

## A PARAMETER-UNIFORM NUMERICAL METHOD FOR A SYSTEM OF SINGULARLY PERTURBED BOUNDARY VALUE PROBLEMS

DEVENDRA KUMAR

(Received September 11, 2012)

**Abstract.** A parameter-uniform numerical method based on Shishkin mesh is constructed and analyzed for a weakly coupled system of singularly perturbed second order reaction-diffusion equations. A B-spline collocation method is combined with a piecewise-uniform Shishkin mesh. An asymptotic error bound (independent of the perturbation parameter  $\varepsilon$ ) in the maximum norm is established theoretically. To illustrate the theoretical results, two test problems have been carried out.

### 1. Introduction

In this paper, we consider a numerical method for a system of two weakly coupled singularly perturbed ordinary differential equations. These systems of equations have applications in fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, gas porous electrodes theory, predator-prey population dynamics etc. Asymptotic analysis of these problems can describe the layer structure of the solution for small values of  $\varepsilon$ . It is well-known fact that the solution of these problems exhibit a multiscale character *i.e.*, there are regions in which solution change rapidly and other than these regions solution behaves uniformly and varies slowly. Therefore, the classical numerical methods fail to produce good approximations to the exact solutions for these problems. There are a good number of papers on non-classical methods which cover scalar second order ordinary differential equations. But only a few authors have developed numerical methods for vector singularly perturbed order ordinary differential equations. Some of these can be find in [2, 3, 8, 12, 14] and the references therein. Matthews *et al.* [10] provided a method for the numerical solution of system of equations of the form (1a)-(1b) on Shishkin mesh using classical finite difference scheme. Madden and Stynes [7] presented a uniformly convergent numerical method for system of reaction-diffusion BVPs. Linß and Madden [5] studied the system of reaction-diffusion problems and obtained numerical solution on Shishkin mesh.

Within the literature on the asymptotic solution of singularly perturbed vector problems of second order, there are few papers dealing with strongly coupled systems where the first derivatives are coupled. The design and analysis of appropriate numerical methods for singularly perturbed differential equations is an area of current interest (see for example [1, 6, 12, 16, 17] and the references therein).

---

2010 *Mathematics Subject Classification* 65L10, 65L11, 65L20, 65L60, 65D07.

*Key words and phrases:* Singular perturbation, Weakly coupled system, Boundary value problems, Boundary layer, B-spline collocation method.

The advent of Shishkin meshes has fueled significant advances into the broad area of singularly perturbed differential equations. However, there has been little work done on parameter-robust numerical methods for systems of singularly perturbed differential equations.

The parameter multiplying the highest derivatives characterize the diffusion coefficient of the substances. It is well-known [1,12,17] that standard numerical methods are not appropriate for singularly perturbed problems, so we are interested in parameter-uniform numerical methods, whose numerical rate of convergence is independent of the singular perturbation parameter  $\varepsilon$ . Parameter-uniform numerical methods for a scalar reaction-diffusion equation have been examined extensively in the literature (see, for example [12,15,17] and the references therein).

We shall examine a parameter-uniform numerical method that combine B-spline collocation with special piecewise-uniform mesh. This paper extends the results for piecewise-uniform meshes applied to a single one-dimensional singularly perturbed reaction-diffusion problem to a system of two coupled reaction-diffusion problems.

The appropriate norm for studying the convergence of numerical solutions to the exact solution of a singular perturbation problem is the following maximum norm

$$\|\Psi\| = \max_{\Omega} |\Psi(x)|, \quad \|\vec{\Psi}^N\| = \max_i \|\Psi_i\|, \quad \vec{\Psi} = (\Psi_1, \Psi_2, \dots, \Psi_m).$$

A sequence of numerical approximations  $\{\vec{U}^N = (U_1^N, U_2^N, \dots, U_m^N)\}$  is said to be convergent, for a system of  $m$  singularly perturbed boundary value problems if an error bound of the form

$$\max_{1 \leq i \leq m} \|u_i - \bar{U}_i^N\| \leq CN^{-p}, \quad p > 0,$$

holds, where  $\vec{u} = (u_1, u_2, \dots, u_m)$  is the solution of the continuous system and  $\bar{U}_i^N$  is the linear interpolant of the nodal values  $\{U_i^N(x_j)\}_{j=0}^N$  generated from the numerical method. Numerical approximations are said to be parameter-uniformly convergent, for a system of singularly perturbed problems if the error constants  $C$  and  $p$  are independent of the singular perturbation parameters and the mesh dimension  $N$ .

