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MULTIPLICATION MODULES AND HOMOGENEOUS IDEALIZATION IV

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Abstract. All rings are commutative with identity and all modules are unital. Let R be a ring, M an R-module and $R(M)$, the idealization of M. Homogeneuous ideals of $R(M)$ have the form $I(+)N$ where I is an ideal of R, N a submodule of M such that $IM \subseteq N$. In particular, $[N : M](+)N$ is a homogeneous ideal of $R(M)$. The purpose of this paper is to investigate how properties of the ideal $[N : M](+)N$ are related to those of N. We determine when $R(M)$ is a μ-ring, strongly Laskerin ring, Hilbert ring or satisfies Property (U) or Property (FU). It is also shown that if all homogeneous ideals of $R(M)$ have a certain prescribed property, then all ideals of $R(M)$ have the same property.

1. Introduction

Let R be a commutative ring and M an R-module. M is a *multiplication* module if every submodule N of M has the form IM for some ideal I of R. Equivalently, $N = [N : M]M$, [16]. A submodule K of M is multiplication if and only if $N \cap K = [N : K]K$ for all submodules N of M, [26, Lemma 1.3]. A submodule N of M is called a *pure* submodule of M if $IN = N \cap IM$ for every ideal I of R, [18]. An ideal I is pure if and only if I is multiplication and idempotent. As a generalization of pure submodules and idempotent ideals, the author and Smith [10] introduced the concept of idempotent submodules: A submodule N of M is *idempotent in* M if $N = [N : M]N$. It is shown [10, Theorem 1.1] that if M is a multiplication module with pure annihilator then N is pure if and only if N is idempotent and multiplication. An R-module M is projective if and only if it is a direct summand of a free R-module. It is proved, [27, Theorems 2.1 and 2.2] that a finitely generated ideal I of R is projective (resp. flat) if and only if I is multiplication and $\mathrm{ann}I = Re$ for some idempotent e of R (resp. $\mathrm{ann}I$ is a pure ideal of R). More generally, if M is a finitely generated multiplication module and $\mathrm{ann}M = Re$ for some idempotent e, then M is projective, [28, Theorem 11], and multiplication modules with pure annihilator are flat, [5, Theorem 8] and [24, Theorem 4.1].

Let R be a ring and M an R-module. Let S be the set of regular elements of R and R_S the total quotient ring of R. For a nonzero ideal I of R, let $I^{-1} = \{x \in R_S : xI \subseteq R\}$. I is an invertible ideal if $II^{-1} = R$. Let

$$T = \{t \in S : tm = 0 \text{ for some } m \in M \text{ implies } m = 0\}.$$

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T is a multiplicatively closed subset of S, and if M is torsion-free then $T = S$. In particular, if M is faithful multiplication then $T = S$, [17, Lemma 4.1]. Let N be a nonzero submodule of M and let $N^{-1} = \{x \in R_T : xN \subseteq M\}$. N^{-1} is an R-submodule of R_T, $R \subseteq N^{-1}$ and $NN^{-1} \subseteq M$. Following [25], N is *invertible in M* if $NN^{-1} = M$. It is shown, [7, Proposition 2.1] that if N is an invertible submodule of a finitely generated faithful multiplication R-module, then N is multiplication. It is also shown if N is a submodule of a multiplication module M and $[N : M]$ is an invertible ideal of R then N is invertible in M. The converse is true if we assume further that M is finitely generated and faithful, [7, Proposition 2.1] and [25, Remark 3.2 and Lemma 3.3].

Let N be a submodule of M and I an ideal of R. The residual submodule N by I is $[N :_M I] = \{m \in M : Im \subseteq N\}$, [21]. Obviously $[N : IM]M \subseteq \{N :_M I]$. The reverse inclusion is true if M is multiplication. If M is a faithful multiplication module then $[0 :_M I] = (\mathrm{ann}I)M$.

Let R be a commutative ring with identity and M an R-module. The R-module $R(M) = R(+)M$ becomes a commutative ring with identity under the product $(r, m)(r', m') = (rr', r'm + rm')$, called the idealization of M. The idealization of a module is a well-established method to facilitate interaction between a ring on the one hand and a module over a ring on the other. The basic constraction is to embed the module M as an ideal in a ring $R(M)$ which contains R as a subring. This technique was used with great success by Nagata. For a comprehensive survy on idealization, [23], [20], [13], [2], [3] and [4] can be consulted. $0(+)M$ is an ideal of $R(M)$ satisfying $(0(+)M)^2 = 0$, and the structure of $0(+)M$ as an ideal of $R(M)$ is essentially the same as the R-module structure of M. Every ideal contained in $0(+)M$ has the form $0(+)N$ for some submodule N of M, and every ideal contains $0(+)M$ has the form $I(+)M$ for some ideal I of M. Prime (maximal) ideals of $R(M)$ have the form $P(+)M$, where P is a prime (maximal) ideal of R. An ideal H of $R(M)$ is called *homogeneous* if $H = I(+)N$ where I is an ideal of R and N a submodule of M. In this case $I(+)N = (R(+)M)(I(+)N) = I(+)(IM + N)$ gives that $IM \subseteq N$. These ideals play a special role in studying properties of $R(M)$ and showing how these properties are related to those of R and M. Ideals of $R(M)$ need not be homogeneous, [13]. If $I(+)N$ and $J(+)K$ are homogeneous ideals of $R(M)$, then

$$[I(+)N :_{R(M)} J(+)K] = [I : J] \cap [N : K](+)[N :_M J]$$

is homogeneous, [4, Lemma 1]. In particular, $\mathrm{ann}(I(+)N) = (\mathrm{ann}I \cap \mathrm{ann}N)(+)[0 :_M I]$ and if M is faithful multiplication then $\mathrm{ann}(I\ (+)N) = \mathrm{ann}N(+)(\mathrm{ann}I)M$.

Let N be a submodule of M. Then $[N : M](+)N$ is a homogeneous ideal of $R(M)$ since $[N : M]M \subseteq N$. In the first part of this paper we give some conditions under which some properties of $[N : M](+)N$ transfer to N and conversely. We show for example that if M is multiplication and $[N : M](+)N$ is a multiplication ideal of $R(M)$ then N is a multiplication submodule of M. The converse is true if we assume further that M is finitely generated and faithful. We also show that if M is finitely generated faithful multiplication and $[N : M](+)N$ is cancellation (resp. weak cancellation, join principal) then N is cancellation (resp. wesk cancellation, join principal). In the second part we show how properties of $R(M)$ are related to those of R and M. For example we prove that $R(M)$ is a μ-ring if and only if R is a μ-ring and M is a μ-module.

