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A DIFFERENTIAL EQUATION FOR A CELL GROWTH MODEL

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Abstract. We give solutions to the first order differential equation

$$y'(x) + c(x)y(x) = g(x)^{-1} \sum_{i=1}^{I} p_i b\left(\alpha_i x\right) y\left(\alpha_i x\right).$$

In the cell growth model, $y(x)$ is the probability density function for the size of a cell, $b(x)$ is the rate at which a cell of size x divides and creates α_i new cells of size x/α_i with probability p_i, $c(x)$ is a function determined by $b(x)$ and the growth rate of a cell, $g(x)$.

1. Introduction

This paper extends results of Hall and Wake (1989, 1990) and of van-Brunt and Vlieg-Hulstman (2010a, 2010b). Their cell growth model gives the partial differential equation (pde)

$$\partial_t N(x,t) + \partial_x \left[g(x,t)N(x,t)\right] + b(x,t)N(x,t) = \alpha^2 b\left(\alpha x, t\right) N\left(\alpha x, t\right), \tag{1}$$

where $\partial_t = \partial/\partial t$, $N(x,t)$ is the probability density function (pdf) of cells of size x as measured by say volume or mass at time t, $b(x,t)$ is the rate at which cells of size x are dividing and creating α new cells of size x/α at time t, and $g(x,t)$ is the growth rate of the cells at time t. So, their model holds for a fixed α, such as $\alpha = 2$. We give two methods that allow for the more realistic model where α is random. The first is simply to find a solution for fixed α then allow it to vary. The second is to allow α to take a fixed number of values, say

$$\alpha = \alpha_i \text{ with probability } p_i,\ i = 1, \ldots, I \text{ and } \sum_{i=1}^{I} p_i = 1.$$

$N(x,t)$ now becomes a random quantity with mean $n(x,t) = \mathbb{E}\left[N(x,t)\right]$ satisfying

$$\partial_t n(x,t) + \partial_x \left[g(x,t)n(x,t)\right] + b(x,t)n(x,t) = \mathcal{E}\left[b(x,t)n(x,t)\right], \tag{2}$$

where

$$\mathcal{E} f(x,t) = \mathbb{E}\left[\alpha^2 f(\alpha x, t)\right] = \sum_{i=1}^{I} p_i \alpha_i^2 f\left(\alpha_i x, t\right) \tag{3}$$

when this exists. Since (1) is a special case of (3), we now drop the use of $N(x,t)$. We consider solutions of (2) of the form

$$n(x,t) = y(x)N(t),\ b(x,t) = b(x)B(t),\ g(x,t) = g(x)G(t). \tag{4}$$

We shall see that solutions exist only for certain types of these functions.

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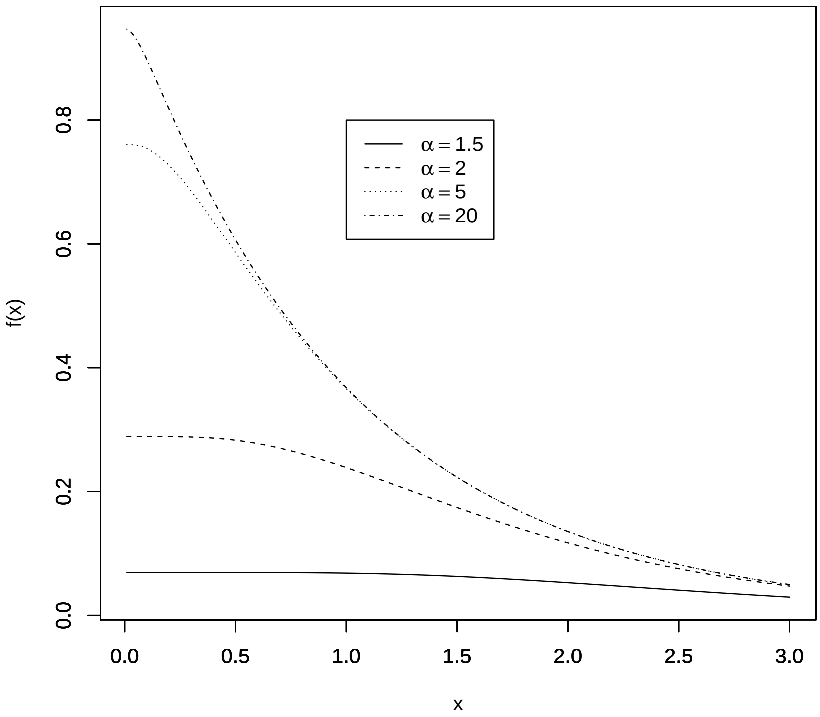

FIGURE 1. Plot of (6) versus x for $\alpha = 1.5, 2, 5, 20$ and $\eta = 1$.

In Section 2, we give a solution that allows $b(x)$ and $g(x)$ to be arbitrary, but only allows *one* of $N(t)$, $B(t)$ and $G(t)$ to be arbitrary. (2) reduces to the form

$$\partial_x y(x) + c(x)y(x) = r(x, y), \tag{5}$$

where

$$r(x, y) = g(x)^{-1} \, \mathcal{E}\left[b(x)y(x)\right].$$

This includes choices of b, g studied by earlier authors, (Examples 2.4 and 2.7), as well as a new choice (Example 2.8), when $I = 1$ based on Kato's fission/fusion function

$$f(x : \alpha, \eta) = \sum_{k=0}^{\infty} e^{-\alpha^k x} \eta^k / \prod_{j=1}^{k} \left(1 - \alpha^j\right) \tag{6}$$

when $y(0) = 0$. Section 2 also gives a new series solution when $y(0) \neq 0$. Examples 2.4 and 2.7 also look at the effect of allowing α to be random.

Figure 1 shows the behavior of (6) versus x for selected α and $\eta = 1$. We can see that $f(x : \alpha, \eta)$ is a monotonically decreasing function of x and a monotonically increasing function of α.

Section 3 defines a multivariate form of Kato's function to find solutions when $I \geq 1$ and either b, g are constants or $g(x) = g_0 x$, $b(x) = b_0 x^n$, $n > 0$, or $g(x) = g_0$, $b(x) = b_0 x^n$.

Section 4 gives a new type of solution: either $b(x)$ or $g(x)$ is arbitrary - but not both, while *two* of $N(t)$, $B(t)$ and $G(t)$ can be arbitrary. If we accept *both* types of restrictions, then any linear combination of the solutions of Sections 2 and 4 is possible.

Earlier papers found solutions when $y(0) = 0$. We shall also give solutions when $y(0) \neq 0$. Models with $\alpha > 1$ are sometimes referred to as *fission* models, and those with $\alpha < 1$ as *fusion* models. For any function $f(x)$, set

$$I_f(x) = \int_0^x f(z)dz \tag{7}$$

when this exists.

The problem of this paper was analyzed for continuous pdf's in Hall *et al.* (1991). A future work is to compare our results and ensure that if the pdf for division in the solutions in Hall *et al.* (1991) takes the form $p(\xi, \alpha) = \sum w_i \delta(\xi/\alpha_i - x)$ for suitable parameter values, where $\delta(\cdot)$ is the Dirac-delta function.

2. Two Solutions to the PDE (2)

Substituting the separation of variables solution (4), into (2), and then dividing by $y(x)N(t)B(t)$, we find

$$q(t) + \beta(t)\gamma(x) = \delta(x), \tag{1}$$

where

$$q(t) = [\partial_t N(t)] \,/\, [B(t)N(t)]\,,\ \beta(t) = G(t)/B(t),$$
$$\gamma(x) = \{\partial_x [g(x)y(x)]\} \,/y(x),\ \delta(x) = -b(x) + y(x)^{-1}\ \mathcal{E}\,[b(x)y(x)]$$

for $\mathcal{E}$ of (3). Multiplying (1) by $\partial_x\partial_t$ gives $\partial_t\beta(t) = 0$ or $\partial_x\gamma(x) = 0$. So, either (i): $\beta(t) = \beta$ is a constant, or (ii):

$$\gamma(x) = \gamma \text{ is a constant,} \tag{2}$$

a situation we deal with in Section 3. In this section, we suppose that $\beta(t) = \beta$ is a constant, and that $b(x)$, $g(x)$ are given. So, $q = q(t) = \delta(x) - \beta\gamma(x)$ is a constant, and $N(t)$, $G(t)$ are given using the notation of (7), by

$$N(t) = N(0)\exp\left[qI_B(t)\right],\ G(t) = \beta B(t). \tag{3}$$

So, surprisingly, only one of $N(t)$, $B(t)$ and $G(t)$ can be chosen arbitrarily.