Parameter-uniform numerical methods, based on piecewise-uniform meshes, for systems of singularly perturbed reaction-diffusion problems were examined theoretically by Shishkin in [18]. However, no numerical results were provided there in [18]. In the case of a system of two singularly perturbed reaction-diffusion problems, with diffusion coefficients  $\varepsilon_1, \varepsilon_2$ , the following three separate cases were identified in [18]:

Case (i)  $\varepsilon_1 = \varepsilon_2 = \varepsilon$

Case (ii)  $\varepsilon_1 = \varepsilon, \varepsilon_2 = 1$  and

Case (iii)  $\varepsilon_1, \varepsilon_2$  are arbitrary.

The first case was examined by Matthews in [11], where the parameter-uniform convergence of essentially first order was established. An analogous result for the second case was established by Matthews in [9]. The third case was examined by Maddan and Stynes in [7], where first order uniform convergence was established.

In this paper, we consider the following two-point boundary value problem for the singularly perturbed system of ordinary differential equations

$$L_\varepsilon \vec{u}(x) \equiv \begin{pmatrix} -\varepsilon \frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon \frac{d^2}{dx^2} \end{pmatrix} \vec{u}(x) + A(x) \vec{u}(x) = \vec{f}(x), \quad x \in \Omega = (0, 1), \quad (1a)$$

$$\vec{u}(0) = \vec{u}_0, \quad \vec{u}(1) = \vec{u}_1, \quad (1b)$$

where  $\varepsilon$  is a small positive parameter,

$$\vec{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \quad A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}, \quad \vec{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}.$$

The functions  $a_{11}(x)$ ,  $a_{12}(x)$ ,  $a_{21}(x)$ ,  $a_{22}(x)$ ,  $f_1(x)$ ,  $f_2(x)$  are sufficiently smooth and satisfying the following conditions:

$$a_{12}(x) \leq 0, \quad a_{21}(x) \leq 0, \quad a_{11}(x) > |a_{12}(x)|, \quad a_{22}(x) > |a_{21}(x)|, \quad \forall x \in \bar{\Omega}. \quad (1c)$$

In general the problem (1a)–(1b) possess two boundary layer regions each of width  $O(\sqrt{\varepsilon})$ . The conditions (1c) are used to derive the standard maximum principle.

Now we shall give some preliminary results where we use  $C$  as a generic positive constant independent of the mesh sizes and the perturbation parameter  $\varepsilon$ .

**Lemma 1.1.** (*Maximum Principle*) Assuming (1a)–(1c); then if  $\vec{\psi}(0) \geq \vec{0}$ ,  $\vec{\psi}(1) \geq \vec{0}$  and  $L_\varepsilon \vec{\psi}(x) \geq \vec{0}$ ,  $\forall x \in \Omega$  then  $\vec{\psi}(x) \geq \vec{0}$ ,  $\forall x \in \bar{\Omega}$ .

**Proof.** Let  $\vec{\psi}(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$  and  $\vec{\psi}(x) \not\geq \vec{0}$  for some  $x \in \bar{\Omega}$ . Without loss of generality assume  $\psi_1(z) = \min_{x \in [0,1]} \psi_1(x) < 0$  and  $\psi_1(z) \leq \psi_2(z)$ . Note that  $z \neq 0, 1$ . The first component of  $L_\varepsilon \vec{\psi}(z)$  is

$$-\varepsilon \psi_1''(z) + a_{11}\psi_1(z) + a_{12}\psi_2(z) = -\varepsilon \psi_1''(z) + (a_{11} + a_{12})\psi_1(z) + a_{12}(\psi_2(z) - \psi_1(z)) < 0.$$

Thus we have a contradiction and hence  $\vec{\psi}(x) \geq \vec{0}$ ,  $\forall x \in \bar{\Omega}$ .  $\square$

**Lemma 1.2.** (*Stability Lemma*) Let  $\vec{u}(x)$  be the solution of (1a)–(1b). Then  $\vec{u}(x)$  satisfies the following stability bound

$$\|\vec{u}\| \leq \frac{1}{\gamma} \|\vec{f}\| + \|\vec{u}(0)\| + \|\vec{u}(1)\|,$$

where  $\gamma = \min_{x \in \Omega} \{a_{11}(x) + a_{12}(x), a_{21}(x) + a_{22}(x)\}$ .