All rings are assumed to be commutative with 1 and all modules are unital. For the basic concepts used, we refer the reader to [18]-[23].

2. Some Properties of the Ideal $[N : M](+)N$

Let R be a ring, M an R-module and N a submodue of M. Albu and Smith [1] proved that N is irreducible (resp. completely irreducible) if and only if the ideal $[N : M](+)N$ is irreducible (resp. completely irreducible). It is also shown that N is a primal submodule of M with adjoint prime ideal P if and only if $[N : M](+)N$ is a primal ideal of $R(M)$ with adjoint prime ideal $P(+)M$. In this section we give some conditions under which the properties of the homgeneous ideal $[N : M](+)N$ transfer to the submodule N and conversely.

Theorem 1. *Let R be a ring, M an R-module and N a submodule of M.*

(1) Let M be cyclic. If $[N : M](+)N$ is a principal ideal of $R(M)$ then N is a cyclic submodule of M. The converse is true if we assume further that M is faithful.

(2) Let M be finitely generated. If $[N : M](+)N$ is a finitely generated ideal of $R(M)$ then N is a finitely generated submodule of M. The converse is true if we assume further that M is faithful and multiplication.

(3) Let M be multiplication. If $[N : M](+)N$ is a multiplication ideal of $R(M)$ then N is a multiplication submodule of M. The converse is true if we assume further that M is finitely generated and faithful.

(4) If $[N : M](+)N$ is an invertible ideal of $R(M)$ then N is an invertible submodule of M. The converse is true if we assume that M is finitely generated faithful multiplication.

(5) Let M be faithful. If $[N : M](+)N$ is a faithful ideal of $R(M)$ then N is a faithful submodule of M. The converse is true if we assume further that M is multiplication.

Proof. (1) Let $[N : M](+)N = R(M)(a, n) = Ra(+)(Rn + aM)$ for some $a \in R$, $n \in M$. Then $[N : M] = Ra$. Since M is cyclic (hence multiplication), $N = aM$ is cyclic. Conversely, let N be cyclic and M faithful cyclic. It follows by [28, Proposition 13] that $[N : M]$ is a principal ideal of R. Let $[N : M] = Ra$ for some $a \in R$. Then $[N : M](+)N = [N : M](+)[N : M]M = Ra(+)aM = R(M)(a, 0)$ is a principal ideal of $R(M)$.

(2) Let $[N : M](+)N = \sum_{i=1}^{n} R(M)(a_i, n_i)$ for some $a_i \in [N : M]$ and $n_i \in N$. Since

$$\begin{aligned} R(M)(a_i, n_i) &= R(M)((a_i, 0) + (0, n_i)) \subseteq R(M)(a_i, 0) + R(M)(0, n_i) \\ &= Ra_i(+)a_iM + 0(+)Rn_i = Ra_i(+)Rn_i + a_iM, \end{aligned}$$

$[N:M](+)N \subseteq \sum_{i=1}^{n} Ra_i(+)\sum_{i=1}^{n} Rn_i + a_iM$. Hence $N \subseteq \sum_{i=1}^{n} Rn_i + a_iM$. Since $n_i \in N$ and $a_iM \subseteq [N:M]M \subseteq N$, we get that $N = \sum_{i=1}^{n} Rn_i + a_iM$. As M is finitely generated, N is finitely generated. Conversely, suppose N is a finitely generated submodule of a finitely generated faithful multiplication R-module M. It follows by [28, Theorem 10] and [8, Proposition 2.2] that $[N:M]$ is a finitely generated ideal of R. Let $[N:M] = \sum_{i=1}^{n} Ra_i$ for some $a_i \in R$. Then

$$\begin{aligned}[N:M](+)N &= [N:M](+)[N:M]M = \sum_{i=1}^{n} Ra_i(+)\sum_{i=1}^{n} Ra_iM \\ &= \sum_{i=1}^{n} Ra_i(+)a_iM = \sum_{i=1}^{n} R(M)(a_i,0),\end{aligned}$$

so that $[N:M](+)N$ is finitely generated.

(3) Let M be multiplication and $[N:M](+)N$ be a multiplication ideal of $R(M)$. Let K be a submodule of N. Then $[K:M](+)K$ is an ideal of $R(M)$ that is contained in $[N:M](+)N$. Hence $[K:M](+)K = H([N:M](+)N)$ for some ideal H of $R(M)$. It follows that

$$[K:M](+)K+0(+)N = H([N:M](+)N)+0(+)N = (H+0(+)M)([N:M](+)N).$$

Let $H+0(+)M = I(+)M$ for some ideal I of R. Then

$$[K:M](+)N = (I(+)M)([N:M](+)N) = I[N:M](+)N.$$

This gives that $[K:M] = I[N:M]$ and hence $K = [K:M]M = I[N:M]M = IN$, so N is multiplication. For the converse, if N is multipication then $[N:M]$ is a multiplication ideal of R, [28, Theorem 10] and [8, Proposition 2.2]. It is shown in [4, Theorem 9] that if $I(+)N$ is a homogeneous ideal of R such that I is a multiplication ideal of R and N a multiplication submodule of M such that $\mathrm{ann}I + [IM:N] = R$ then $I(+)N$ is multiplication. Using this fact we have

$$\mathrm{ann}[N:M] + [[N:M]M:N] = \mathrm{ann}N + [N:N] = R,$$

so $[N:M](+)N$ is multiplication.

(4) Assume $[N:M](+)N$ is invertible, then it is multiplication. We show that $[N:M]$ is a multiplication ideal of R. Let $I \subseteq [N:M]$ be an ideal of R. Then $IM \subseteq N$, and hence $I(+)IM \subseteq [N:M(+)N$. There exists an ideal H of $R(M)$ such that $I(+)IM = H([N:M(+)N)$. It follows that

$$I(+)N = I(+)IM+0(+)N = H([N:M](+)N)+0(+)N = (H+0(+)M)([N:M](+)N).$$

Let $H+0(+)M = A(+)M$ for some ideal A of R. Then

$$I(+)N = (A(+)M)([N:M](+)N) = A[N:M](+)N.$$

Hence $I = A[N:M]$, and hence $[N:M]$ is a multiplication ideal of R. Also $[N:M](+)N$ has a regular element, say (a,m) for some $a \in [N:M]$ and $m \in M$. It follows that a is a regular element and hence $[N:M]$ is an invertible ideal of R. Hence $N = [N:M]M$ is an invertible submodule of M, [7, Proposition 2.1].