Example 2.1. *If cell division or growth is constant over time, then $B(t) = B$, $G(t) = \beta B$, and by (3), $N(t) = N(0)\exp(qt)$ grows (if $q > 0$), or dies (if $q < 0$), exponentially with time.*

Most of the work is solving for $y(x)$, *the cell pdf*, at a given time: it satisfies

$$\overline{g}(x)\partial_x y(x) + c_0(x)y(x) = \mathcal{E}\left[b(x)y(x)\right],$$

where $c_0(x) = q + b(x) + \partial_x\overline{g}(x)$ and $\overline{g}(x) = \beta g(x)$. We now absorb β into $g(x)$, that is, we take $\beta = 1$. So, (5) holds with

$$c(x) = g(x)^{-1}c_0(x) = s(x) + g(x)^{-1}\partial_x g(x) = g(x)^{-1}\left[q + b(x) + \partial_x g(x)\right],$$
$$s(x) = g(x)^{-1}\left[q + b(x)\right]. \tag{4}$$

So, if $g(x_0) \neq 0$ at $x_0 = 0$ say, then

$$I_c(x) = \int_0^x c(z)dz = \log\ [g(x)/g(0)] + I_s(x),\ g(x)^{-1}e^{I_c(x)} = g(0)^{-1}e^{I_s(x)}, \quad (5)$$

$$Y(x) = e^{I_c(x)}y(x) \quad (6)$$

satisfies

$$\partial_x Y(x) = e^{I_c(x)} r(x,y),$$
$$e^{I_c(x)}y(x) - y(0) = Y(x) - Y(0) = \int_0^x e^{I_c(z)} r(z,y)dz.$$

Set $\mathbb{Z} = \{0,1,2,\ldots\}$. For $k \in \mathbb{Z}^I$, set

$$\alpha^k = \prod_{i=1}^{I} \alpha_i^{k_i}. \quad (7)$$

Let us try for a solution of (5) of the form

$$y(x) = \sum_{k=0}^{\infty} q_k(x),$$

that is,

$$Y(x) = \sum_{k=0}^{\infty} Q_k(x),\ Q_k(x) = e^{I_c(x)} q_k(x), \quad (8)$$

where the sum from zero to infinity is over $k = (k_1, \ldots, k_I) \geq \mathbf{0}$ in $\mathbb{Z}^I$. Let E_i be the ith unit vector in $\mathbb{Z}^I$. Set

$$Q_k(x) = 0 \text{ unless } k \geq \mathbf{0}. \quad (9)$$

Substituting into (6) gives

$$\begin{aligned}\sum_{k=\mathbf{0}}^{\infty} \partial_x Q_k(x) &= e^{I_c(x)} g(x)^{-1}\ \mathcal{E}\left[b(x)e^{-I_c(x)} \sum_{k=\mathbf{0}}^{\infty} Q_k(x)\right] \\ &= e^{I_c(x)} g(x)^{-1} \sum_{i=1}^{I} p_i \alpha_i^2 b\left(\alpha_i x\right) e^{-I_c(\alpha_i x)} \sum_{k=\mathbf{0}}^{\infty} Q_{k-E_i}\left(\alpha_i x\right)\end{aligned}$$

since

$$\sum_{k=\mathbf{0}}^{\infty} Q_k(x) = \sum_{k=\mathbf{0}}^{\infty} Q_{k-E_i}\left(\alpha_i x\right).$$

So, (8) is a solution if $\{Q_k\}$ satisfy the recurrence relation

$$\partial_x Q_k(x) = e^{I_c(x)} g(x)^{-1} \sum_{i=1}^{I} p_i \alpha_i^2 b\left(\alpha_i x\right) e^{-I_c(\alpha_i x)} Q_{k-E_i}\left(\alpha_i x\right),\ k \geq \mathbf{0}. \quad (10)$$

This is true whether or not each $\alpha_i > 1$. Integrating using (5) gives

$$Q_k(x) - Q_k(0) = \sum_{i=1}^{I} \mathcal{A}_i Q_{k-E_i}(x), \quad (11)$$

where

$$\begin{aligned}\mathcal{A}_i f(x) &= \int_0^x g(z)^{-1} p_i \alpha_i^2 e^{I_c(z)-I_c(\alpha_i z)} b(\alpha_i z) f(\alpha_i z)\, dz \\ &= g(0)^{-1} p_i \alpha_i^2 \int_0^x e^{I_s(z)-I_c(\alpha_i z)} b(\alpha_i z) f(\alpha_i z)\, dz\end{aligned}$$

by (5). Putting $k = \mathbf{0}$ gives $Q_{\mathbf{0}}(x) = Q_{\mathbf{0}}(0)$, that is, $Q_{\mathbf{0}}(x)$ is a constant.

First consider *the case* $I = 1$. Then, $Q_k(x) = Q_k(0) + \mathcal{A}_1 Q_{k-1}(x)$, giving

$$\begin{aligned}&Q_1(x) = Q_1(0) + Q_0(0)\ \mathcal{A}_1 1, \\ &Q_2(x) = Q_2(0) + \mathcal{A}_1 Q_1(x) = Q_2(0) + Q_1(0)\ \mathcal{A}_1 1 + Q_0(0)\ \mathcal{A}_1^2 1, \\ &Q_k(x) = \sum_{i=0}^{k} Q_{k-i}(0)\ \mathcal{A}_1^i 1.\end{aligned}$$

Summing (11) over $k \geq 1$ and putting $j = i + 1$ gives Theorem 2.1.

Theorem 2.1. *Consider the case $I = 1$. Then, the solution for $Y(x)$ can be expressed as*

$$Y(x) = Y(0) + \sum_{k=1}^{\infty} \sum_{j=1}^{k} Q_{k-j}(0) \mathcal{A}_1^j 1 = Y(0) \sum_{j=0}^{\infty} \mathcal{A}_1^j 1 = Y(0) \left(\mathbf{I} - \mathcal{A}_1\right)^{-1} 1$$

when

$$\sum_{i=0}^{\infty} \mathcal{A}_1^i 1$$

converges, where $\mathbf{I}$ *is the identity operator.*

The method giving Theorem 2.1 can be easily adapted if we choose $x_0 \neq 0$.

Example 2.2. *Suppose that $I = 1$, $b(x) = b$, $g(x) = g$ are constants. Then,*

$$\begin{aligned}&c(x) = s(x) = g^{-1}(q + b) = s \text{ say}, \\ &I_s(z/\alpha) - I_c(z) = Sz, \\ &\mathcal{A}_1 f(x) = \theta \int_0^{\alpha x} e^{-Sz} f(z) dz, \\ &\mathcal{A}_1 e^{-\lambda x}/\theta = (S + \lambda)^{-1} \left[1 - e^{-\alpha(S+\lambda)x}\right],\end{aligned}$$

where $S = s(\alpha^{-1} - 1)$ and $\theta = \alpha b/g$. If $s = 0$ then $\mathcal{A}_1^i 1 = (\theta x)^i \alpha^{\binom{i+1}{2}}$ so that

$$Y(x) = Y(0) \sum_{i=0}^{\infty} (\theta x)^i \alpha^{\binom{i+1}{2}}.$$

This is convergent if $|\alpha| < 1$ or if $\alpha = 1 < \theta x$. Suppose that $s \neq 0$. We give a solution in terms of θ,

$$\begin{aligned}&\lambda_0 = 0,\ \lambda_j = \alpha\left(S + \lambda_{j-1}\right) = S \sum_{k=1}^{j} \alpha^k = S\alpha\left(\alpha^j - 1\right)/(\alpha - 1), \\ &u_j = \left(S + \lambda_j\right)^{-1} = S^{-1}(\alpha - 1)/\left(\alpha^{j+1} - 1\right),\ j \geq 0.\end{aligned}$$