**Proof.** Define two barrier functions  $\vec{\psi}^\pm(x) = \frac{1}{\gamma} \|\vec{f}\| + \|\vec{u}(0)\| + \|\vec{u}(1)\| \pm \vec{u}(x)$ , then it is easy to see  $\vec{\psi}^\pm(0) \geq 0$ ,  $\vec{\psi}^\pm(1) \geq 0$  and  $L_\varepsilon \vec{\psi}^\pm(x) \geq 0$ . A consequence of Lemma 1.1 gives the required estimate.  $\square$

The following lemma gives the classical bounds on the seminorms of the solution of the system, in terms of powers of  $\varepsilon$ .

**Lemma 1.3.** For  $k = 1, 2$ , we have the following bounds:

$$|u_i|_k \leq C(1 + \varepsilon^{-k/2}), \quad |u_i|_{k+2} \leq C\varepsilon^{-(k+2)/2}, \quad i = 1, 2.$$

**Proof.** The proof can be obtained by using Lemma 1.2 and applying the arguments given in [13].  $\square$

The sharper bounds on the derivatives of the solution can be obtained by decomposing the solution  $\vec{u}(x)$  of (1a)-(1b) into the smooth and singular components as  $\vec{u}(x) = \vec{v}(x) + \vec{w}_l(x) + \vec{w}_r(x)$ , where  $\vec{v}(x)$  is the smooth component and  $\vec{w}_l(x)$  and  $\vec{w}_r(x)$  are the left and right singular components respectively.

**Lemma 1.4.** *The smooth and singular components and their derivatives satisfy the following estimates*

$$\begin{aligned} \|v_i^{(k)}\| &\leq C(1 + \varepsilon^{\frac{2-k}{2}}), \quad 0 \leq k \leq 3, \quad i = 1, 2, \\ |w_{l,i}^{(k)}(x)| &\leq C\varepsilon^{-k/2} \exp(-x\sqrt{\gamma/\varepsilon}), \quad x \in \bar{\Omega}, \quad i = 1, 2, \\ |w_{r,i}^{(k)}(x)| &\leq C\varepsilon^{-k/2} \exp(-(1-x)\sqrt{\gamma/\varepsilon}), \quad x \in \bar{\Omega}, \quad i = 1, 2. \end{aligned}$$

## 2. Discrete Problem

Since on equidistant meshes no classical method can attain convergence at all mesh points uniformly in  $\varepsilon$ , unless one uses an unacceptably large number of grid points. Therefore, unless we use a specially chosen mesh, we shall not be able to get  $\varepsilon$ -uniform convergence at all the mesh points. The simplest possible non-uniform mesh, namely a piecewise-uniform mesh proposed by Shishkin is sufficient for the construction of an  $\varepsilon$ -uniform method. Shishkin mesh is fine near layers but coarser otherwise.

The fitted piecewise uniform mesh  $\bar{\Omega}^N$  is constructed by dividing  $\bar{\Omega}$  into three non overlapping subintervals  $[0, \sigma]$ ,  $(\sigma, 1 - \sigma)$  and  $[1 - \sigma, 1]$ . The transition parameter  $\sigma$  is chosen to be  $\sigma = \min\{1/4, \sigma_0\sqrt{\varepsilon} \ln N\}$ , where  $\sigma_0 \geq 1/\sqrt{\gamma}$ . In the numerical results considered in this paper we have taken  $\sigma_0 = 1/\sqrt{\gamma}$ . We place  $N/4$  grid points in each of the subintervals  $[0, \sigma]$  and  $[1 - \sigma, 1]$  which are the subdomains where the solution takes high gradients and  $N/2$  grid points are placed in the subinterval  $(\sigma, 1 - \sigma)$ . It is assumed that  $N$  is a multiple of 4, which guarantees that there is at least one point in the boundary layer regions. Then we have  $\bar{\Omega}^N = \{x_i\}_{i=0}^N$ , where