Conversely, suppose M is a finitely generated faithful multiplication module and N invertible. By [7, Proposition 2.1], N is multiplication and by (3), $[N:M](+)N$ is a multiplication ideal of $R(M)$. Since N is invertible, we infer from [7, Proposition 2.1] and [25, Lemma 3.2] that $[N:M]$ is invertible. Let $a \in [N:M]$ be a regular element. It follows by [4, Lemma 6] that $(a,0) \in [N:M](+)N$ is a regular element. So $[N:M](+)N$ is an invertible ideal of $R(M)$.

(5) If M is faithful, then by [4, Lemma 1] we have that

$$0 = \mathrm{ann}([N:M](+)N) = \mathrm{ann}[N:M] \cap \mathrm{ann}N(+)[0:_M [N:M]] = \mathrm{ann}N(+)[0:_M [N:M]].$$

So $\mathrm{ann}N = 0$. If M is faithful multiplication and $\mathrm{ann}N = 0$, then again by [4, Lemma 1] $0 = \mathrm{ann}N(+)(\mathrm{ann}N)M = \mathrm{ann}([N:M](+)N)$. □

The next theorem shows how the purity, idempotent and direct sum properties transfer from $[N:M](+)N$ to N and conversely.

Theorem 2. *Let R be a ring, M an R-module and N a submodule of M.*

(1) If $[N:M](+)N$ is an idempotent ideal of $R(M)$ then N is an idempotent submodule of M. The converse is true if M is finitely generated faithful and multiplication.

(2) If $[N:M](+)N$ is a pure ideal of $R(M)$ then N is a pure submodule of M. The converse is true if M is finitely generated faithful and multiplication.

(3) Let M be faithful multiplication. If $[N:M](+)N$ is a direct summand in $R(M)$ then N is a direct summand in M. The converse is true if we assume further that M is finitely generated.

Proof. (1) Let $[N:M](+)N$ be idempotent. Then

$$[N:M](+)N = ([N:M](+)N)^2 = [N:M]^2(+)[N:M]N,$$

so that $N = [N:M]N$, and hence N is idempotent. Conversely, let M be finitely generated faithful and multiplication. Then $[N:M] = [[N:M]N:M] = [N:M]^2$. Hence

$$[N:M](+)N = [N:M]^2(+)[N:M]N = ([N:M](+)N)^2,$$

and hence $[N:M](+)N$ is idempotent.

(2) Let $[N:M](+)N$ be a pure ideal of $R(M)$. Let I be an ideal of R. Then

$$\begin{aligned} I[N:M](+)IN &= (I(+)IM)([N:M](+)N) \\ &= (I(+)IM) \cap ([N:M](+)N) = I \cap [N:M](+)IM \cap N. \end{aligned}$$

Hence $IN = IM \cap N$ and this shows that N is pure in M. Conversely, let M be finitely generated, faithful and multiplication. If N is pure in M, then by [10, Theorem 1.1], N is multiplication and idempotent. It follows by part (1) and Threorem 1(3) that $[N:M](+)N$ is idempotent and multiplication. So it is pure by [10, Theorem 1.1].

(3) Let $[N:M](+)N$ be a direct summand in $R(M)$. Then $R(M) = [N:M](+)N \oplus H$ for some ideal H of $R(M)$. It follows that $R(M) = [N:M](+)N + H + 0(+)M$. Assume that $H + 0(+)M = I(+)M$ for some ideal I of R. Then $R(M) = [N:M] + I(+)M$, and hence $R = [N:M] + I$. It follows that $M =$

$N + IM$. Next, since $R(M) = [N : M](+)N + H$ is multiplication, we infer from [9, Theorem 2.1] that

$$0(+)M = 0(+)M + ([N : M](+)N \cap H) = [N : M](+)M \cap H + 0(+)M$$
$$= [N : M](+)M \cap I(+)M = ([N : M] \cap I)(+)M.$$

Hence $0 = [N : M] \cap I$. As M is faithful multiplication, we infer from [17, Corollary 1.7] that $0 = ([N : M] \cap I)M = N \cap IM$. Hence $M = N \oplus IM$ and N is a direct summand in M. Conversely, let M be finitely generated faithful and multiplication. If N is direct summand in M, then $M = N \oplus K$ for some submodule K of M. Hence $M = N + K$ and $0 = N \cap K$. It follows by [28, Proposition 4] and [9, Corollary1.2] that

$$R = [M : M] = [(N + K) : M] = [N : M] + [K : M].$$

Also

$$0 = [0 : M] = [(N \cap K) : M] = [N : M] \cap [K : M].$$

This implies that $R = [N : M] \oplus [K : M]$. So

$$R(M) = [N : M] + [K : M](+)N + K = [N : M](+)N + [K : M](+)K.$$

Finally

$$[N : M](+)N \cap [K : M](+)K = [N : M] \cap [K : M](+)N \cap K = 0.$$

Hence $R(M) = [N : M](+)N \oplus [K : M](+)K$. So $[N : M](+)N$ is a direct summand in $R(M)$. This finishes the proof of the theorem. □

The next result shows how projectivity and flatness of the ideal $[N : M](+)N$ transfer to N and conversely.

Theorem 3. *Let R be a ring, M an R-module and N a submodule of M.*

(1) Let M be locally cyclic projective. If $[N : M](+)M$ is a projective ideal of $R(M)$ then N is a projective submodule of M.

(2) Let M be finitely generated faithful multiplication. Then $[N : M](+)N$ is a finitely generated projective ideal of $R(M)$ if and only if N is a finitely generated projective submodule of M.

(3) Let M be finitely generated faithful multiplication. Then $[N : M](+)N$ is a finitely generated flat ideal of $R(M)$ if and only if N is a finitely generated flat submodule of M.

