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FACTORIZING A MATRIX QUADRATIC POLYNOMIAL

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Abstract. Let A_1, A_2 lie in $\mathbb{C}^{r\times r}$ and t in $\mathbb{C}$. For the matrix quadratic polynomial,

$$I + A_1 t + A_2 t^2 = (I - \alpha_1 t)\,(I - \alpha_2 t)\,,$$

we give explicit solutions for (α_1, α_2) in $\mathbb{C}^{r\times r}$. We consider two cases: i) A_1 and A_2 commute; ii) A_1 and A_2 do not commute.

1. Introduction

Consider the quadratic equation: $ax^2 + bx + c = 0$, where a, b and c are real numbers. It is well known that its solutions are $x = (1/(2a))\{-b \pm \sqrt{b^2 - 4ac}\}$. Here, we consider the problem of factorizing and inverting a matrix quadratic polynomial (Gohberg *et al.*, 2009). Matrix quadratic polynomials have applications in many areas. We mention: exact analysis of production lines with no intermediate buffers (Papadopoulos and Okelly, 1993); packet delay analysis for cellular digital packet data (Massey and Srinivasan, 1997); analysis of a finite FIFO buffer in an advanced packet-switched network (Krieger *et al.*, 1998); analysis of edge waves (Fu, 2002); truncation and augmentation of level-independent QBD processes (Latouche and Taylor, 2002); MAP/M/c queues with constant impatient time (Choi *et al.*, 2004); analytical study of non-linear transport across a semiconductor-metal junction (Peres, 2009); approximations to quasi-birth-and-death processes (Beuerman and Coyle, 1989; Latouche and Ramaswami, 1993; Bean and Latouche, 2010); analysis of semi-infinite periodic structures (Fallahi and Hafner, 2010); and, dynamic element vibration analysis (Gupta and Lawson, 2010).

Consider the matrix quadratic polynomial:

$$Q(t) = I + A_1 t + A_2 t^2 = (I - \alpha_1 t)\,(I - \alpha_2 t) \tag{1.1}$$

for $t \in \mathbb{C}$, where $I = I_r$ is the identity matrix in $\mathbb{C}^{r\times r}$. We seek α_1, α_2 such that

$$\alpha_1 + \alpha_2 = -A_1,\ \alpha_1\alpha_2 = A_2. \tag{1.2}$$

In fact, the factorization in (1.1) will hold if and only if we have $\alpha_1 + \alpha_2 = -A_1$ and $\alpha_1\alpha_2 = A_2$. In particular, if $A_1 = 0$, we have $\alpha_1^2 = \alpha_2^2 = -A_2$. So, when $A_1 = 0$, the factorization exists if we can obtain a square root for A_2.

Sections 2 and 3 give solutions to (1.1) when A_1, A_2 do and do not commute. To the best of our knowledge, the results in the short note are new and original. Higham and Kim (2000) did consider solutions of $I + A_1 T + A_2 T^2 = 0$ for A_1, A_2, T in $\mathbb{C}^{r\times r}$. However, they did not provide any analytical result except to show that the solutions can be represented as a Schur decomposition, see their Theorem 3.

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Higham and Kim (2000) mainly provided various numerical techniques for solving $I + A_1T + A_2T^2 = 0$. A problem of inverting $Q(t)$ in (1.1) using a matrix power series representation is considered in Section 4.

2. A Solution when A_1, A_2 Commute

Theorem 2.1 gives solutions to (1.1) when A_1, A_2 are $r \times r$ matrices that commute and have the same Jordan block structure. Since they commute they have common left and right eigenvectors, say

$$A_j = L\Lambda_j R^*,$$

where $LR^* = I$.

Theorem 2.1. *If both Λ_j are diagonal (for example, if both A_j have distinct eigenvalues) then (1.1) has solution*

$$\alpha_1, \alpha_2 = L\left(-\Lambda_1 \pm D^{1/2}\right) R^*/2, \tag{2.1}$$

where $D = \Lambda_1^2 - 4\Lambda_2$. Otherwise, we can write

$$\Lambda_j = \text{diag}\left(J_{m_i}\left(\lambda_{ji}\right),\ 1 \le i \le p\right),\ \sum_{i=1}^{p} m_i = r,\ J_m(\lambda) = \lambda I_m + U_m, \tag{2.2}$$

where $(U_m)_{ij} = \delta_{i+1,j}$ and $J_m(\lambda)$ denotes the Jordan blocks, where $\delta_{ai} = 1$ or 0 for $a = i$ or $a \neq i$. Then (1.1) has solution