$$x_i = \begin{cases} ih_i, & i = 0, 1, \dots, N/4, \\ \sigma + (i - N/4)h_i, & i = N/4 + 1, \dots, 3N/4, \\ 1 - \sigma + (i - 3N/4)h_i, & i = 3N/4 + 1, \dots, N, \end{cases}$$

where

$$h_i = \begin{cases} 4\sigma/N, & i = 1, 2, \dots, N/4, \\ 2(1 - 2\sigma)/N, & i = N/4 + 1, N/4 + 2, \dots, 3N/4, \\ 4\sigma/N, & i = 3N/4 + 1, 3N/4 + 2, \dots, N. \end{cases}$$

Thus the piecewise-uniform mesh spacing  $h$  is  $4\sigma/N$  for the interval  $[0, \sigma]$  and  $[1 - \sigma, 1]$  while  $2(1 - 2\sigma)/N$  for the interval  $(\sigma, 1 - \sigma)$ .

The B-spline collocation method described in [4] for scalar problems, can be extended to the vector problems considered in this paper with the same order of convergence  $O(N^{-2} \ln^3 N)$ . The only difference is that in [4], we have used a single B-spline approximation, here we use two distinct approximations corresponding to each solutions  $u_1$  and  $u_2$ . Thus we have the following  $\varepsilon$ -uniform error estimate

**Theorem 2.1.** *Let  $S_1(x)$  be the collocation approximation from the space of cubic splines  $\phi_3(\bar{\Omega}^N)$  to the solution  $u_1(x)$  and let  $S_2(x)$  be the collocation approximation from the space of cubic splines  $\phi_3(\bar{\Omega}^N)$  to the solution  $u_2(x)$  of the boundary value*

problem (1a)–(1b). If  $f_1, f_2 \in C^2[0, 1]$ , then the parameter-uniform error estimate is given by

$$\sup_{0 < \varepsilon \leq 1} \max_{0 \leq j \leq N} |u_i(x_j) - S_i(x_j)| \leq CN^{-2} \ln^3 N, \quad i = 1, 2.$$

### 3. Numerical Experiments and Conclusion

A numerical method is  $\varepsilon$ -uniform of order  $(p_1, p_2)$  on a mesh  $\bar{\Omega}^N$  if  $|u_i - U_i^N| \leq CN^{-p_i}$ ,  $i = 1, 2$ , where  $u_i$  is the  $i$ th component of the solution to the continuous problem, and  $U_i^N$  is the  $i$ th component of the solution to the discrete problem,  $C$  and  $p_i > 0$  are constants independent of  $\varepsilon$  and  $N$ . If the exact solution of the problem is unknown the pointwise error is determined using the double mesh principle i.e.,  $E_{\varepsilon,i}^N = \|U_i^N - U_i^{2N}\|_{\bar{\Omega}^N}$ ,  $i = 1, 2$ .

The computed maximum pointwise errors  $E_{\varepsilon}^N = \max_{i=1,2} E_{\varepsilon,i}^N$  are given in Tables 1 and 2. Also for each value of  $N$  the  $\varepsilon$ -uniform error  $E^N = \max_{\varepsilon=2^{-2}, 2^{-4}, \dots, 2^{-28}} E_{\varepsilon}^N$  and the numerical rate of convergence  $p^N$  defined by  $p^N = \log_2 \frac{E^N}{E^{2N}}$  are given in Table 3.