Proof. (1) Assume $[N : M](+)M$ is projective. Then $F = [N : M](+)M \oplus H$ for some ideal H of $R(M)$ and some free ideal F of $R(M)$. Hence $F = [N : M](+)M + H$ and $0 = [N : M](+)M \cap H$. Now $F + 0(+)M = [N : M](+)M + H + 0(+)M$. Let $F + 0(+)M = I(+)M$ and $H + 0(+)M = J(+)M$ for some ideals I and J of R. Then $I(+)M = [N : M] + J(+)M$, and hence $I = [N : M] + J$. Since $F = [N : M](+)N + H$ is free (hence multiplication), we infer from [9, Theorem 2.1] that

$$0(+)M = 0(+)M + ([N : M](+)M \cap H) =$$
$$[N : M](+)M \cap H + 0(+)M = [N : M] \cap J(+)M.$$

Hence $0 = [N : M] \cap J$, and hence $I = [N : M] \oplus J$. To prove that $[N : M]$ is a projective ideal of R, we need to show that I is a free ideal of R. Since

$0(+)M \subseteq [N : M](+)M \subseteq F$, $F = F + 0(+)M = I(+)M$. Since F is free, it follows by [3, Theorem 9] that I is free. Finally, since M is locally cyclic projective, M is multiplication and projective, [8, Theorem 3.4] and [11, Theorem 1.3]. Hence $N = [N : M]M \cong [N : M] \otimes M$ is a projective submodule of M.

(2) Let $[N : M](+)N$ be a finitely generated projective (hence multiplication) ideal of $R(M)$. It follows by Theorem 1 that N is a finitely generated multiplication submodule of M. By [27, Theorem 2.1] and [4, Lemma 1] we have that

$$Re(+)eM = R(M)(e,0) = \mathrm{ann}([N : M](+)N) = \mathrm{ann}N(+)(\mathrm{ann}N)M.$$

for some idempotent e of R. So $Re = \mathrm{ann}N$ and by [28, Theorem 11] N is finitely generated projective. Conversely, let M be a finitely generated faithful multiplication module. Since N is finitely generated projective, it follows by [8, Proposition 3.7] that N is finitely generated multiplication. By Theorem 1, $[N : M](+)N$ is a finitely generated multiplication ideal of $R(M)$. Also by [8, Proposition 3.7], $[N : M]$ is a finitely generated projective ideal of R. Hence $\mathrm{ann}N = \mathrm{ann}[N : M] = Re$ for some idempotent e of R. As M is faithful multiplication, we get from [4, Lemma 1] that

$$R(M)(e,0) = Re(+)eM = \mathrm{ann}N(+)(\mathrm{ann}N)M = \mathrm{ann}([N : M](+)N).$$

By [13, Theorem 3.7] and [4, Lemma 6], $(e,0)$ is an idempotent element in $R(M)$. So $[N : M](+)N$ is a finitely generated projective ideal of $R(M)$.

(3) Suppose $[N : M](+)N$ is a finitely generated flat ideal of $R(M)$. Then $[N : M)(+)N$ is a finitely generated multiplication ideal of R, and by Theorem 1, N is a finitely generated multiplication submodule of M. Moreover, $\mathrm{ann}([N : M](+)N)$ is a pure ideal of $R(M)$. Since M is faithful multiplication, $\mathrm{ann}([N : M](+)N) = \mathrm{ann}N(+)(\mathrm{ann}N)M$. It follows by Theorem 2 that $\mathrm{ann}N$ is a pure ideal of R. So N is a flat submodule of M, [24, Theorem 4.1] and [5, Theorem 8]. Conversely, suppose M is finitely generated faithful multiplication. Since N is flat, N is multiplication by [8, Theorem 3.7] and by Theorem 1, $[N : M](+)N$ is finitely generated multiplication. Moreover, $[N : M]$ is a finitely generated flat ideal of R and hence $\mathrm{ann}N = \mathrm{ann}[N : M]$ is a pure ideal of R. As M is faithful multiplication $\mathrm{ann}([N : M](+)N) = \mathrm{ann}N(+)(\mathrm{ann}N)M$ and by Theorem 2 $\mathrm{ann}([N : M](+)N)$ is a pure ideal of $R(M)$. This finally shows that $[N : M](+)N$ is a flat ideal of $R(M)$. □

Generalizing the case for ideals, an R-module M is called cancellation (resp. weak cancellation) if $IM = JM$ for some ideals I and J of R then $I = J$ (resp. $I+\mathrm{ann}M = J+\mathrm{ann}M$). Equivalentely $[IM : M] = I$ (resp. $[IM : M] = I+\mathrm{ann}M$) for every ideal I of R. An R-module M is cancellation if and only if M is faithful weak cancellation. Examples of cancellation modules include free modules and finitely generated faithful multiplication modules, [28, Corollary to Theorem 9]. A submodule N of an R-module M is called join principal if $[(IN+K) : N] = I+[K : N]$ for every ideal I of R and every submodule K of M, [12]. We now give a result showing how the cancellation (resp. weak cancellation, join principal) properties transfer from $[N : M](+)N$ to N.

Theorem 4. *Let R be a ring, M finitely generated faithful multiplication R-module and N a submodule of M.*

(1) If $[N:M](+)N$ is a cancellation ideal of $R(M)$ then N is a cancellation submodule of M.

(2) If $[N:M](+)N$ is a weak cancellation ideal of $R(M)$ then N is a weak cancellation submodule of M.

(3) If $[N:M](+)N$ is a join principal ideal of $R(M)$ then N is a join principal submodule of M.

(4) N is a cancellation multiplication submodule of M if and only if $[N:M](+)N$ is a cancellation multiplication ideal of $R(M)$.

Proof. (1) Let I be an ideal of R. Then

$$\begin{aligned} I(+)M &= \left[(I\,(+)\,M)\,([N:M]\,(+)N) :_{R(M)} [N:M]\,(+)N\right] \\ &= \left[(I\,[N:M]\,(+)N) :_{R(M)} [N:M]\,(+)N\right] \\ &= [I\,[N:M]:[N:M]]\,(+)M \end{aligned}$$

Since M is finitely generated faithful multiplication module,

$$I = [I\,[N:M]:[N:M]] = [IN:N]\,,$$

and hence N is cancellation.

(2) Let I be an ideal of R. Then

$$\begin{aligned} (I+\mathrm{ann}N)\,(+)M &= I(+)M + \mathrm{ann}(\,[N:M]\,(+)N) \\ &= \left[(I(+)M)\,([N:M]\,(+)N) :_{R(M)} [N:M]\,(+)N\right] \\ &= [I\,[N:M]:[N:M]]\,(+)\,M. \end{aligned}$$

Since M is finitely generated faithful multiplication,

$$I+\mathrm{ann}N = [I\,[N:M]:[N:M]] = [IN:N]\,,$$

and hence N is weak cancellation.

