NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 43 (2013), 31-36

ON A CONSTRUCTIVE CHARACTERIZATION OF CLASSES OF HARMONIC FUNCTIONS

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(Received 22 February, 2013)

Abstract. Let Γ be a quasiconformal curve in the complex plane $\mathbb{C}$. In this study, a constructive characterization of classes of harmonic functions with singularities on closed quasiconformal curves is obtained.

1. Introduction and the Main Results

A function $u = u(x, y)$ is called harmonic if it is real valued having continuous partial derivatives of order one and two, and satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

One usually defines the *Laplace* (differential) *operator*

$$\Delta = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2$$

and so u is harmonic if and only if $\Delta u = 0$ (and u is of class C^2).It is known that real part of an analytic function is harmonic.

Harmonic functions play an important role in many areas of applied mathematics and mechanics. It is actually the approximation of these functions by rational functions or some other functions, which can be found easily. Direct and inverse problems related to the approximation of harmonic functions have been studied in references [**2**]-[**10**], [**12**]-[**18**], [**21**], [**22**] and [**24**]. If direct and inverse theorems are in full accordance, it is common to say that a class has a *constructive characterization.*

In this study the constructive characterization of classes of harmonic functions with singularities on a quasiconformal curves is studied. To prove the inverse theorem here, we use the standard scheme for the proofs of inverse theorem [**9**] , [**13**] and [**23**].

Let $\mathbb{C}$ be the complex plane and let $\Gamma \subset \mathbb{C}$ be an arbitrary closed Jordan curve with its complements $\Omega = C\Gamma = \Omega_1 \cup \Omega_2$ $(0 \in \Omega_1, \infty \in \Omega_2)$. Consider the function $w = \phi_i(z)$, $(i = 1, 2)$ that conformaly and univalently maps Ω_i onto Ω_i', respectively $\left(\Omega_1' = \{w : |w| < 1\}, \Omega_2' = \{w : |w| > 1\}\right)$ with normalization

2010 *Mathematics Subject Classification* 30E10, 31C05, 41A20, 41A27.
Key words and phrases: Conformal mapping, harmonic functions, harmonic rational approximation, quasiconformal curve, direct and inverse theorems.
The author wishes to express deep gratitude to the referee for valuable suggestions.

$$\phi_1(0)=0,\ \phi_1^{'}(0)>0,\ \phi_2(\infty)=\infty,\ \lim_{z\to\infty}\phi_2(z)/z>0\ .$$

The function, inverse to $w=\phi_i(z)$, is denoted by $z=\psi_i(w),(i=1,2)$.

For arbitrary natural n and $\delta>0$ we set

$$\Gamma_{1+\frac{(-1)^i}{n}}:=\left\{\zeta:\zeta\in\Omega_i,\ |\phi_i(\zeta)|=1+\frac{(-1)^i}{n}\right\},\ i=1,2,$$

$$\rho_{1+\frac{(-1)^i}{n}}(z):=\inf_{\zeta\in\Gamma_{1+\frac{(-1)^i}{n}}}|\zeta-z|,\ \rho_{1/n}(z):=\min\left\{\rho_{1+\frac{(-1)^i}{n}}(z)\right\},\ i=1,2$$

$$D(z,\delta):=\{\zeta:|\zeta-z|<\delta\},\quad z\in\mathbb{C},$$

$$G_n:=\bigcup_{\zeta\in\Gamma}D\left(\zeta,\frac{1}{2}\rho_{1/n}(\zeta)\right).$$

We denote by $c,c_1,c_2...$ positive constants, different in different relations, in general, and depending, if not specifically said otherwise, on the curve Γ or on other quantities inessential for the problems we are interested in. We shall also employ the symbol $A\preccurlyeq B$, denoting that $A\le CB$, where $C=const>0$ does not depend on A or B.

Let $\omega(\delta)$, $\delta>0$, be a function of the type of modulus of continuity, i.e. a positive nondecreasing (with $\omega(+0)=0$) function satisfying for some $c=const>0$ the condition $\omega(t\delta)\le ct\omega(\delta)$, $\delta>0$, $t>1$.

In the present paper we are interested in the case when Γ is a quasiconformal curve. The convenient geometrical quasiconformal definition of the curve is as follows (see, [20, p.100])

Let us consider a Jordan curve Γ and two arbitrary points z_1 and z_2 on it. By $\Gamma(z_1,z_2)$ we denote one of the two curves (with less diameter) on which the points z_1 and z_2 divide the curve Γ. A necessary and sufficient condition for the curve Γ to be quasiconformal is that the relation

$$diam\ \Gamma(z_1,z_2)\preccurlyeq|z_1-z_2|$$

holds.

As P.P.Belinskii's example shows (see, [11, p. 42]), a quasiconformal curves may be unrectifiable at any point. Detailed information on the quasiconformal curves can be found in the books [**1**], [**9**], [**11**] and [**20**].