**Example 3.1.** Consider the following system of SPBVP [11]:

$$\begin{pmatrix} -\varepsilon \frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon \frac{d^2}{dx^2} \end{pmatrix} \vec{u}(x) + \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \vec{u}(x) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad x \in (0, 1),$$

$$\vec{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \vec{u}(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

**Example 3.2.** Consider the following system of SPBVP:

$$\begin{pmatrix} -\varepsilon \frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon \frac{d^2}{dx^2} \end{pmatrix} \vec{u}(x) + \begin{pmatrix} 4 & -2 \\ -1 & 3 \end{pmatrix} \vec{u}(x) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x \in (0, 1),$$

$$\vec{u}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \vec{u}(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We also compute the constant in the error estimate, i.e., since we have the theoretical error bound  $E^N \leq CN^{-2} \ln^3 N$  from Theorem 2.1 we approximate the constant in the error estimate by  $C^N = E^N N^2 / \ln^3 N$  and these values are given in Table 3.

It is clear that the method is parameter-uniform for these problems with the errors for each  $N$  stabilizing as  $\varepsilon \rightarrow 0$ . For the given examples, the graphs of the solutions  $u_1$  and  $u_2$  are given in Figures 1-2 for different values of  $\varepsilon$  using  $N = 16$ . The error distributions in the solutions are given in Figures 3-4. From the graphs it is clear that the error is maximum near the transition point and it is almost zero in outer region. One can see our numerical experiments produced practically the same results as given in the Theorem 2.1, which confirm that the method is parameter-uniform.

TABLE 1.  $E_\epsilon^N$  for Example 3.1

$\epsilon$	Number of mesh points $N$						
	16	32	64	128	256	512	1024
$2^{-2}$	8.325E-04	2.075E-04	5.184E-05	1.296E-05	3.239E-06	8.098E-07	2.025E-07
$2^{-4}$	2.349E-03	5.807E-04	1.448E-04	3.617E-05	9.041E-06	2.260E-06	5.651E-07
$2^{-6}$	8.872E-03	2.156E-03	5.358E-04	1.337E-04	3.341E-05	8.351E-06	2.088E-06
$2^{-8}$	<b>1.884E-02</b>	6.473E-03	2.154E-03	5.355E-04	1.336E-04	3.339E-05	8.346E-06
$2^{-10}$	1.884E-02	<b>6.488E-03</b>	2.319E-03	7.879E-04	<b>2.569E-04</b>	8.121E-05	<b>2.506E-05</b>
$2^{-12}$	1.879E-02	6.484E-03	<b>2.320E-03</b>	<b>7.880E-04</b>	2.569E-04	8.121E-05	2.506E-05
$2^{-14}$	1.877E-02	6.478E-03	2.319E-03	7.880E-04	2.569E-04	<b>8.122E-05</b>	2.506E-05
$2^{-16}$	1.876E-02	6.475E-03	2.319E-03	7.879E-04	2.569E-04	8.122E-05	2.506E-05
$2^{-18}$	1.876E-02	6.475E-03	2.319E-03	7.879E-04	2.569E-04	8.121E-05	2.506E-05
$2^{-20}$	1.876E-02	6.474E-03	2.319E-03	7.879E-04	2.569E-04	8.121E-05	2.506E-05
$2^{-22}$	1.876E-02	6.474E-03	2.319E-03	7.879E-04	2.569E-04	8.121E-05	2.506E-05
$2^{-24}$	1.876E-02	6.474E-03	2.319E-03	7.879E-04	2.569E-04	8.121E-05	2.506E-05
$2^{-26}$	1.876E-02	6.474E-03	2.319E-03	7.879E-04	2.569E-04	8.121E-05	2.506E-05
$2^{-28}$	1.876E-02	6.474E-03	2.319E-03	7.879E-04	2.569E-04	8.121E-05	2.506E-05