If $\alpha > 1$ and $S > 0$ (that is, $s < 0$,) then $u_j \downarrow 0$, $0 < \lambda_j \uparrow \infty$ as $j \uparrow \infty$. Then, we obtain an expansion of the form

$$(\mathcal{A}_1/\theta)^i \, 1 = \sum_{j=1}^{i} v_{ij} \left(1 - e^{-\lambda_j x}\right) = -\sum_{j=0}^{i} v_{ij} e^{-\lambda_j x}, \ i \geq 0,$$

where the coefficients v_{ij} are given by the recurrence relations

$$v_{00} = -1, \ v_{i+1,0} = \sum_{j=0}^{i} v_{ij} u_j, \ v_{i+1,j} = -v_{i,j-1} u_{j-1}$$

for $1 \leq j \leq i+1$. If $y(0) \neq 0$, this gives

$$Y(x)/Y(0) = 1 + \sum_{j=1}^{\infty} w_j \left(1 - e^{-\lambda_j x}\right),$$

where

$$w_j = \sum_{i=j}^{\infty} \theta^i v_{ij}.$$

Also

$$w_0 = -1 + \theta \sum_{j=0}^{\infty} u_j w_j, \ w_j = -\theta u_{j-1} w_{j-1}$$

for $j \geq 1$, giving

$$\begin{aligned} w_j &= (-\theta)^j U_j w_0, \ j \geq 1, \\ w_0 &= -(1+T)^{-1}, \\ Y(x)/Y(0) &= 1 + w_0 \sum_{j=1}^{\infty} (-\theta)^j U_j \left(1 - e^{-\lambda_j x}\right), \end{aligned}$$

where

$$U_j = \prod_{i=0}^{j-1} u_i, \ T = \sum_{j=1}^{\infty} (-\theta)^j U_j.$$

Alternatively, by the definition of the partial ordinary Bell polynomial, $\widehat{B}_{ji}(U)$, tabled on pages 307-308 of Comtet (1974),

$$T^i = \sum_{j=i}^{\infty} (-\theta)^j \widehat{B}_{ji}(U), \tag{12}$$

so that

$$w_0 = -\sum_{j=0}^{\infty} (-\theta)^j \gamma_j,$$

$$Y(x)/Y(0) = 1 - \sum_{i=1}^{\infty} (-\theta)^i \beta_i(x),$$

where

$$\gamma_j = \sum_{i=0}^{j} (-1)^i \widehat{B}_{ji}(U),$$
$$\beta_i(x) = \sum_{j=1}^{i} \gamma_{i-j} U_j \left(1 - e^{-\lambda_j x}\right).$$

This gives the solution $y(x) = e^{-sx}Y(x)$ for any (α, θ, s) for which this converges, that is

$$y(x) = Y(x)e^{-sx} = Y(0)\left\{e^{-sx} - \sum_{j=1}^{\infty} w_j \left[e^{-sx} - e^{-sx\left(\alpha^j - 2\right)}\right]\right\}$$

since $s + \lambda_j = -s(\alpha^j - 2)$. If $\alpha > 1$ and $s < 0$ then as x increases,

$$Y(x)/Y(0) \to 1 - T/(1+T) = -w_0$$

and $y(x)/(-w_0 y(0)) \approx e^{-sx} \uparrow \infty$, so that $y(x)$ cannot be a pdf. If $\alpha > 1$ and $s > 0$ then for $j \geq 1$, $\lambda_j < 0$, and also $s + \lambda_j < 0$ if $\alpha^j > 2$, so that $y(x)$ is unlikely to be a pdf, in contrast to the method of Example 2.4.

This example can be extended to $b(x)$ and $g(x)$ exponentials, as we now illustrate.

Example 2.3. *Suppose that $I = 1$, $b(x) = b_0 e^{b_1 x}$, $g(x) = g_0 e^{g_1 x}$. So,*

$$s(x) = g_0^{-1} e^{-g_1 x}\left(q + b_0 e^{b_1 x}\right) = \sum_{i=0}^{1} S_i e^{n_i x}, \text{ say},$$

$$c(x) = s(x) + g_1, \; I_c(z) = \sum_{i=0}^{1} S_i G_0\left(n_i, z\right),$$

$$I_s(z/\alpha) - I_c(z) = I_s(z/\alpha) - I_s(z) - g_1 z = \sum_{i=0}^{1} S_i\left[G_0\left(n_i, z/\alpha\right) - G_0\left(n_i, z\right)\right] - g_1 z$$
$$= a_0 z + \sum_{i=1}^{4} a_i e^{m_i z} \text{ say},$$

$$\mathcal{A}_1 = \theta \mathcal{A},$$

where

$$G_0(m,x) = \int_0^x e^{mz}dz = \begin{cases} (e^{mx}-1)/m, & m \neq 0, \\ x, & m = 0, \end{cases}$$

$$\mathcal{A}f(x) = \int_0^{\alpha x} \left(a_0 z + \sum_{i=1}^4 a_i e^{m_i z} \right) e^{b_1 z} f(z) dz,$$

$$\mathcal{A}e^{mx} = a_0 G_1(m+b_1, \alpha x) + F_1(m,x),$$

$$\mathcal{A}1 = a_0(\alpha x)^2/2 + F_1(0,x),$$

$$F_1(m,x) = \sum_{i=1}^4 a_i G_0(m+m_i+b_1, \alpha x),$$

$$G_1(m,x) = \int_0^x z e^{mz} dz = \begin{cases} x m^{-1} e^{mx} + m^{-2}(1-e^{mx}), & m \neq 0, \\ x^2/2, & m = 0. \end{cases}$$

In this way $\mathcal{A}^j e^{mx}$ *(and so* $\mathcal{A}^j 1$*) can be written in terms of* $\{G_k(m,x), k = 0, \ldots, j\}$*, where*

$$\begin{aligned} G_r(m,x) &= \int_0^x z^r e^{mz} dz \\ &= \partial_m^r G_0(m,x) \\ &= \begin{cases} m^{-1}\left[x^r e^{mx} - r G_{r-1}(m,x)\right], & m \neq 0, r \geq 1, \\ x^{r+1}/(r+1), & m = 0 \end{cases} \\ &= \sum_{i=0}^r \binom{r}{i} \left(x^{r-i} e^{mx} - \delta_{ij} \right) M_i \end{aligned}$$

by Leibniz's rule, where $M_i = \partial_m^i m^{-1} = (-1)^i i! m^{-1-i}$*. We omit further details.*

Theorem 2.2 extends Theorem 2.1 for *general* $I \geq 1$. A further extension of Theorem 2.2 is given by Theorem 2.3.

Theorem 2.2. *Set*

$$U_i Q_k = Q_{k-E_i}, \ \mathcal{V} = \sum_{i=1}^I \mathcal{A}_i U_i.$$

By (11),

$$Q_k(x) = Q_k(0) + \mathcal{V} Q_k(x), \tag{13}$$

so that $(\mathbf{I} - \mathcal{V})Q_k(x) = Q_k(0)$, $(\mathbf{I} - \mathcal{V})Y(x) = Y(0)$*, and*

$$Y(x) = Y(0) \sum_{r=0}^\infty \mathcal{V}^r 1 = Y(0) (\mathbf{I} - \mathcal{V})^{-1} 1$$

if this converges.