Let us denote by $C_{\Delta}^{\omega}(\Gamma)$ the class of real-valued, continuous in $\mathbb{C}$, harmonic in $\bar{\mathbb{C}}\backslash\Gamma$ functions u satisfying, for any z and $\zeta\in\mathbb{C}$, the condition

$$|u(z)-u(\zeta)|\le c\omega(|z-\zeta|),\ c=c(u)=const>0.$$

Let $B_{\Delta}^{\omega}(\Gamma)$ be the class of real-valued, continuous in $\bar{\mathbb{C}}$, harmonic in $\bar{\mathbb{C}}\backslash\Gamma$ functions such that, for any $n\in\mathbb{N}$, there is a harmonic rational function

$$R_n(z)=Re\sum_{j=-n}^{n}a_jz^j,\ n=1,2,...,\ a_j\in\mathbb{C}\tag{1.1}$$

satisfying the relation

$$|u(z)-R_n(z)|\le c_1\omega\left[\rho_{1/n}(z)\right],\quad z\in G_n.$$

The main results of this work are as follows:

Theorem 1. Let Γ be a quasiconformal curve and $f \in B_{\Delta}^{\omega}(\Gamma)$ .Then $f \in C_{\Delta}^{\mu}(\Gamma)$, where

$$\mu(\delta) = \delta \int_{\delta}^{1} \frac{\omega(t)}{t^2} dt, \ 0 < \delta < 1/2 .$$

Corollary 1. If

$$\delta \int_{\delta}^{1} \frac{\omega(t)}{t^2} dt \le c_2 \omega(\delta), \quad 0 < \delta < 1/2$$

then

$$C_{\Delta}^{\omega}(\Gamma) = B_{\Delta}^{\omega}(\Gamma).$$

2. Proofs of Main Results

We need the following theorem:

Theorem 2 [**15**]. Let Γ be a closed quasiconformal curve, $u(z) \in C_{\Delta}^{\omega}(\Gamma)$. Then for each natural $n = 1, 2, ...$ there exists a harmonic rational function $R_n(z)$, that for $z \in G_n$ the inequality

$$|u(z) - R_n(z)| \le c_3 \omega \left[\rho_{1/n}, (z)\right]$$

where the constant $c_3 > 0$ is independent of z and n, holds.

The following lemma is proved easily with the help of the Schwarz's formula (see, for example, [19, Paragraph 44])

Lemma 1. Let Γ be closed quasiconformal curve. If a rational harmonic function of type (1.1), $n \in \mathbb{N}$, satisfies the inequality

$$|q_n(z)| \le \omega \left[\rho_{1/n}, (z)\right], \quad \forall z \in G_n, \tag{2.1}$$

then, for $z \in \underset{\zeta \in \Gamma}{\cup} D\left(\zeta, \frac{1}{4}\rho_{1/n}(\zeta)\right)$, the estimate

$$|grad q_n(z)| \le c_4 \omega \left[\rho_{1/n}(z)\right] / \rho_{1/n}(z) \tag{2.2}$$

is valid, where the constant c_4 is independent of z and n.

Proof of Lemma 1. Suppose that the condition (2.1) is satisfied. We determine the rational function $R_n(z)$ for which $Re R_n(z) = q_n(z)$. As seen in [9, p.160], [**4**] for small $u > 0$ we may construct the curves $\Gamma^*_{1+(-1)^j u}$, $j = 1, 2$ and there is an $\varepsilon > 0$ for which the following inclusion holds:

$$\Gamma^*_{1+(-1)^j \frac{\varepsilon}{n}} \subset \underset{\zeta \in \Gamma}{\cup} D\left(\zeta, \frac{1}{4}\rho_{1/n}(\zeta)\right).$$

We set $\rho(z) = \rho_{1/n}(z)$. By consideration of Schwarz's formula (see, [**19**], Paragraph 44) and (2.1) for $z \in \Gamma^*_{1+(-1)^j \frac{\varepsilon}{n}}$ we obtain

$$\left|R_n'(z)\right| = \frac{1}{\pi} \left| \int_{\partial D(z, \varepsilon^* \rho(z))} \frac{q_n(\varsigma)}{(\varsigma - z)^2} \right| \le c_5 \omega \left[\rho(z)\right] / \rho(z), \tag{2.3}$$

where ε^* is chosen so that the inclusion

$$D\left(z,\varepsilon^*\rho\left(z\right)\right)\subset\underset{\zeta\in\Gamma}{\cup}D\left(\zeta,\frac{1}{2}\rho_{1/n}\left(\zeta\right)\right)$$

is valid. Thus, the desired inequality (2.2) follows from (2.3).