TABLE 2.  $E_\epsilon^N$  for Example 3.2

$\epsilon$	Number of mesh points $N$						
	16	32	64	128	256	512	1024
$2^{-2}$	5.468E-04	1.363E-04	3.404E-05	8.510E-06	2.127E-06	5.318E-07	1.330E-07
$2^{-4}$	1.550E-03	3.830E-04	9.549E-05	2.387E-05	5.967E-06	1.492E-06	3.729E-07
$2^{-6}$	5.703E-03	1.424E-03	3.552E-04	8.850E-05	2.212E-05	5.529E-06	1.382E-06
$2^{-8}$	8.185E-03	2.309E-03	7.827E-04	2.621E-04	8.500E-05	2.211E-05	5.526E-06
$2^{-10}$	8.611E-03	3.134E-03	9.383E-04	2.662E-04	8.558E-05	2.696E-05	8.306E-06
$2^{-12}$	1.045E-02	4.294E-03	1.376E-03	3.900E-04	1.063E-04	2.859E-05	8.313E-06
$2^{-14}$	1.105E-02	4.871E-03	1.693E-03	4.943E-04	1.334E-04	3.558E-05	9.509E-06
$2^{-16}$	1.120E-02	5.066E-03	1.862E-03	5.791E-04	1.586E-04	4.132E-05	1.081E-05
$2^{-18}$	1.124E-02	5.119E-03	1.921E-03	6.267E-04	1.812E-04	4.759E-05	1.210E-05
$2^{-20}$	1.125E-02	5.132E-03	1.938E-03	6.439E-04	1.945E-04	5.367E-05	1.370E-05
$2^{-22}$	1.125E-02	5.136E-03	1.942E-03	6.487E-04	1.993E-04	5.732E-05	1.534E-05
$2^{-24}$	1.125E-02	<b>5.137E-03</b>	<b>1.943E-03</b>	6.499E-04	2.007E-04	5.867E-05	1.634E-05
$2^{-26}$	<b>1.126E-02</b>	5.137E-03	1.943E-03	6.502E-04	<b>2.011E-04</b>	5.906E-05	1.672E-05
$2^{-28}$	1.126E-02	5.137E-03	1.943E-03	<b>6.503E-04</b>	2.011E-04	<b>5.916E-05</b>	<b>1.682E-05</b>

TABLE 3.  $E^N, p^N$  and  $C^N$  for Examples 3.1 and 3.2

$N$	Example 3.1			Example 3.2		
	$E^N$	$p^N$	$C^N$	$E^N$	$p^N$	$C^N$
16	1.884E-02	1.54	0.23	1.126E-02	1.13	0.14
32	6.488E-03	1.48	0.16	5.137E-03	1.40	0.13
64	2.320E-03	1.56	0.13	1.943E-03	1.60	0.11
128	7.880E-04	1.62	0.11	6.503E-04	1.70	0.10
256	2.569E-04	1.66	0.10	2.011E-04	1.80	0.08
512	8.122E-05	1.70	0.09	5.916E-05	1.80	0.06
1024	2.506E-05	—	0.09	1.682E-05	—	0.05

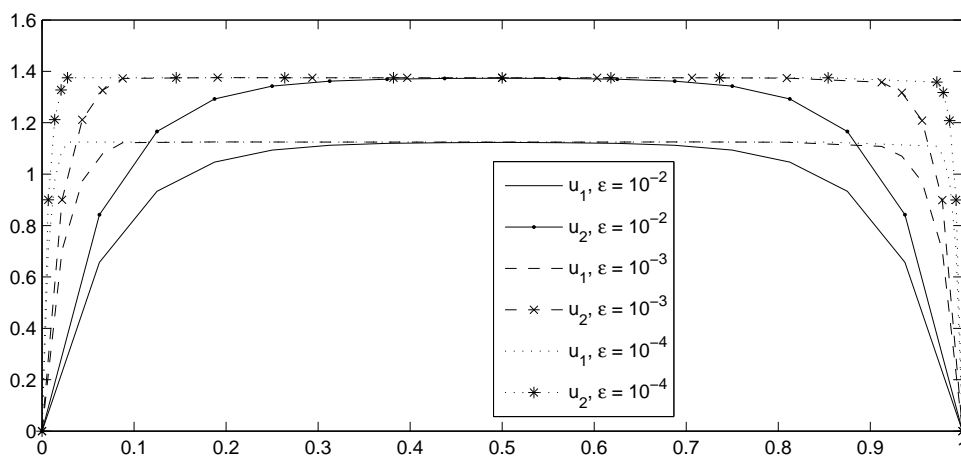


FIGURE 1. Approximate solutions for Example 3.1 for different values of  $\varepsilon$