Theorem 2.3. *Theorem 2.2 holds with* $\mathcal{V}$ *replaced by*

$$\mathcal{S} = \sum_{i=1}^I \mathcal{A}_i.$$

Proof: By the multinomial theorem,

$$\mathcal{V}^r = \sum \left\{ \binom{r}{\nu} \mathcal{A}^\nu U^\nu : \ |\nu| = r \right\},$$

where

$$\nu = (\nu_1, \ldots, \nu_I), \ |\nu| = \sum_{i=1}^{I} \nu_i, \ \mathcal{A}^\nu = \mathcal{A}_1^{\nu_1} \cdots \mathcal{A}_I^{\nu_I}, \ U^\nu = U_1^{\nu_1} \cdots U_I^{\nu_I}.$$

Moreover,

$$\mathcal{V}^r Q_k(x) = \sum_\nu \left\{ \binom{r}{\nu} \mathcal{A}^\nu Q_{k-\nu}(x) : \ |\nu| = r \right\} = \begin{cases} 0, & \text{if } r > |k|, \\ \binom{r}{k} Q_0(x), & \text{if } r = |k| \end{cases} \tag{14}$$

by (9). By (13),

$$Q_k(x) = \sum_{r=0}^{J-1} \mathcal{V}^r Q_k(0) + \mathcal{V}^J Q_k(x).$$

Putting $J = |k| + 1$ gives

$$Q_k(x) = \sum_{r=0}^{|k|} \mathcal{V}^r Q_k(0)$$

and

$$\begin{aligned} Y(x) &= \sum_{r=0}^{\infty} \mathcal{V}^r \sum_{r \le |k|} Q_k(0) \\ &= \sum_{0 \le r = |\nu| \le |k| \le \infty} \binom{r}{\nu} Q_{k-\nu}(0) \ \mathcal{A}^\nu 1 \\ &= \sum_{r=0}^{\infty} \sum_{|\nu|=r}^{\infty} \binom{r}{\nu} \mathcal{A}^\nu 1 \ T_\nu, \end{aligned}$$

where

$$T_\nu = \sum_{|k| \ge |\nu|} Q_{k-\nu}(0) = \sum_{k \ge \nu} Q_{k-\nu}(0) = Y(0)$$

by (14). This gives

$$Y(x) = Y(0) \, (I - \mathcal{S})^{-1} \, 1 = Y(0) \sum_{r=0}^{\infty} \mathcal{S}^r 1,$$

$$y(x) = y(0) e^{-I_c(x)} \sum_{r=0}^{\infty} \mathcal{S}^r 1$$

if this converges, by (6). The proof is complete. □

Suppose now that $g(0)b(x) \ge 0$ for $x \ge 0$. Then, if $f(x) \ge 0$ when $x \ge 0$, $\mathcal{A}_i f(x) \ge 0$, $\mathcal{A} f(x) \ge 0$, $\mathcal{A}^r f(x) \ge 0$, and $y(x)/y(0) \ge 0$ for $x \ge 0$ if $y(0) \ne 0$. Hall and Wake (1989, 1990) looked for $y(x)$ a *steady size* pdf for the size of a cell, that is, one satisfying $y(x) \ge 0$, $\int_0^\infty y(x) dx = 1$. Their results are included in the examples.

We now give extensions of earlier results. Under the assumption that the functions b and g depend only on x, Hall and Wake (1989, 1990) and van-Brunt and Vlieg-Hulstman (2010a, 2010b) considered (2) and (4) for the special case $I = 1$, $B(t) = G(t) = 1$. (This is equivalent to $I = 1$, $B(t) = B_0$, $G(t) = G_0$ since B_0, G_0 can be absorbed into $b(x)$, $g(x)$.) By (3),

$$N(t) = N(0)e^{qt}. \tag{15}$$

(5) becomes

$$\partial_x y(x) + c(x)y(x) = \alpha^2 b(\alpha x)y(\alpha x)/g(x), \tag{16}$$

where $c(x) = [q + b(x) + \partial_x g(x)]/g(x)$.

Example 2.4. *Consider again Example 2.2: $I = 1$, $b(x) = b$, $g(x) = g$ are constants. So, $c(x) = (q + b)/g = c$ say. Then,*

$$\partial_x y(x) + cy(x) = a\alpha y(\alpha x), \tag{17}$$

where $a = \alpha b/g$. This is the pantograph *equation studied by Kato and McLeod (1971) for $\alpha < 1$ as well as for $\alpha > 1$. (They replace $(\alpha, a\alpha, c)$ by $(\lambda, a, -b)$.) van-Brunt and Vlieg-Hulstman (2010a, 2010b) give references for various applications of it. Suppose that $\alpha \neq 1$. A solution of (10) is*

$$Q_k(x) = e^{-\gamma_k x} r_k,$$

where $\gamma_k = (\alpha^k - 1)c$ and r_k are constants such that

$$r_k/r_{k-1} = -a\alpha/\gamma_k = \eta/\left(1 - \alpha^k\right) = \nu_k$$

say, where

$$a = g^{-1}b\alpha,\ \eta = a\alpha/c,\ r_k/r_0 = \prod_{j=1}^{k} \nu_j,$$

interpreted as one if $k = 0$. This gives

$$y(x) = e^{-cx}Y(x) = r_0 \sum_{k=0}^{\infty} e^{-\alpha^k cx} \prod_{j=1}^{k} \nu_j = r_0 f\left(cx : \alpha, \eta\right), \tag{18}$$

where f is Kato's function (6). This is the solution given by Theorem 9 (i), page 923 of Kato and McLeod (1971). (It is also the solution given by their Theorem 5, page 909 when $\alpha < 1$.) As they noted, $f(x : \alpha, \eta)$ has an exponential upper tail with scale parameter α^{-1}: $f(x : \alpha, \eta) \approx e^{-\alpha x}$ as $x \to \infty$. Also,

$$\begin{aligned} \partial_x f(x : \alpha, \eta) &= -f(x : \alpha, \alpha\eta), \\ \int_x^{\infty} f(x : \alpha, \eta)dx &= f(x : \alpha, \theta), \end{aligned} \tag{19}$$

where $\theta = \eta/\alpha = \alpha b/(q + b)$. As in equation (36) of Hall and Wake (1990), we set

$$K(\alpha, \eta) = f(0 : \alpha, \eta) = \sum_{k=0}^{\infty} \eta^k / \prod_{j=1}^{k} \left(1 - \alpha^j\right) = y(0)/r_0. \tag{20}$$

Hall and Wake (1990, equation (96)) showed that

$$K(\alpha, \eta) = \prod_{n=1}^{\infty} (1 - \eta/\alpha^n) . \tag{21}$$

Taking logs of (21) gives

$$\log \ K(\alpha, \eta) = -\sum_{k=1}^{\infty} (k-1)! \eta^k \left(\alpha^k - 1\right)^{-1} .$$

This is absolutely convergent if $|\eta|\alpha < 1$, but diverges if $\eta = 1$. When $\alpha > 1$, Hall and Wake (1989) argue that $f(x : \alpha, \eta) > 0$ for $x > 0$. So,

$$f_{\alpha,\eta}(x) = f(x : \alpha, \eta)/K(\alpha, \theta) \tag{22}$$

is a pdf on $[0, \infty)$ of a random variable, say X if $K(\alpha, \theta) \neq 0$ (that is, if $\eta \neq \alpha^{n+1}$, $n \geq 1$), and

$$\text{Prob}(X > x) = f(x : \alpha, \theta)/K(\alpha, \theta), \ x \geq 0. \tag{23}$$

Kato and McLeod (1971, page 923) note that $f(x : \alpha, \eta)$ is analytic in η for $c < 0$ so that $f_{\alpha,\eta}(0) = 0$ only for exceptional η. But $f(0 : \alpha, \eta) = K(\alpha, \eta) = 0$ implies $\eta = \alpha^n$ for some $n \geq 1$.

Figure 2 shows the behavior of (22) versus x for selected α and $\eta = 1$. We can see that $f_{\alpha,\eta}(x)$ is a monotonically decreasing function of x. The upper tail of $f_{\alpha,\eta}(x)$ becomes thinner with increasing values of α.