(3) Suppose I is an ideal of R and K a submodule of M. Then

$$\begin{aligned} &[((I(+)M)([N:M](+)N) + [K:M])(+)M :_{R(M)} [N:M](+)N] \\ &\qquad = I(+)M + [[K:M](+)M :_{R(M)} [N:M](+)N]. \end{aligned}$$

But

$$\begin{aligned} &[((I(+)M)([N:M](+)N) + [K:M])(+)M :_{R(M)} [N:M](+)N] \\ &\qquad = [I[N:M] + [K:M](+)M :_{R(M)} [N:M](+)N] \\ &\qquad = [(I[N:M] + [K:M]) : [N:M](+)M, \end{aligned}$$

and

$$\begin{aligned} &I(+)M + [[K:M](+)M :_{R(M)} [N:M](+)N] \\ &\qquad = I(+)M + [[K:M]:[N:M]](+)M = I + [[K:M]:[N:M]](+)M. \end{aligned}$$

Since M is finitely generated faithful multiplication,

$$\begin{aligned} [(IN+K):N] &= [(I[N:M] + [K:M]) : [N:M]] \\ &= I + [[K:M]:[N:M]] = I + [K:N]. \end{aligned}$$

Hence N is join principal.

(4) Suppose $[N:M](+)N$ is multiplication and cancellation. It follows by Theorem 1 and the first part of this theorem that N is multiplication and cancellation.

For the converse, let H_1 and H_2 be ideals of $R(M)$ such that $H_1([N : M](+)N) = H_2([N : M](+)N)$. Hence

$$\begin{aligned}(H_1 + 0(+)M)([N : M](+)N) &= H_1([N : M](+)N) + 0(+)N \\ &= H_2([N : M](+)N) + 0(+)N = (H_2 + 0(+)M)([N : M](+)N).\end{aligned}$$

Let $H_1+0(+)M = I(+)M$ and $H_2+0(+)M = J(+)M$ for some ideals I and J of R. It follows that $I[N : M](+)N = J[N : M](+)N$, and hence $I[N : M] = J[N : M]$. This implies that $IN = JN$. Since N is cancellation, we get that $I = J$ and hence $H_1 + 0(+)M = H_2 + 0(+)M$. Next, since N is cancellation, N is faithful and by Theorem 1, $[N : M](+)N$ is faithful. Moreover, N is multiplication and hence $[N : M](+)N$ is multiplication. It follows by [15, Corollary 1.7] that

$$\begin{aligned}(H_1 \cap 0(+)M)([N : M](+)N) &= H_1([N : M](+)N) \cap (0(+)M)([N : M](+)N) \\ &= H_1([N : M](+)N) \cap 0(+)N.\end{aligned}$$

Since $H_1([N : M](+)N) = H_2([N : M](+)N)$, we infer that

$$(H_1 \cap 0(+)M)([N : M](+)N) = (H_2 \cap 0(+)M)([N : M](+)N).$$

Let $H_1 \cap 0(+)M = 0(+)K$ and $H_2 \cap 0(+)M = 0(+)L$ for some submodules K and L of M. It follows that

$$0(+)[N : M]K = (0(+)K)(N : M](+)N) = (0(+)L)([N : M](+)N) = 0(+)[N : M]L.$$

Hence $[N : M]K = [N : M]L$ and hence $[N : M][K : M] = [N : M][L : M]$. As N is cancellation and M finitely generated faithful multiplication, $[N : M]$ is a cancellation ideal of R and hence $[K : M] = [L : M]$. This gives that $K = L$, and hence $0(+)K = 0(+)L$. So $H_1 \cap 0(+)M = H_2 \cap 0(+)M$. Finally, using the modular law, one gets that

$$\begin{aligned}H_1 &= (H_1 + 0(+)M) \cap H_1 = (H_2 + 0(+)M) \cap H_1 \\ &= H_2 + (H_1 \cap 0(+)M) = H_2 + (H_2 \cap 0(+)M) = H_2.\end{aligned}$$

Hence $[N : M](+)N$ is a cancellation ideal of $R(M)$. □

The dual notion of the concept of multiplication modules was introduced by Ansari-Toroghy and Farshadifar in [14] and some properties of this class of modules have been considered. An R-module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = [0 :_M I]$. It is shown that M is a comultiplication module if and only if for each submodule N of M, we have $N = [0 :_M \mathrm{ann}N]$. It is clear that if M is a comultiplication module then every submodule of M is comultiplication. An ideal I of a ring R is comultiplication if $I = \mathrm{ann}(\mathrm{ann}I)$. We end this section by a result showing how the comultiplication property transfers from $I(+)N$ to its components I and N and conversely.

Theorem 5. *Let R be a ring, M faithful multiplication R-module, I an ideal of R and N a submodule of M such that $IM \subseteq N$.*

(1) $0(+)N$ is a comultiplication ideal of $R(M)$ if and only if N is a comultiplication submodule of M.

(2) I is a comultiplication ideal of R if and only if $I(+)IM$ is a comultiplication ideal of $R(M)$.

(3) $I(+)N$ is a comultiplication ideal of $R(M)$ if and only if I is a comultiplication ideal of R and N is a comultiplication submodule of M.

(4) Assuming further that M is finitely generated. Then N is a comultiplication submodule of M if and only if $[N : M](+)N$ is a comultiplication ideal of $R(M)$.

Proof. (1) Suppose N is a comultiplication submodule of M. Since M is faithful multiplication, we infer that $N = [0 :_M \mathrm{ann}N] = \mathrm{ann}(\mathrm{ann}N)M$. It follows that

$$\begin{aligned}[0 :_{R(M)} \mathrm{ann}(0(+)N)] &= [0 :_{R(M)} \mathrm{ann}N(+)M] \\ &= \mathrm{ann}(\mathrm{ann}N) \cap \mathrm{ann}M(+)[0 :_M \mathrm{ann}N] \\ &= 0(+)\mathrm{ann}(\mathrm{ann}N)M = 0(+)N.\end{aligned}$$

Hence $0(+)N$ is a comultiplication ideal of $R(M)$. The statement is reversible.