Proof of Theorem 1. Let $f\in B_{\Delta}^{\omega}\left(\Gamma\right)$ and $\delta>0$ be an arbitrary sufficiently small number. Let $z_0\in\Gamma$ be an arbitrary point. Let $\left\{R_n\left(z\right)\right\}_{n=1}^{\infty}$ be a sequence of rational functions, deg $R_n\left(z\right)\leq n$, such that for $z\in G_n$ and $n=1,2,...$ the following inequality holds:

$$\left|u\left(z\right)-R_n\left(z\right)\right|\leq c_6\omega\left[\rho_{1/n},\left(z\right)\right]. \tag{2.4}$$

We define a subsequence $\left\{R_{k^j}\left(z\right)\right\}_{j=1}^{\infty}$ from the approximating sequence $\left\{R_n\left(z\right)\right\}_{n=1}^{\infty}$. We choose natural number p so that the following inequality holds:

$$\rho_{1/k^{p+2}}\left(z\right)\leq 4\delta\leq\rho_{1/k^{p+1}}\left(z\right).$$

The function $u(z)$ can be represented in the form

$$u(z)=\sum_{j=0}^{N-1}V_j\left(z\right)+\left[u(z)-T_{2^N}\left(z\right)\right], \tag{2.5}$$

where

$$V_0\left(z\right)=T_k(z),\qquad V_j\left(z\right)=T_{k^{j+1}}(z)-T_{k^j}(z),\ j=0,...,(N-1).$$

Use of equation (2.5) for $z\in D\left(z_0,\delta\right)$ we have

$$u\left(z\right)-u\left(z_0\right)=\sum_{j=0}^{N-1}V_j\left(z\right)+\left[u(z)-T_{k^N}\left(z\right)\right]-\sum_{j=0}^{N-1}V_j\left(z_0\right)+\left[u(z_0)-T_{k^N}\left(z_0\right)\right]$$

$$\leq c_6\delta+\sum_{j=1}^{N-1}\int\limits_{z_0}^{z}\left|gradV_j\left(\zeta\right)\right|\left|d\zeta\right|+\left|u(z)-T_{k^N}\left(z\right)\right|+\left|u(z_0)-T_{k^N}\left(z_0\right)\right|. \tag{2.6}$$

Consideration of (2.4) leads to

$$\left|u(z_0)-T_{k^N}\left(z_0\right)\right|\leq c_7\omega\left[\rho_{1/k^N}\left(z_0\right)\right]\preccurlyeq\omega\left(\delta\right), \tag{2.7}$$

$$\left|u(z)-T_{k^N}\left(z\right)\right|\leq c_8\omega\left[\rho_{1/k^N}\left(z\right)\right]\preccurlyeq\omega\left(\delta\right). \tag{2.8}$$

We have, for $z\in G_n$

$$\begin{aligned}\left|V_j\left(z\right)\right| &\leq \left|f(z)-T_{k^{j+1}}\left(z\right)\right|+\left|f(z)-T_{k^j}\left(z\right)\right|\\ &\leq c_9\omega\left[\rho_{1/k^{j+1}}\left(z\right)\right]+c_{10}\omega\left[\rho_{\frac{1}{k^j}}\left(z\right)\right]\preccurlyeq\omega\left[\rho_{\frac{1}{k^{j+1}}}\left(z\right)\right].\end{aligned}$$

Then by Lemma 1 for $z\in D\left(z_0,\delta\right)$,

$$\left|gradV_j\left(z\right)\right|\leq c_{11}\frac{\omega\left[\rho_{\frac{1}{k^{j+1}}}\left(z\right)\right]}{\rho_{\frac{1}{k^j}}\left(z\right)}\leq\frac{\omega\left[\rho_{\frac{1}{k^{j+1}}}\left(z\right)\right]}{\left[\rho_{\frac{1}{k^j}}\left(z\right)\right]}\left[1-\frac{\rho_{\frac{1}{k^{j+1}}}\left(z\right)}{\rho_{\frac{1}{k^j}}\left(z\right)}\right].$$

$$\preccurlyeq \frac{\omega\left[\rho_{\frac{1}{k^{j+1}}}(z)\right]}{\left[\rho_{\frac{1}{k^{j}}}(z)\right]^{2}}\left[\rho_{\frac{1}{k^{j}}}(z)-\rho_{\frac{1}{k^{j-1}}}(z)\right] \preccurlyeq \int\limits_{\rho_{\frac{1}{k^{j+1}}}(z)}^{\rho_{\frac{1}{k^{j}}}(z)} \frac{\omega(t)}{t^{2}}dt. \tag{2.9}$$

Taking into account expressions (2.6)- (2.9), we obtain

$$|u(z)-u(z_0)| < c_{12}\delta + \delta \int\limits_{\rho_{\frac{1}{k^{j+1}}}(z)}^{\rho_{\frac{1}{k^{j}}}(z)} \frac{\omega(t)}{t^{2}}dt + c_{13}\omega(\delta) + c_{14}\omega(\delta) \le c_{15}\delta \int\limits_{\delta}^{1} \frac{\omega(t)}{t^{2}}dt,$$

where $0 < \delta < 1/2$. The proof of Theorem 2 is completed.

Using Theorem1 and 2, we obtain Corallary 1.

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