If $\eta = \alpha$ then $f_{\alpha,\eta}(0) = 0$. Hall and Wake (1989) considered this case and assumed in their equation (3), that $q = q_1$, where $\eta = \alpha^n$ if and only if $q = q_n$, where $q_n = (\alpha^{2-n} - 1)b$, $n = 1, 2, \ldots$ since $c = (q + b)/g$. ($n = 2$ gives us the no growth case, $q = 0$.) We have

$$\int_0^{\infty} y(x) dx = 1$$

if and only if $K(\alpha, \theta) = c/r_0 = (q + b)/(gr_0)$; $q = q_1$ if and only if $c = a$ if and only if $\theta = 1$ if and only if $\eta = \alpha$. So, $K(\alpha, 1) = \alpha b/(gr_0)$ and $y(0) = 0$. So, for the special case $\eta = \alpha$, $c = a$ that $q = q_1$, $\theta = 1$, $y(0) = 0$ and $y(x)$ is the pdf of a positive random variable $Y = X/a$ with moments given by

$$\mathbb{E}\left[X\right] = d_1^{-1} = \mu \ \textit{say}, \ \mathbb{E}\left[X^r\right] = r!/d_r, \ d_r = \prod_{k=1}^{r} \left(1 - \alpha^{-k}\right) = d_r(\alpha) \ \textit{say}, \tag{24}$$

$$\text{var}(X) = \mathbb{E}\left[(X - \mu)^2\right] = \left(1 - \alpha^{-2}\right)^{-1}, \ \mathbb{E}\left[(X - \mu)^3\right] = 2\left(1 - \alpha^{-3}\right)^{-1},$$

and by (19) or (23), X has distribution

$$\text{Prob}(X > x) = K(\alpha, 1)^{-1} \sum_{k=0}^{\infty} e^{-\alpha^k x} / \prod_{j=1}^{k} \left(1 - \alpha^j\right) .$$

Comtet (1974, problem 10, page 159) shows that any polynomial in $(x_1, x_2, \ldots)$, where $x_j = 1/(1 - \alpha^j)$, can be written as the sum of polynomials in x_j.

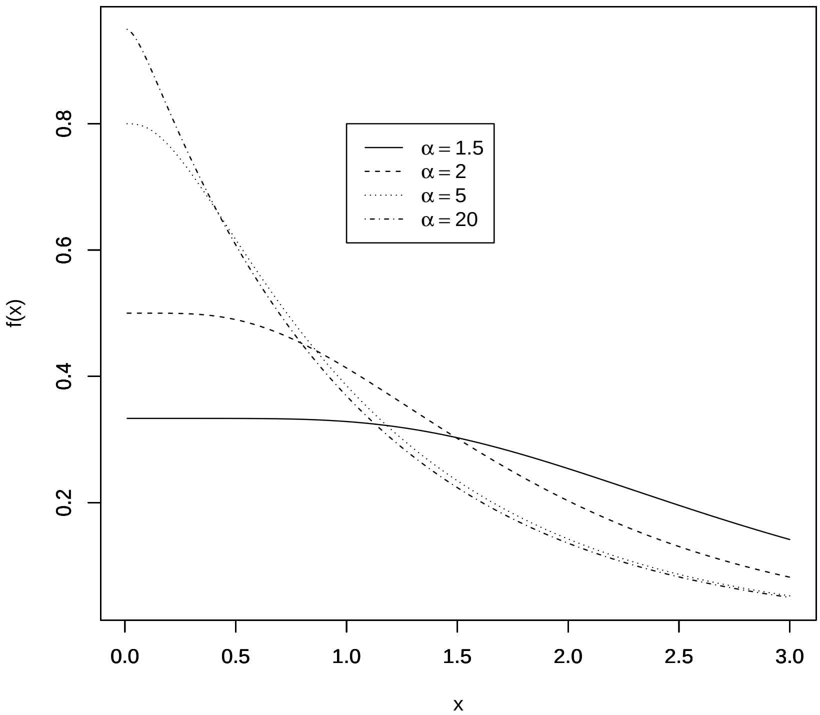

FIGURE 2. Plot of (22) versus x for $\alpha = 1.5, 2, 5, 20$ and $\eta = 1$.

Now consider the more general case of any α, θ *such that* $K(\alpha, \theta) \neq 0$. *That is, we no longer require that* $y(0) = 0$. *By (17), the nth moment of* $X = cY$, $m_n = \mathbb{E}\left[X^n\right]$, *satisfies*

$$-nm_{n-1} + cm_n = \eta \int^x y(\alpha x)dx = \eta\alpha^{-n-1}m_n,\ m_n/m_{n-1} = n/e_n,$$

where $e_n = 1 - \eta\alpha^{-n-1} = 1 - \theta\alpha^{-n}$. *So,*

$$m_j = j!/\prod_{n=1}^{j} e_n,\ \mathbb{E}\left[e^{Xt}\right] = 1 + S(t),\ S(t) = \sum_{j=1}^{\infty} t^j \nu_j,$$

where

$$\nu_j = 1/\prod_{n=1}^{j} e_n.$$

Also, since

$$S(t)^k = \sum_{j=k}^{\infty} t^k \widehat{B}_{jk}(\nu),$$

the jth cumulant, $\kappa_j = \kappa_j(X)$, is given by

$$\kappa_j/j! = \sum_{k=1}^{j} (-1)^{k-1} k^{-1} \widehat{B}_{jk}(\nu).$$

(For a power series in η for m_j, see the last equation on page 118 of Comtet (1974).) These moments are conditional on α. Consider again the case $\eta = \alpha$. By (17), $Y = X/a = Y_0/a_0$, where $a_0 = b/g$, $Y_0 = X/\alpha$. Now suppose that α is **random** *but not (b, g). Then, $a = a_0\alpha$ and Y_0 has* conditional *and* unconditional *moments*

$$\mathbb{E}\left[(Y_0^r|\alpha)\right] = r!\alpha^{-r}/d_r, \ \mathbb{E}\left[Y_0^r\right] = r!\mathbb{E}\left[\alpha^{-r}/d_r(\alpha)\right].$$

Suppose that α has pdf $f_\alpha(u)$ on $(1, \infty)$. The means, $\mathbb{E}[Y_0] = \mathbb{E}\left[(\alpha-1)^{-1}\right]$ and $\mathbb{E}[Y] = a_0^{-1}\mathbb{E}[Y_0]$, are infinite if $f_\alpha(1+) > 0$, but finite if $\alpha = 1$ is unlikely in the sense that $f_\alpha(u) = (u-1)^\gamma[1+o(1)]$ as $u \downarrow 1$, where $\gamma > 0$, which we now assume. Similarly,

$$a_0^{-r}\mathbb{E}\left[Y^r\right] = \mathbb{E}\left[Y_0^r\right] = r!\mathbb{E}\left[\alpha^{\binom{r}{2}} / \prod_{k=1}^{r}\left(\alpha^k - 1\right)\right]$$

is finite if and only if $r < \gamma + 1$. For example, if $\gamma > 1$, then Y_0 has finite variance

$$\mathrm{var}\,(Y_0) = 2\mathbb{E}\left[\alpha/\prod_{k=1}^{2}\left(\alpha^k - 1\right)\right] - \left\{\mathbb{E}\left[(\alpha-1)^{-1}\right]\right\}^2, \ \mathrm{var}(Y) = a_0^{-2}\mathrm{var}\,(Y_0).$$

The unconditional distribution of Y_0 is

$$\begin{aligned} \mathrm{Prob}\,(Y_0 > y) &= \mathbb{E}\left[K(\alpha,\theta)^{-1}\sum_{k=0}^{\infty} e^{-\alpha^{k+1}y/\theta}\theta^k / \prod_{j=1}^{k}\left(1-\alpha^j\right)\right] \\ &\approx \mathbb{E}\left[e^{-\alpha y}/K(\alpha,\theta)\right] \end{aligned}$$

as $y \to \infty$. To find its behavior for large y, we use the partial fraction expansion

$$f(x)^{-1} = \sum_{k=1}^{J} c_k^{-1}\left(x - \alpha_k\right)^{-1},$$

where

$$f(x) = \prod_{k=1}^{J}\left(x-\alpha_k\right), \ c_k = f_{\cdot 1}\left(\alpha_k\right) = \prod_{1\le j\le J,\ j\ne k}\left(\alpha_k - \alpha_j\right).$$

See, for example, equation (2.102) in Gradshteyn and Ryzhik (2000). So, putting $x = 1$, $J = \infty$, $\alpha_k = \alpha^{-k}$,

$$K(\alpha,\theta)^{-1} = \sum_{k=1}^{\infty} c_k^{-1}\left(1-\alpha_k\right)^{-1},$$

where

$$c_k = \prod\left\{\left(\alpha_k - \alpha_j\right) : j \ge 1, \ j \ne k\right\}.$$

The next example gives a general result based on Example 2.4.

