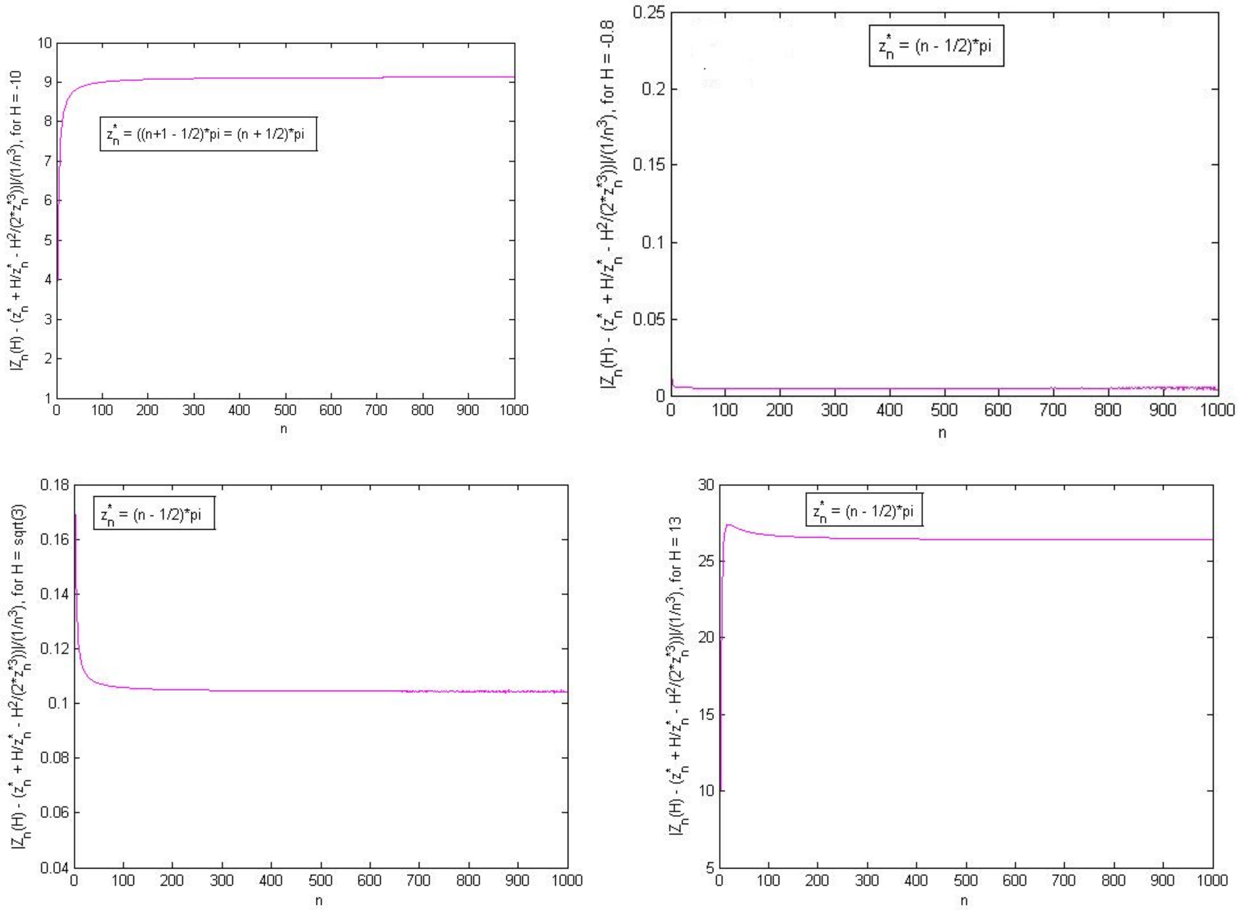

FIGURE 2. Illustration of (1.5), (1.6) for $H = -10$, $H = -0.8$, $H = \sqrt{3}$, $H = 13$ (top-bottom, left-right). The ratio $\frac{|Z_n(H) - \left((n \mp \frac{1}{2})\pi + \frac{H}{(n \mp \frac{1}{2})\pi} - \frac{H^2}{2(n \mp \frac{1}{2})^3 \pi^3}\right)|}{1/n^3}$ flattens out as $n \to \infty$, as predicted by these formulas. Numerically, we found $-H_0 \approx -1$.

References

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