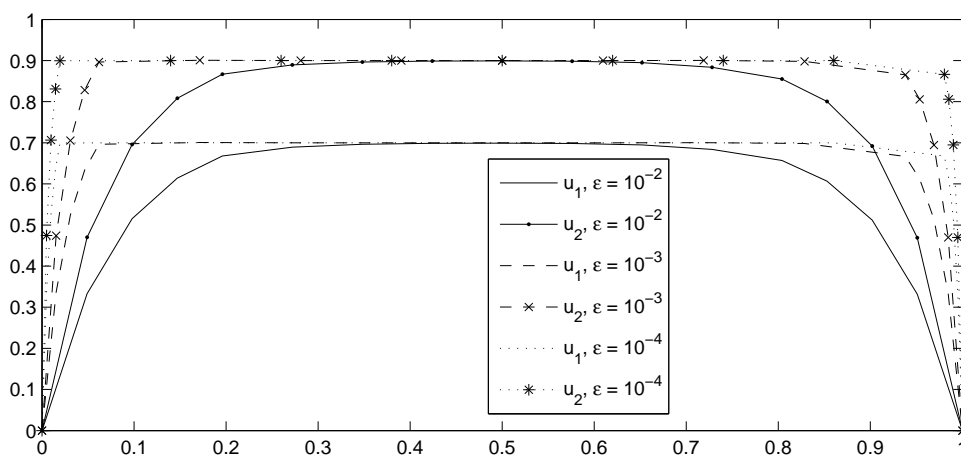


FIGURE 2. Approximate solutions for Example 3.2 for different values of  $\varepsilon$



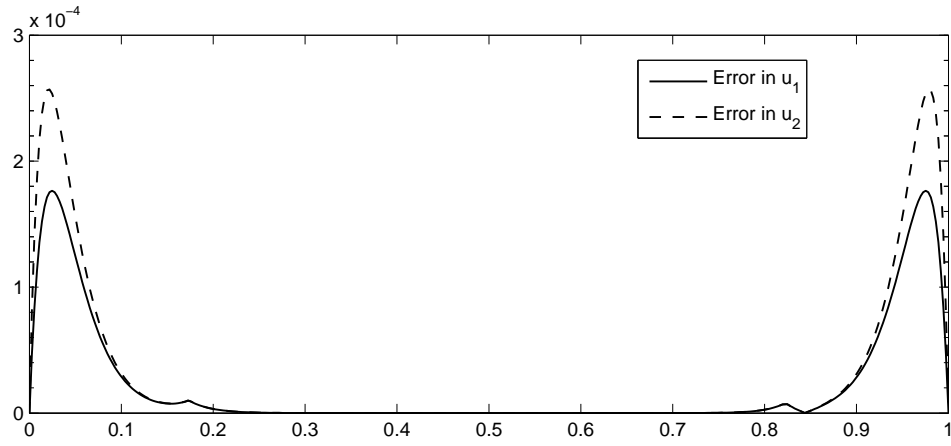


FIGURE 3. Error distribution for Example 3.1 for  $\varepsilon = 10^{-3}$  using  $n = 256$

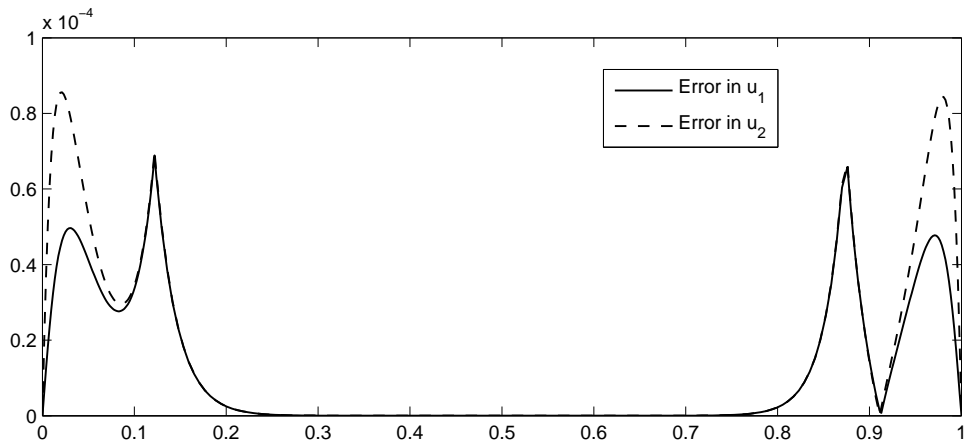


FIGURE 4. Error distribution for Example 3.2 for  $\varepsilon = 10^{-3}$  using  $n = 256$