Example 2.5. *Suppose that a random variable has pdf*

$$f_\alpha(u) = (u-1)^\gamma \widetilde{f}(u),$$

where

$$\widetilde{f}(u) = \sum_{j=0}^{\infty} f_j (u-1)^j, \ \gamma > 0.$$

Then,

$$\mathbb{E}\left[e^{-\alpha y}/K(\alpha,1)\right] = e^{-y}\Gamma(\gamma)\sum_{j=0}^{\infty}(\gamma)_j y^{\gamma-j}\sum_{j=0}^{i} P_{ij}(f) C_{j-i-1},$$

where

$$C_j = \sum_{k=1}^{\infty} k^{j-i} c_k^{-1} k^j, \ (\gamma)_j = \Gamma(\gamma+j)/\Gamma(\gamma) = \gamma(\gamma+1)\cdots(\gamma+j-1),$$

$P_{00} = f_0$*, and* $P_{ij}(f)$ *is a polynomial in* $(f_0, f_1, \ldots)$*. This asymptotic expansion follows by applying the saddlepoint expansion given in Withers and Nadarajah (2013) with* $X = 1$*,* $\lambda = \gamma - 1$*,* $F(\alpha) = \alpha y$*,* $M = 1$*,* $G(\alpha) = \widetilde{f}(\alpha)(\alpha-1)/(\alpha^k - 1)$*. We omit details.*

The next two examples allow $g(0)$ to be zero.

Example 2.6. *See Section 1.1 of Hall and Wake (1990) for the case* $I = 1$*,* $b(x) = b$*,* $g(x) = g_0 x^{1-k}$*,* $k > 0$*.*

Example 2.7. *Section 2 of Hall and Wake (1990) studied the case* $I = 1$*,* $g(x) = g_0 x$*,* $b(x) = b_0 x^n$*,* $n > 0$*, so that*

$$I_b(x) = \int_0^x b(y)dy = Bx^N,$$

where $N = n+1$*,* $B = b_0/N$*, and gave references to others who have studied that case. (van-Brunt and Vlieg-Hulstman (2010b) also studied this case.) Setting*

$$a_0 = b_0/g_0, \ Z(x) = x^\gamma y(x), \ \gamma = 1 + q/g_0, \ a_2 = a_0 \alpha^{n+2-\gamma} \tag{25}$$

transforms (16) to

$$\partial_x Z(x) + a_0 x^{n-1} Z(x) = a_2 x^{n-1} Z(\alpha x). \tag{26}$$

Hall and Wake (1990) made the choice $\gamma = 2$*, that is,* $q = g_0$*, and assumed that*

$$[g(x)y(x)]_0^\infty = 0, \ \int_0^\infty y(x)dx = 1.$$

They showed that there is then a unique solution with

$$y(x) = Cx^{-2}\sum_{k=0}^{\infty}\alpha^{kn} e^{-I_a(\alpha^k x)} / \prod_{j=1}^{k}\left(1-\alpha^{jn}\right),$$

$$C^{-1}(n/a_0)^{1/n} = \begin{cases} K(\alpha^n, 1)\log\ \alpha \left[\prod_{j=1}^{m}\left(\alpha^{jn}-1\right)\right]/m!, & \textit{if } m = n^{-1} \in \mathbb{I}^+, \\ K(\alpha^n, \alpha)\,\Gamma\left(1-n^{-1}\right), & \textit{if } n^{-1} \notin \mathbb{I}^+, \end{cases}$$

where $\mathbb{I}^+ = \{1, 2, \ldots\}$ *and* $K(\cdot, \cdot)$ *is given by (20). By (21),*

$$K(\alpha^n, \alpha) = \prod_{k=1}^{\infty} \left(1 - \alpha^{1-kn}\right).$$

They showed that $y(x)$ *is the pdf of a positive random variable* $Y = X/a_0$*, and gave the moments* $\mathbb{E}\left[X^{in}\right]$, $i \geq 1$*. For example, if* $n = 1$ *then*

$$\mathbb{E}[X] = (\alpha - 1)^{-1}, \ \mathbb{E}\left[X^2\right] = (\alpha - 1)^{-1}(\log\ \alpha)^{-1},$$
$$\mathbb{E}[X^r] = (r-2)!/\left[d_{r-2}(\alpha - 1)(\log\ \alpha)\right]$$

for $r > 2$ *and* $d_r = d_r(\alpha)$ *of (24) above. If we allow* $\alpha > 1$ *to be* random, *we can write down the* unconditional *moments* $\mathbb{E}\left[X^{in}\right]$, $i \geq 1$*; for example, if* $n = 1$ *then*

$$\mathbb{E}[X] = \mathbb{E}\left[(\alpha - 1)^{-1}\right], \ \mathbb{E}\left[X^2\right] = \mathbb{E}\left[(\alpha - 1)/(\log\ \alpha)\right],$$
$$\mathbb{E}[X^r] = (r-2)!\mathbb{E}\left\{1/\left[d_{r-2}(\alpha)(\alpha - 1)(\log\ \alpha)\right]\right\}$$

for $r > 2$*. Returning to fixed* α*, we now give a solution which does* not *require their choice* $q = g_0$ *in (15). Their transformations*

$$z = x^n/n, \ Y(z) = Z(x) \tag{27}$$

give

$$\partial_z Y(z) + a_0 Y(z) = a_1 \alpha^n Y(\alpha^n z),$$

where $a_1 = \alpha^{2-\gamma} a_0$*. This is just (17) with* (x, y, c, a, α) *changed to* $(z, Y, a_0, a_1, \alpha^n)$*. So, in terms of* f *of (18), we obtain the solution*

$$y(x) = x^{-\gamma} Z(x), \tag{28}$$

where

$$Z(x) = r_0 f\left(a_0 x^n/n : \alpha^n, \alpha^{n+2-\gamma}\right)$$

for γ *of (25). Clearly, if* $r_0 \neq 0$, $y(0) = \infty$ *unless* $0 = K(\alpha^n, \alpha^{n+2-\gamma})$*, that is, by (21), unless* $(2-\gamma)/n$ *is a non-negative integer. But it is not clear how* $f(x, \alpha, \eta)$ *behaves near* $x = 0$*. Also, to choose* r_0 *so that* $y(x)$ *is a pdf, one can write* $\int_x^\infty y(x)dx$ *as a series if* $(1-\gamma)/n$ *is a non-negative integer.*

Does this method extend to $g(x) = g_0 x^{g_1}$?

Example 2.8. *Suppose that* $I = 1$, $g(x) = g_0 x^{g_1}$, $b(x) = b_0 x^{b_1}$*. By (4),* $c(x) = s(x) + g_1 x^{-1}$*, where* $s(x) = g_0^{-1} x^{-g_1}(q + b_0 x^{b_1})$*. Set* $Z(x) = x^\gamma y(x)$*. Then, (5) becomes*

$$\partial_x Z(x) + \overline{c}(x) Z(x) = a_1 \alpha^{2+b_1-\gamma} x^{b_1-g_1-\gamma} Z(\alpha x),$$

where

$$\overline{c}(x) = c(x) - \gamma x^{-1}, \ a_1 = g_0^{-1} b_0.$$

So, $\overline{c}(x)$ *is a weighted sum of powers of* x *with exponents* $(-1, -g_1, b_1 - g_1)$*. This reduces to the exponents* $(-1, b_1 - g_1)$ *if* $q = 0$ *or* $b_1 = 0$ *or* $g_1 = 1$*, and to the single exponent* -1 *if* $b_1 - g_1 = -1$ *and either* $g_1 = 1$ *(considered in Example 2.7), or* $q = 0$*.*