(2) Let I be a comultiplication ideal R. It follows that $I = \mathrm{ann}(\mathrm{ann}I)$. Since M is a faithful multiplication module, we obtain that

$$\begin{aligned}\big[0 :_{R(M)} \mathrm{ann}(I(+)IM)\big] &= \big[0 :_{R(M)} \mathrm{ann}I(+)(\mathrm{ann}I)M\big] \\ &= \mathrm{ann}(\mathrm{ann}I) \cap \mathrm{ann}((\mathrm{ann}IM)(+)\mathrm{ann}(\mathrm{ann}I)M \\ &= \mathrm{ann}(\mathrm{ann}I)(+)\mathrm{ann}(\mathrm{ann}I)M = I(+)IM.\end{aligned}$$

So $I(+)IM$ is a comultiplication ideal of $R(M)$. The statement is reversible.

(3) Suppose $I(+)N$ is a comultiplication ideal of $R(M)$. Then each of $0(+)N$ and $I(+)IM$ is comultiplication. So by the first two parts of the theorem we have that N is a comultiplication submodule of M and I is a comultiplication ideal of R. Conversely assume that N is a comultiplication submodule of M and I is a comultiplication ideal of R. It follows that $0(+)N$ and $I(+)IM$ are comultiplication ideals of $R(M)$. Now $0(+)N \cap I(+)IM = 0(+)IM$ and

$$\begin{aligned}[0 :_{R(M)} &\mathrm{ann}(0(+)IM)] = [0 :_{R(M)} \mathrm{ann}IM(+)M] \\ &= [0 :_{R(M)} \mathrm{ann}I(+)M] = \mathrm{ann}(\mathrm{ann}I) \cap \mathrm{ann}M(+)[0 :_M \mathrm{ann}I] \\ &= 0(+)\mathrm{ann}(\mathrm{ann}I)M = 0(+)IM.\end{aligned}$$

This shows that $0(+)N \cap I(+)IM$ is a comultiplication ideal of $R(M)$, and by [15, Theorem 2.15], $I(+)N = 0(+)N + I(+)IM$ is a comultiplication ideal of $R(M)$.

(4) Suppose N is a comultiplication submodule of M. Then $N = [0 :_M \mathrm{ann}N] = \mathrm{ann}(\mathrm{ann}N)M$. Since M is finitely generated faithful multiplication , it follows that $[N : M] = [\mathrm{ann}(\mathrm{ann}N)M : M] = \mathrm{ann}(\mathrm{ann}N) = \mathrm{ann}(\mathrm{ann}[N : M])$. Hence $[N : M]$ is a comultiplication ideal of R. This gives that $[N : M](+)N$ is a comultiplication ideal of $R(M)$. The converse follows immediately by part (3) of the theorem. □

3. Some Properties of the Ring $R(M)$

In this section we investigate how properties of $R(M)$ are related to those of R and M. A well-known property possessed by each commutative ring is that if an ideal I of R is contained in the union of the prime ideals P_i of R, then I is contained in a particular $P_{i.}$ As a strong version of this result, we call a ring R to be a *μ-ring* if $I, A_{1,}A_2, ..., A_n$ are ideals of R such that $I \subseteq \cup A_i$, then I is contained in some A_i, [20]. As a generalization of this concept to the module case, we say that an R-module M is a *μ-module* if $N, K_1, ..., K_n$ are submodules of M such that $N \subseteq \cup K_i$ then $N \subseteq K_r$ for some r. For porperties of μ-rings, see [20, p. 87].

Theorem 6. *Let R be a ring and M an R-module. Then $R(M)$ is a μ-ring if and only if R is a μ-ring and M is a μ-module.*

Proof. Suppose $R(M)$ is a μ-ring. Let $I, A_1, ..., A_n$ be ideals of R such that $I \subseteq \cup A_i$. Then $I(+)M \subseteq \cup A_i(+)M = \cup(A_i(+)M)$. It follows that $I(+)M \subseteq A_k(+)M$ for some k and hence $I \subseteq A_k$. So R is a μ-ring. Next, let $N, K_1, ..., K_n$ be submodules of M such that $N \subseteq \cup K_i$. Then $0(+)N \subseteq 0(+) \cup K_i = \cup 0(+)K_i$. Hence there exists m such that $0(+)N \subseteq 0(+)K_m$ and hence $N \subseteq K_m$, so M is a μ-module. Conversely let $H, H_1, ..., H_n$ be ideals of $R(M)$ such that $H \subseteq \cup H_i$. Let $H_i' = H \cap H_i$. Then $H = \cup H_i'$. To show that $R(M)$ is a μ-ring, it it enough to show that $H = H_n'$ for some n. Now

$$H + 0(+)M = (\cup H_i') + 0(+)M = \cup(H_i' + 0(+)M).$$

Let $H + 0(+)M = I(+)M$ and $H_i' + 0(+)M = A_i(+)M$ for some ideals I and A_i of R. Then $I(+)M = \cup A_i(+)M$ and hence $I = \cup A_i$. So there exists k such that $I \subseteq A_k \subseteq \cup A_i = I$. Hence $I = A_k$ and this gives that

$$H + 0(+)M = I(+)M = A_k(+)M = H_k' + 0(+)M.$$

On the other hand $H \cap 0(+)M = \cup H_i' \cap 0(+)M = \cup(H_i' \cap 0(+)M)$. Let $H \cap 0(+)M = 0(+)N$ and $H_i' \cap 0(+)M = 0(+)K_i$ for some submodules N and K_i of M. It follows that $0(+)N = \cup 0(+)K_i = 0(+) \cup K_i$. Hence $N = \cup K_i$. There exists l such that $N \subseteq K_l$ and this gives that $N \subseteq K_l \subseteq \cup K_i = N$. So $N = K_l$ and hence

$$H \cap 0(+)M = 0(+)N = 0(+)K_l = H_l' \cap 0(+)M.$$

This implies that

$$H + 0(+)M = H_k' + 0(+)M \subseteq (H_k' + H_l') + 0(+)M \subseteq H + 0(+)M,$$

so that $H + 0(+)M = (H_k' + H_l') + 0(+)M$. Similarly,

$$H \cap 0(+)M = H_l' \cap 0(+)M \subseteq (H_k' + H_l') \cap 0(+)M \subseteq H \cap 0(+)M,$$

and hence $H \cap 0(+)M = (H_k' + H_l') \cap 0(+)M$. Using the modular law, one obtains that

$$\begin{aligned} H &= (H + 0(+)M) \cap H = (H_k' + H_l' + 0(+)M) \cap H = (H_k' + H_l') + (H \cap 0(+)M) \\ &= (H_k' + H_l') + ((H_k' + H_l') \cap 0(+)M) = H_k' + H_l', \end{aligned}$$

and this shows that $R(M)$ is a μ-ring. $\square$

Matsuda defines two properties for a ring R : R satisfies *Property* (U) if each regular ideal of R is a union of regular principal ideals of R, and R satisfies *Property* (FU) if $\mathrm{Reg}(I) \subseteq \bigcup_{i=1}^{n} J_i$ implies $I \subseteq \bigcup_{i=1}^{n} J_i$ for each finite family of regular ideals $I, J_1, ..., J_n$ of R, where $\mathrm{Reg}(I)$ denotes the set of regular elements of I. He shows that Property (U) implies Property (FU) but not conversely, see [20, p. 195].