$$\alpha_1, \alpha_2 = \left(-A_1 \pm \delta^{1/2}\right)/2$$

with

$$\delta = L\Delta R^*,\ \delta^{1/2} = L\Delta^{1/2}R^*, \tag{2.3}$$

where

$$\begin{aligned}
&\Delta = \text{diag}\left(\Delta_{m_i}\left(\lambda_{1i}, \lambda_{2i}\right),\ 1 \le i \le p\right),\\
&\Delta_m\left(\lambda_1, \lambda_2\right) = J_m\left(\lambda_1\right)^2 - 4J_m\left(\lambda_2\right) = a_0 I_m + \epsilon,\\
&\epsilon = a_1 U_m + U_m^2,\ a_0 = \lambda_1^2 - 4\lambda_2,\ a_1 = 2\lambda_1 - 1,
\end{aligned}$$

and

$$\begin{aligned}
&\Delta^{1/2} = \text{diag}\left(\Delta_{m_i}^{1/2},\ 1 \le i \le p\right),\\
&\Delta_m^{1/2} = a_0^{1/2}\left(I_m + a_0^{-1}\epsilon\right)^{1/2} = \sum_{k=0}^{\infty}\binom{1/2}{k} a_0^{1/2-k}\epsilon^k = \sum_{i=0}^{m-1}\gamma_i U_m^i,\\
&\gamma_i = \sum_{i/2 \le k}\binom{1/2}{k} a_0^{1/2-k} a_1^{2k-i}.
\end{aligned}$$

For $\delta^{1/2}$ in (2.3) to exist, one must have $a_0 \neq 0$ for every Jordan block $J_m(\lambda)$ of A_j.

Proof: Since A_1, A_2 commute (1.1) has solution

$$\alpha_1, \alpha_2 = \left(-A_1 \pm \delta^{1/2}\right)/2,\ \delta = A_1^2 - 4A_2.$$

So, if both Λ_j are diagonal then (2.1) follows. Otherwise, (2.2) holds and the Jordan blocks commute since

$$J_m(\lambda_1)\, J_m(\lambda_2) = \lambda_1\lambda_2 I_m + (\lambda_1 + \lambda_2)\, U_m + U_m^2 = J_m(\lambda_2)\, J_m(\lambda_1).$$

So, (2.3) follows. The proof is complete. □

3. A Solution when A_1, A_2 do not Commute

To find a solution in this case, suppose that α_2 has diagonal Jordan form: $\alpha_2 = L\Lambda R^*$, where $\Lambda = \text{diag}\ (\lambda_1, \ldots, \lambda_r)$. By (1.2), we seek $\alpha = \alpha_2 \in \mathbb{C}^{r\times r}$ satisfying $\alpha^2 + A_1\alpha + A_2 = 0$, that is, $L\Lambda^2R^* + A_1 L\Lambda R^* + A_2 = 0$, that is, $L\Lambda^2 + A_1 L\Lambda + A_2 L = 0$. Write $L = (L_{ij})$ and consider first the case when the Jordan form is diagonal, $\Lambda = \text{diag}\ (\lambda_1, \ldots, \lambda_r)$, with $\{\lambda_j\}$ distinct.

Theorem 3.1. *The (i, j) element in $L\Lambda^2R^* + A_1 L\Lambda R^* + A_2 = 0$ is*

$$L_{ij}\lambda_j^2 + A_{1ia}L_{aj}\lambda_j + A_{2ia}L_{aj} = 0,$$

where we use the tensor summation of implicit summation of the repeated subscript a over its range $1, 2, \ldots, r$. That is,

$$M_{ia}(\lambda_j)\, L_{aj} = 0,$$

where

$$M_{ia}(\lambda) = \delta_{ai}\lambda^2 + A_{1ia}\lambda + A_{2ia}.$$

That is, $M(\lambda_j)L = 0$. But $\det\,(L) \neq 0$ *so that $\{\lambda_j\}$ are the roots of* $\det\, M(\lambda) = 0$. *The left hand side is a polynomial of degree $2r$ in λ with roots $\lambda_1, \ldots, \lambda_{2r}$ say. We can choose a subset of r of them, say $\{\lambda_{n_j}\}$ with corresponding right-eigenvectors $\{l_{n_j}\}$ in $N_r = \binom{2r}{r}$ ways. For each of these N_r choices we have a different solution*

$$\Lambda = \text{diag}\,(\lambda_{n_1}, \ldots, \lambda_{n_r}),\ L = (l_{n_1}, \ldots, l_{n_r}),\ R^* = L^{-1}$$

for $\alpha_2 = L\Lambda R^*$, $\alpha_2 = -A_1 - \alpha_1$.