The case $b_1 - g_1 = -1$**, that is,** $g(x) = g_0 x^{g_1}$ **and** $b(x) = b_0 x^{g_1 - 1}$**.** *Choose*

$$\gamma = c_1 + g_1, \tag{29}$$

where $c_1 = g_0^{-1} q$, *so that* $\overline{c}(x) = c_1 x^{-g_1}$. *The method of the previous example needs* $c_1 = 1 + \gamma$, *that is* $a_1 = -1$, *that is,* $g_0 = -b_0$, *a case not applicable to cell growth. In this case,*

$$\partial_x Z(x) - x^{n-1} Z(x) = a_1 \alpha^2 x^{n-1} Z(\alpha x),$$

where $n - 1 = -g_1$. *This is just (26) with* (a_0, a_2) *changed to* $(-1, a_1 \alpha^2)$. *So, in terms of the pantograph function of (18) and* γ *of (29), we obtain the solution*

$$y(x) = r_0 x^{-\gamma} f\left(c_1 x^{1-g_1} / (1 - g_1) : \alpha^n, -a_1 \alpha^2\right).$$

The case $q = 0$**, that is,** $N(t) = N(0)$**,** $b(x) = b_0 x^{b_1}$**, and** $g(x) = g_0$**.** *Taking* $\gamma = 0$, *(26) holds with* (a_0, n, a_2) *replaced by* $(a_1, b_1 + 1, a_1 \alpha^{b_1 + 2})$. *So, by (28), a solution is*

$$y(x) = r_0 f\left(b_0 x^{b_1+1} / \left[g_0 \left(b_1 + 1\right)\right] : \alpha^{b_1+1}, \alpha^{b_1+3}\right).$$

In Section 4, we extend this case to $I \geq 1$.

The Mellin transform method only applies if $b(x) = b_0 x^n$. Hall and Wake (1990) gave a method for $b(x) = b_0 x^k I(x \geq x_1)$ in their Section 3, while their Section 4 gives a method for $a(x) = (x_2 - x)^{-1} I(x_1 \leq x \leq x_2)$.

3. A Multivariate Form of Kato's Function

Examples 2.4, 2.7, 2.8 gave solutions when $I = 1$ based on Kato's function (6). Suppose now that $I \geq 1$.

Example 3.1. *Consider Example 2.4, that is with constant* $g(x) = g$, $b(x) = b$, *but allow* $I \geq 1$. *So, (5) takes the form*

$$\partial_x y(x)/c + y(x) = \sum_{i=1}^{I} w_i y\left(\alpha_i x\right), \tag{1}$$

where $c = g^{-1}(q + b)$ *and* $w_i = (b/cg) p_i \alpha_i^2$. *A solution is*

$$y(x) = \sum_{k \geq 0_I} e^{-\alpha^k c x} r(k)$$

since (10) has solution

$$Q_k(x) = e^{-\gamma_k x} r(k), \ \gamma_k = \left(\alpha^k - 1\right) c,$$

where α^k *is defined by (7) and* $r(k)$ *are constants given by the recurrence equation*

$$r(k) = \nu(k) \sum \left\{ w_i r\left(k - E_i\right) : \ 1 \leq i \leq I, \ E_i \leq k \right\}, \ k \neq 0_I,$$

where $\nu(k) = (1 - \alpha^k)^{-1}$ *and* 0_I *is the I-vector of zeros. (Recall that* E_i *is the ith unit vector in* $\mathbb{R}^I$ *and k is now an integer vector.) One can show that* $r(k)$ *has the form*

$$r(k) = r\left(0_I\right) w^k f(k)$$

and $f(k)$ *is given by the recurrence equation*

$$f(k) = \nu(k) \sum \left\{ f\left(k - E_i\right) : \ 1 \leq i \leq I, \ E_i \leq k \right\}, \ k \neq 0_I, \tag{2}$$

where

$$w^k = \prod_{i=1}^{I} w_i^{k_i}.$$

That is,

$$y(x)/r\left(0_I\right) = f\left(cx : \alpha, w\right),$$

where

$$f(x : \alpha, w) = \sum_{k \geq 0_I} e^{-\alpha^k x} r(k : \alpha, w), \;\; r(k : \alpha, w) = r(k)/r\left(0_I\right). \tag{3}$$

We call this the **multivariate Kato function**. *If $I = 1$ it reduces to $f(x : \alpha, w)$ with $w = \eta$ of Example 2.4. f_k, the solution of (2), can be built up from the case $I = 1$, one dimension at a time. The first step uses*

$$f\left(k_i E_i\right)/f\left(\left(k_i - 1\right) E_i\right) = \nu\left(k_i E_i\right), \;\; k_i \geq 1$$

so that

$$f\left(n_i E_i\right) = \prod_{a=1}^{n_i} \nu\left(a E_i\right) = \prod_{a=1}^{n_i} \left(1 - \alpha_i^a\right)^{-1} = F\left(\alpha_i, n_i\right) \;\; say,$$

for $n_i \geq 1$. Next, get the two dimensional solutions using the recurrence equation in k_1,

$$f\left(k_1 E_1 + k_2 E_2\right) = \nu\left(k_1 E_1 + k_2 E_2\right)\left[f\left(k_1 E_1 + \left(k_2 - 1\right) E_2\right) + f\left(\left(k_1 - 1\right) E_1 + k_2 E_2\right)\right],$$

that is, setting $n = k_1$,

$$z[n] = b[n]\left(z[n-1] + a[n]\right),$$

where

$$\begin{aligned} &z\left[k_1\right] = f\left[k_1, k_2\right], \;\; f\left[k_1, k_2\right] = f\left(k_1 E_1 + k_2 E_2\right), \\ &b\left[k_1\right] = \nu\left[k_1, k_2\right] = \nu\left(k_1 E_1 + k_2 E_2\right), \;\; a\left[k_1\right] = f\left[k_1, k_2 - 1\right] \end{aligned}$$

with solution

$$z[n] = b[0] \cdots b[n]\left(a[0] + a[1]/b[0] + a[2]/b[0]b[1] + \cdots + a[n]/b[0] \cdots b[n-1]\right)$$

for $n \geq 0$. Now get the two dimensional solutions using the recurrence equation in k_1 for $z[k_1] = f[k_1, k_2, k_3] = f(k_1 E_1 + k_2 E_3 + k_2 E_3)$ for fixed k_2, k_3; and so on.

Just as the univariate Kato function was also used to give solutions to Examples 2.7 and 2.8 when $I = 1$, its multivariate version can be used to give solutions to these examples when $I > 1$, as we now show.

Example 3.2. *Consider Example 2.7, that is with constant $g(x) = g_0 x$, $b(x) = b_0 x^n$, $n > 0$, but allow $I \geq 1$. So, for a_0 of (25), (5) takes the form*

$$\partial_x y(x) + c(x) y(x) = a_0 x^{n-1} \sum_{i=1}^{I} p_i \alpha_i^{2+n} y\left(\alpha_i x\right),$$

where

$$c(x) = a_0 x^{n-1} + \left(1 + q g_0^{-1}\right) x^{-1}.$$

Transforming with $Z(x)$, γ of (25) gives

$$\partial_x Z(x) + a_0 x^{n-1} Z(x) = a_0 x^{n-1} \sum_{i=1}^{I} w_i Z(\alpha_i x), \tag{4}$$

where

$$w_i = p_i \alpha_i^{n+2-\gamma}.$$

The transformation (27) gives

$$\partial_z Y(z) + a_0 Y(z) = a_0 \sum_{i=1}^{I} w_i Y(\alpha_i z).$$

This is just (1) with (y, x, c, α_i) replaced by (Y, z, a_0, α_i^n). So, by (3),

$$Y(z)/r(0_I) = f(a_0 z : A_n, w), \tag{5}$$

where $(A_n)_i = \alpha_i^n$, that is,

$$y(x)/r(0_I) = x^{-1-q/g_0} f(a_0 x^n / n : A_n, w).$$

Example 3.3. *Suppose, (as in the last part of Example 2.8 with b_1 replaced by n), that $g(x) = g_0$, $b(x) = b_0 x^n$, $q = 0$, but $I \geq 1$. So, for a_0 of (25), (5) takes the form*

$$\partial_x y(x) + c(x) y(x) = a_0 x^n \sum_{i=1}^{I} w_i y(\alpha_i x),$$

where $c(x) = a_0 x^n$ and $w_i = p_i \alpha_i^{n+2}$. This is just (4) with $(Z, n-1, \gamma)$ replaced by $(y, n, 0)$. So, by (5), a solution is

$$y(x)/r(0_I) = f\left(a_0 x^N / N : A_N, w\right),$$

where $N = n + 1$.