The next Theorem shows how Properties (U) and (FU) transfer from $R(M)$ to R and conversely.

Theorem 7. *Let R be a ring and M an R-module.*

(1) Let M be faithful multiplication. If $R(M)$ satisfies Property (U) then R satisifes Property (U). The converse is true if we assume further that M is divisible.

(2) Let M be faithful multiplication. If $R(M)$ satisfies Property (FU) then R satisifes Property (FU). The converse is true if we assume further that M is divisible.

Proof. (1) Suppose $R(M)$ satisfies Property (U). Let I be a regular ideal of R. Since M is faithful multiplication, it follows by [4, Lemma 6] that $I(+)M$ is a regular ideal of $R(M)$. Hence $I(+)M = \cup R(M)(a_\alpha, m_\alpha)$, where $R(M)(a_\alpha, m_\alpha)$ is a regular principal ideal of $R(M)$. It follows that

$$I(+)M = \cup R(M)(a_\alpha, m_\alpha) \subseteq \cup(Ra_\alpha(+)Rm_\alpha + a_\alpha M) = \cup Ra_\alpha(+)\cup(Rm_\alpha + a_\alpha M),$$

and hence $I \subseteq \cup Ra_\alpha \subseteq I$. So $I = \cup Ra_\alpha$. Finally, since (a_α, m_α) is regular, a_α is regular and this shows that R satisfies Property (U). Conversely, assume R satisfies Property(U). Let H be a regular ideal of $R(M)$. Since M is divisible, [13, Theorem 3.9] shows that H is homogeneous, and has the form $I(+)M$ for some ideal I of R such that $I \cap S \neq \phi$ where $S = R - (Z(R) \cup Z(M))$, $Z(R)$ is the set of zero divisors of R and $Z(M)$ the set of zero divisors on M. Note that $Z(M) = \{t \in R : tm = 0$ for some nonzero $m \in M\}$ and if M is faithful multiplication, hence torsion free, [17, Lemma 4.1], we get that $Z(M) \subseteq Z(R))$. Since $H = I(+)M$ is regular, it follows that I is a regular ideal of R. Let $a \in I$ be regular. Since M is divisble, $M = aM \subseteq IM \subseteq M$, so that $M = IM$ and hence $H = I(+)IM$. Let $I = \cup Ra_\alpha$ for some regular principal ideals Ra_α of R. It follows that

$$H = I(+)IM = \cup Ra_\alpha(+) \cup a_\alpha M = \cup Ra_\alpha(+)a_\alpha M = \cup R(M)(a_\alpha, 0).$$

Since a_α is regular and M faithful multiplication, we infer from [4, Lemma 6] that $(a_\alpha, 0)$ is regular and this shows that $R(M)$ satisfies Property (U).

(2) Suppose $R(M)$ satisfies Property (FU). Let I be a regular ideal of R and $\{g_\alpha\} = \mathrm{Reg}(I)$. Since M is faithful multiplication, $I(+)M$ is a regular ideal of $R(M)$, [4, Lemma 6], and $\{(g_\alpha, m_\alpha)\}$ are the regular elements of $I(+)M$, where $m_\alpha \in M$. For if (b, k) is any regular element of $I(+)M$, then b is a regular element of I and $b \in \{g_\alpha\}$. Assume now $\{g_\alpha\} = \mathrm{Reg}(I) \subseteq \bigcup_{i=1}^{n} J_i$, where J_i are regular ideals of R. Hence $\{(g_\alpha, m_\alpha)\} = \mathrm{Reg}(I(+)M) \subseteq \bigcup_{i=1}^{n} J_i(+)M$, where $J_i(+)M$ are regular ideals of $R(M)$, [4, Lemma 6]. Since $R(M)$ satisfies Property (FU), $I(+)M \subseteq \bigcup_{i=1}^{n} J_i(+)M$, and hence $I \subseteq \bigcup_{i=1}^{n} J_i$. So R satisifes Property (FU). Conversely, assume R satisfies Property (FU). Let H be a regular ideal of $R(M)$. Since M is divisible, $H = I(+)IM$ for some regular ideal I of R such that $I \cap S \neq \phi$, where $S = R - Z(R)$. Assume $\{(g_\alpha, n_\alpha)\}$ be the set of regular elements of $I(+)IM$ such that $\{(g_\alpha, n_\alpha)\} \subseteq \bigcup_{i=1}^{n} H_i$ for some regular ideals H_i of R. Again, since M is divisible, $H_i = J_i(+)J_iM$ for some regular ideals J_i or R. Since (g_α, n_α) is regular in $R(M)$, g_α is regular in R. Hence $\{g_\alpha\} = \mathrm{Reg}(I)$, and hence $\{g_\alpha\} \subseteq \bigcup_{i=1}^{n} J_i$. As R satisfies

Property (FU), $I \subseteq \bigcup_{i=1}^{n} J_i$ and hence $H = I(+)IM \subseteq \bigcup_{i=1}^{n} J_i(+)J_iM = \bigcup_{i-1}^{n} H_i$. Hence $R(M)$ satisfies Property (FU). $\square$

A ring R is called *Laskerian* if every ideal of R is a finite intersection of primary ideals of R and it is called *strongly Laskerian* if R is Laskerian and for every prime ideal P of R, there exists a positive integer n such that $(\sqrt{P})^n \subseteq P$. R is said to be *primary* if R contains at most one proper prime ideal of R. On the other hand, a ring R is called an *RM-ring* (restricted minimum condition) if R is one dimensional Noetherian ring. For details about Laskerian, primary and RM-rings, see [19]. The next result shows how the Laskerian, primary and RM-properties transfer from $R(M)$ to R and conversely.

Proposition 8. *Let R be a ring and M an R-module.*

(1) If $R(M)$ is strongly Laskerian then R is strongly Laskerian. The converse is true if every ideal of $R(M)$ contains $0(+)M$.
(2) $R(M)$ is a primary ring if and only if R is.
(3) If $R(M)$ is an RM-ring then so too is R and the converse in true if M is finitely generated.