Next, we consider the case when $M(\lambda_j)$ has K_j Jordan blocks, say $J_{m_{jk}}(0) = U_{m_{jk}}$ for $k = 1, \ldots, K_j$ and $j = 1, \ldots, 2r$.

Theorem 3.2. *For each j and $k = 1, \ldots, K_j$, there exists a non-trivial vector l_{jk} such that $M(\lambda_j)l_{jk} = \mathbf{0}$. We have $K = \sum_{j=1}^{2r} K_j$ solutions, $\binom{\lambda}{l} = \binom{\lambda_j}{l_{jk}}$, to $M(\lambda)l = \mathbf{0}$. We can choose r of them to form $L \in \mathbb{C}^{r\times r}$ in $\binom{K}{2r}$ ways. This then is the number of solutions α_2 to (1.2). The case $K = 2r$ only holds if $K_j \equiv 1$, that is, if for $j = 1, \ldots, 2r$, $M(\lambda_j)$ has only one Jordan block with an eigenvalue zero.*

In Theorem 3.1, we have assumed that $M(\lambda_j)l_j = 0$ holds for $j = 1, \ldots, 2r$ with each l_j a non-zero vector in $\mathbb{C}^r$. In Theorem 3.2, we have assumed that the vectors l_{jk}, $k = 1, \ldots, K_j$ are linearly independent for each j.

Example 3.1. *Suppose that $r = 2$. Then $\lambda_1, \ldots, \lambda_4$ are the roots of*

$$0 = \det\, M(\lambda) = \lambda^4 + \lambda^3\text{trace}\,(A_1) + \lambda^2\,(\det A_1 + \text{trace}\,(A_2)) + \lambda\,[A_1, A_2] + \det A_2,$$

where $[A_1, A_2] = A_{111}B_{222} - A_{112}B_{221} - A_{121}B_{212} + A_{122}B_{211}$.

4. Inversion of $Q(t)$

Here, we consider inverting $Q(t)$ given by (1.1). For example, if

$$\sum_{n=0}^{\infty} x_n t^n = Q(t)^{-1} = \left(I - \alpha_2 t\right)^{-1} \left(I - \alpha_1 t\right)^{-1} \tag{4.1}$$

then

$$x_n = \sum_{n_1+n_2=n} \alpha_2^{n_2} \alpha_1^{n_1} = \sum_{n_2=0}^{n} \alpha_2^{n_2} \alpha_1^{n-n_2}. \tag{4.2}$$

So, if α_i commute then

$$x_n = \left(\alpha_1 - \alpha_2\right)^{-1} \left(\alpha_1^{n+1} - \alpha_2^{n+1}\right).$$

More generally, if $Q(t)$ is a polynomial, how can the partial fraction expansion for its inverse be obtained when $r > 1$? For $r = 1$ see, for example, Gradshteyn and Ryzhik (2000).

By (1.1), $Q(t) = F_1(t)F_2(t)$, where $F_i(t) = I_r - \alpha_i t$, so that $Q(t)^{-1} = F_2(t)^{-1}F_1(t)^{-1}$, and x_n is given by (4.2). We can get a compact solution if we can find matrices X, Y such that

$$Q(t)^{-1} = XF_1(t)^{-1} + F_2(t)^{-1}Y. \tag{4.3}$$

This would give the coefficient of t^n in (4.1) simply as

$$x_n = X\alpha_1^n + \alpha_2^n Y. \tag{4.4}$$

Multiplying (4.3) on the left by $F_2(t)$ and on the right by $F_1(t)$ gives $I_r = F_2(t)X + YF_1(t)$, that is, $X + Y = I_r$, $\alpha_2 X + Y\alpha_1 = 0$, that is,

$$\alpha_2 X + \left(I_r - X\right)\alpha_1 = 0,$$

that is, $X\alpha_1 - \alpha_2 X = \alpha_1$. We now show how to solve the more general equation