## References

- [1] P. A. Farrell, A. F. Hegarty, J. J. H. Miller, E. O’Riordan and G. I. Shishkin, *Robust Computational Techniques for Boundary Layers*. Chapman & Hall/CRC, Boca Raton, 2000.
- [2] M. G. Gasparo and M. Macconi, *Initial-value methods for second order singularly perturbed boundary value problems*. J. Optim. Theory Appl., **66** (1990), 197–210.
- [3] M. G. Gasparo and M. Macconi, *Parallel initial-value algorithms for singularly perturbed boundary value problems*. J. Optim. Theory Appl., **73** (1992), 501–517.
- [4] D. Kumar and M. K. Kadalbajoo, *A parameter-uniform numerical method for time-dependent singularly perturbed differential-difference equations*. Appl. Math. Model., **35** (2011), 2805–2819.
- [5] T. Linß and N. Madden, *Accurate solution of a system of coupled singularly perturbed reaction-diffusion equations*. Computing., **73** (2004), 121–133.
- [6] T. Linß, *Layer-adapted meshes for convection-diffusion problems*. Comput. Methods Appl. Mech. Engrg., **192** (2003), 1061–1105.
- [7] N. Madden and M. Stynes, *A uniformly convergent numerical method for a coupled system of two singularly perturbed linear reaction-diffusion problems*. IMA J. Numer. Anal., **23** (2003), 627–644.
- [8] R. E. O’Malley Jr., *Singular Perturbation Methods for Ordinary Differential Equations*. Springer-Verlag, New York, 1990.
- [9] S. Matthews, *Parameter robust numerical methods for a system of two coupled singularly perturbed reaction-diffusion equations*. Master Thesis, School of Mathematical Sciences, Dublin City University, 2000.
- [10] S. Matthews, E. O’Riordan and G. I. Shishkin, *A numerical method for a system of singularly perturbed reaction-diffusion equations*. J. Comput. Appl. Math., **145** (2002), 151–166.
- [11] S. Matthews, J. J. H. Miller, E. O’Riordan and G. I. Shishkin, *Parameter-robust numerical methods for a system of reaction-diffusion problems with boundary layers*. in: J.J.H. Miller, G.I. Shishkin, L. Vulkoe (Eds.), *Analytical and Numerical Methods for Convection-Dominated and Singularly Perturbed Problems*, Nova Science Publishers, Inc, New York, USA, pp. 219–224 (2000).
- [12] J. J. H. Miller, E. O’Riordan and G. I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems*. World Scientific, Singapore, 1996.
- [13] J. J. H. Miller, E. O’Riordan and G. I. Shishkin, *Fitted mesh methods for the singular perturbation reaction diffusion problem*, in *Proceedings of the Fifth International Colloquium on Numerical Analysis*. 1996, Plovdiv, Bulgaria, Academic Publications, pp. 99–105 (1997).
- [14] C. E. Pearson, *On a differential equation of boundary layer type*. J. Math. Phys., **47** (1968), 134–154.
- [15] E. O’Riordan and M. Stynes, *A uniformly accurate finite-element method for a singularly perturbed one-dimensional reaction-diffusion problem*. Math. Comput. **47** (1986), 555–570.
- [16] H. G. Roos, *Layer-adapted grids for singular perturbation problems*. Z. Angew. Math. Mech., **78** (1998), 291–309.

- [17] H. G. Roos, M. Stynes and L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations*. Springer, New York, 1996.
- [18] G. I. Shishkin, *Mesh approximation of singularly perturbed boundary-value problems for systems of elliptic and parabolic equations*. Comput. Maths. Math. Phys. **35** (1995), 429–446.

Devendra Kumar  
Department of Mathematics  
Birla Institute of Technology & Science,  
Pilani-333031, India  
dkumar2845@gmail.com