4. Other Solutions for the PDE (2)

Here, we suppose that (2) holds. We show that (a) $b(x)$ and $g(x)$ cannot both be arbitrary: one determines the other; (b) $b(x)$ or $g(x)$ determines $y(x)$; (c) we are free to choose any two of $N(t)$, $B(t)$, $G(t)$ to determine the other. By (2),

$$\partial_x y(x) + c(x) y(x) = 0, \; q(t) + \beta(t)\gamma = \delta = \delta(x) \tag{1}$$

for some constant δ, where $c(x) = (\partial_x g(x) - \gamma)/g(x)$. That is,

$$\partial_t \log N(t) = \delta \; B(t) - \gamma G(t), \; [b(x) + \delta] \; y(x) = \mathcal{E}[b(x) y(x)]. \tag{2}$$

So, $N(t)$ is given in terms of $B(t)$ and $G(t)$ by $N(t) = N(0) \exp\{\delta \; I_B(t) - \gamma I_G(t)\}$, again using the notation (7). Also, $y(x)$ is given by

$$y(x) = e^{-I_c(x)} y(0) = e^{\gamma I_h(x)} h(x) g(0) y(0), \tag{3}$$

where $h(x) = g(x)^{-1}$. But $y(x)$ must also satisfy (2). So, we have Theorem 4.1.

Theorem 4.1. *With notation as above, either $y(x) \equiv 0$, or if $y(0) \neq 0$, then $g(x)$ and $b(x)$ must satisfy*

$$[b(x) + \delta]\, e^{-I_c(x)} = \mathcal{E}\left[b(x) e^{-I_c(x)}\right]. \tag{4}$$

Putting $x = 0$ gives

$$\delta = \left\{\mathbb{E}\left[\alpha^2\right] - 1\right\} b(0),$$

where

$$\mathbb{E}\left[\alpha^2\right] = \sum_{i=1}^{I} p_i \alpha_i^2.$$

Theorem 4.2 expresses $g(x)$ in terms of $b(x)$.

Theorem 4.2. *Consider the case $I = 1$. Then,*

$$g(x) = e^{I_c(x)} \left[g(0) + \gamma \int_0^x e^{-I_c(z)} dz\right], \tag{5}$$

where

$$I_c(x) = \sum_{k=0}^{\infty} D\left(\alpha^{-k} x\right) \tag{6}$$

when this converges, where

$$D(x) = \log \left\{\alpha^2 \widetilde{b}(x) / \left[\widetilde{b}\left(\alpha^{-1} x\right) + \alpha^2 - 1\right]\right\}, \ \widetilde{b}(x) = b(x)/b(0).$$

Proof: Write (4) as

$$\begin{aligned} &\exp\left[I_c(\alpha x) - I_c(x)\right] = \alpha^2 b(\alpha x) / \left[b(x) + \delta\right] = \exp\left[D(\alpha x)\right], \qquad (7)\\ &I_c(x) - I_c\left(\alpha^{-1} x\right) = D(x). \end{aligned}$$

So, $D(0) = 0$. Replace x by $\alpha^{-k} x$ and sum from $k = 0$ to infinity to obtain (6). By (1), we have (5). □

An alternative approach to Theorem 4.2 is to assume that $b(x)$ has a Taylor series about zero and to obtain a similar series for $g(x)$.

Example 4.1. *Suppose that $b(x) = b$, a constant. Then, $D(x) = 0$ so that $I_c(x)$ is constant, which by (3) must be zero; that is, $c(x) = 0$, $g(x) = g_0 x$. This gives the special case $b(x) = b$, $g(x) = g_0 x$, $y(x) = y(0)$.*

We now express $b(x)$ in terms of $g(x)$. If $g(x)$ has a Taylor series expansion about zero, then one can obtain a Taylor series expansion about zero for $b(x)$ using Bell polynomials. Suppose that $g(x) = \sum_{j=0}^{\infty} g_j x^j = g_0 + T(x)$ say, with $g_0 \neq 0$. Theorem 4.3 shows how to obtain $b(x) = \sum_{j=0}^{\infty} b_j x^j$.

Theorem 4.3. *With notation as above,*

$$b_0 = \left(\alpha^2 - 1\right)^{-1} \delta, \ b_i = \left(\alpha^{i+2} - 1\right)^{-1} \sum_{j=0}^{i-1} b_j S_{i-j}, \ i \geq 1, \tag{8}$$

where

$$S_i = \sum_{r=0}^{i} \widehat{B}_{ir}(s)/r!$$

and $s_i = \left(\alpha^i - 1\right)/i$.

Proof: As in (12), $T(x)^i = \sum_{j=i}^{\infty} \widehat{B}_{ji}(g)x^j$ at $g = (g_1, g_2, \ldots)$, so that

$$g(x)^{-1} = g_0^{-1} \sum_{i=0}^{\infty} (-g_0)^{-i}\, T(x)^i = \sum_{j=0}^{\infty} C_j(g)x^j,$$

where

$$C_j(g) = g_0^{-1} \sum_{i=0}^{j} (-g_0)^{-i}\, \widehat{B}_{ji}(g).$$

Also, by (1), $c(x)g(x) = \partial_x g(x) - \gamma$ so that

$$c(x) = \sum_{j=0}^{\infty} c_j x^j$$

with c_j given by the recurrence relation

$$c_j g_0 + \sum_{i=0}^{j-1} c_i g_{j-i} = c_j \otimes g_j = (j+1)g_{j+1} - \gamma\delta_{j0}.$$

So,

$$c_0 = g_0^{-1}\left(g_1 - \gamma\right), \ c_j = g_0^{-1}\left[(j+1)g_{j+1} - \sum_{i=0}^{j-1} c_i g_{j-i}\right], \ j \geq 1.$$

Now set

$$s_x = I_c(\alpha x) - I_c(x) = \int_x^{\alpha x} c(z)dz = \sum_{i=1}^{\infty} s_i x^i.$$

So,

$$e^{s_x} = \sum_{r=0}^{\infty} S_x^r/r! = \sum_{i=0}^{\infty} S_i x^i.$$

For example, $S_0 = 1$, $S_1 = s_1$ and $S_2 = s_2 + s_1^2/2$. So, by (7), $S_i \otimes (b_i + \delta\delta_{i0}) = b_i\alpha^{i+2}$, giving (8). □

Theorem 4.3 can be extended to $g_0 = 0 \neq g_1$.

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References

[1] L. Comtet, *Advanced Combinatorics.* Reidel, Dordrecht, 1974.

[2] I. S. Gradshteyn and I. M. Ryzhik , *Tables of Integrals, Series and Products*, sixth edition. Academic Press, New York, 2000.

[3] A. J. Hall and G. C. Wake, *A functional differential equation arising in the modelling of cell-growth,* Journal of the Australian Mathematical Society, **30** (B) (1989), pp. 424-435.

[4] A. J. Hall and G. C. Wake, *Functional differential equations determining steady size distributions for populations of cells growing exponentially,* Journal of the Australian Mathematical Society, **31** (B) (1990), pp. 434-453.

[5] A. J. Hall, G. C. Wake and P. W. Gandar, *Steady size distributions for cells in one-dimensional plant tissues,* Journal of Mathematical Biology, **30** (1991), pp. 101-123.

[6] T. Kato and J. B. McLeod, *The functional differential equation $y'(x) = ay(\lambda x) + by(x)$,* Bulletin of the American Mathematical Society, **77** (1971), pp. 891-937.

[7] B. van-Brunt and M. Vlieg-Hulstman, *An eigenvalue problem involving a functional differential equation arising in a cell growth model,* ANZIAM Journal, **51** (a) (2010), pp. 383-393.

[8] B. van-Brunt and M. Vlieg-Hulstman, *Eigenfunctions arising from a first-order functional differential equation in a cell growth model,* ANZIAM Journal, **52** (b) (2010), pp. 46-58.

[9] C. S. Withers and S. Nadarajah, *Saddlepoint expansions in terms of Bell polynomials,* Integral Transforms and Special Functions, **24** (2013), pp. 410-423.

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