$$XA + BX = C, \tag{4.5}$$

where $A, B, C, X \in \mathbb{C}^{r\times r}$, using the vec operator. Write

$$\begin{aligned}
&X = \left(x_1, \ldots, x_r\right), \ A = \left(a_1, \ldots, a_r\right), \ C = \left(c_1, \ldots, c_r\right), \\
&\text{vec } X = x = \left(x_1', \ldots, x_r'\right)', \ c' = \left(c_1', \ldots, c_r'\right) = \left(\text{vec } \alpha_1\right)' \in \mathbb{C}^{r^2}, \\
&Xa_j = \left(\sum_{k=1}^{r} X_{ik} A_{kj}\right) = E_j \text{vec } X, \ E_j \in \mathbb{C}^{r\times r^2}, \ E = \left(E_1', \ldots, E_r'\right)' \in \mathbb{C}^{r^2\times r^2}.
\end{aligned}$$

Given $a \in \mathbb{C}^r$, we can write

$$(Xa)_i = \sum_{k=1}^{r} X_{ik} a_k = f_i' \text{ vec } X,$$

where $f_i' = \left(a_1 e_i', \ldots, a_r e_i'\right) \in \mathbb{C}^{1\times r^2}$ and e_i is the ith unit vector in $\mathbb{C}^r$. This proves Theorem 3.1.

Lemma 4.1. *Let e_i be the ith unit vector in $\mathbb{C}^r$. For $a \in \mathbb{C}^r$, $X \in \mathbb{C}^{r\times r}$,*

$$Xa = F(a)'\text{vec } X,$$

where $F(a) = \left(f_1, \ldots, f_r\right) = \left(a_i e_j\right) \in \mathbb{C}^{r^2\times r}$.

Lemma 4.2. *Set* $G_i = F(a_i)'$, $G(A) = (G_1', \ldots, G_r')'$. *Then,*

$$XA = (Xa_1, \ldots, Xa_r) = (G_1 x, \ldots, G_r x)\,, \text{ vec } (XA) = G(A) \text{ vec } X.$$

Lemma 4.3. *Let* $BX = (Bx_1, \ldots, Bx_r)$. *Then,*

$$\text{vec } (BX) = d(B) \text{vec } X,$$

where $d(B) = \text{diag}(B, \ldots, B)$.

Theorem 4.1. *A solution of (4.5) is given by*

$$\text{vec } X = [G(A) + d(B)]^{-1} \text{vec } C, \ Y = I_r - X. \tag{4.6}$$

Proof: Taking vec of (4.5) gives $[G(A) + d(B)]\text{vec } X = c$. □

Golub and Van Loan (1996, page 372) suggest a different type of solution when $X \in \mathbb{C}^{m \times n}$ and A, B are square with diagonal Jordan forms.

Consider arbitrary $x, c \in \mathbb{C}^s$, $G \in \mathbb{C}^{s \times s}$ for some $s \geq 1$. Does $Gx = c$ have a solution if $\det(G) = 0$? The answer is yes, provided that a certain condition holds, but the solution is not unique. Let $G = L\Lambda R^*$ be its Jordan form and set $y = R^* x$. Consider the case when this Jordan form is diagonal. Then we can write $\Lambda = \text{diag}$ $(\mathbf{0}, \Lambda_2)$, $\Lambda y = \binom{0}{\Lambda_2 y_2} = d = \binom{d_1}{d_2}$, where $\mathbf{0}$ is square, and $d = R^* c$. So, a solution exists provided that $d_1 = \mathbf{0}$, and in this case y_1 is arbitrary, and $y_2 = \Lambda_2^{-1} d_2$. This can be applied to (4.6) with $s = r^2$, $G = G(A) + d(B)$.

Corollary 4.1. *A solution of (4.3) is given by (4.6) with* $A = C = \alpha_1$, $B = \alpha_2$, $Y = I_r - X$. *So, a solution of (4.1) is given by (4.4).*

An extension of (1.1) to a general polynomial: It is not clear how this can be achieved for $r > 1$. If $r = 1$, $\{\alpha_i\}$ are distinct, and $Q(t) = \prod_{i=1}^{I} (1 - \alpha_i t)$. Then, by Gradshteyn and Ryzhik (2000),

$$Q(t)^{-1} = \sum_{i=1}^{I} H_i \left(1 - \alpha_i t\right)^{-1},$$

where

$$H_i = \prod_{j \neq i} \left(1 - \alpha_j / \alpha_i\right).$$

However, even for $I = 2$, it is not clear how to solve

$$Q(t)^{-1} = \sum_{i=1}^{I} H_i \left(I_r - \alpha_i t\right)^{-1}$$

for $\{H_i\}$.

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