Proof. (1) Assume $R(M)$ is strongly Laskerian. Let I be an ideal of R. Then $I(+)M$ is an ideal of $R(M)$ and hence $I(+)M = \bigcap_{i=1}^{n} H_i$ for some primary ideals H_i of $R(M)$. Since $0(+)M \subseteq I(+)M \subseteq H_i$ for all i, $H_i = J_i(+)M$ for some ideals J_i of R. Since $J_i(+)M$ is primary, we obtain from [20, Theorem 25.2] that J_i is primary. Also $I(+)M = \bigcap_{i=1}^{n} J_i(+)M$ gives that $I = \bigcap_{i=1}^{n} J_i$ and this shows that R is Laskerian. Now, let P be a prime ideal of R. Then $P(+)M$ is a prime ideal of $R(M)$. There exists a positive integer n such that $(\sqrt{P(+)M})^n \subseteq P(+)M$. It follows by [13, Theorem 3.2] that $\sqrt{P(+)M} = \sqrt{P}(+)M$ and hence

$$(\sqrt{P})^n(+)(\sqrt{P})^{n-1}M = (\sqrt{P}(+)M)^n \subseteq P(+)M.$$

This shows that $(\sqrt{P})^n \subseteq P$ and hence R is strongly Laskerian. Conversely, let H be an ideal of $R(M)$. Since H contains $0(+)M$, $H = H + 0(+)M = I(+)M$ for some ideal I of R. But R is Laskerian. Thus $I = \bigcap_{i=1}^{n} J_i$ for some primary ideals J_i of R. Hence $H = \bigcap_{i=1}^{n} J_i(+)M$. By [20, Theorem 25.2] , $J_i(+)M$ is primary ideals of R and hence $R(M)$ is Laskerian. Now let $P(+)M$ be a prime ideal of $R(M)$. Then P is a prime ideal of R. There exists a positive integer n such that $(\sqrt{P})^n \subseteq P$. Hence

$$\sqrt{P(+)M})^n = (\sqrt{P}(+)M)^n = (\sqrt{P})^n(+)(\sqrt{P})^{n-1}M \subseteq P(+)M,$$

and this shows that $R(M)$ is strongly Laskerian.

(2) This follows from the fact that P is a proper prime ideal of R if and only if $P(+)M$ is a proper prime ideal of $R(M)$.

(3) Suppose $R(M)$ is an RM-ring Then $R(M)$ is one dimensional ring and hence R is one dimensional ring. For if $0 \neq P$ is a prime ideal of R then $0 \neq P(+)M$

is a prime ideal of $R(M)$. Hence $P(+)M$ is a maximal ideal of $R(M)$ and this implies that P is a maximal ideal of R. Since $R(M)$ is Noetherian, it follows by [2, Proposition 10] and [13, Theorem 4.8] that R is Noetherian and hence R is an RM-ring. Conversely, let R be an RM-ring. Since R is Noetherian and M finitely generated, $R(M)$ is Noetherian, [13, Theorem 4.8]. The fact that $R(M)$ is an RM-ring follows from the fact that $0 \neq P(+)M$ is a prime (maximal) ideal of $R(M)$ if and only if $0 \neq P$ is a prime (maximal) ideal of R. □

A ring R is called *semisimple* if its Jacobson radical is zero. R is called a *Hilbert ring* if for each prime ideal P of R, R/P has a zero Jacobson radical. Equivalently, R is Hilbert if every proper prime ideal of R is the intersection of maximal ideals of R. Examples of Hilbert rings include principal ideal domains with finitely many maximal ideals and zero-dimensional rings with identity. Finally, a ring R is called a *G-ring* if it has a nonzero pesudoradical (the pesudoradical of a ring R is the intersection of nonzero prime ideals of R). For properties of semisimple, Hilbert and G-rings, see [19]. The next result shows how semisimple, Hilbert and G-properties of $R(M)$ are related to those of R.

Proposition 9. *Let R be a ring and M an R-module.*

(1) If $R(M)$ is semisimple, then too is R and in this case $R(M) \cong R$.
(2) R is a Hilbert ring if and only if $R(M)$ is.
(3) If R is a G-ring then so too is $R(M)$.

Proof. (1) Let $R(M)$ be semisimple. Then $\cap P_i(+)M = 0$ where $P_i(+)M$ are the maximal ideals of $R(M)$. It follows that $\cap P_i = 0$ and $M = 0$. So $R \cong R(M)$ is semisimple.

(2) Let $R(M)$ be Hilbert. Let P be a proper ideal. Then $P(+)M$ is a proper prime ideal of $R(M)$. It follows that $R(+)M/P(+)M \cong R/P$ has zero Jacobson radical. So R is Hilbert. The statement is reversible. Equivalently, if $R(M)$ is Hilbert then for each proper prime ideal P of R (and hence each proper prime ideal $P(+)M$ of $R(M)$), $P(+)M = \bigcap_{\mu \text{ maximal}} \mu(+)M$. Therefore $P = \bigcap_{\mu \text{ maximal}} \mu$, and hence R is Hilbert. The converse is now obvious.

(3) Let R be a G-ring. Then $\bigcap_{0 \neq P} P \neq 0$, where the intersection runs over nonzero prime ideals of R. Since for each $0 \neq P$, where P is a prime ideal of R, $0 \neq P(+)M$ is a prime ideal of $R(M)$. Hence $\bigcap_{0 \neq P(+)M} P(+)M \neq 0$, and this gives that $R(M)$ is a G-ring. □

According to [19, p. 32] a ring R is called a *u-ring* if for every proper ideal I of R, $\sqrt{I} \neq R$.

Proposition 10. *Let R be a ring and M an R-module. Then $R(M)$ is a u-ring if and only if R is.*

Proof. Let $R(M)$ be a u-ring. Let I be a proper ideal of R. Then $I(+)M$ is a proper ideal of $R(M)$, and hence $\sqrt{I}(+)M = \sqrt{I(+)M} \neq R(M)$, [13, Theorem 3.2]. Hence $\sqrt{I} \neq R$ and R is a *u-ring*. Conversely, assume that R is a u-ring. Let H be a proper ideal of $R(M)$. Since $0 = (0(+)M)^2 \subseteq H$, we obtain that

$0(+)M \subseteq \sqrt{H}$ and hence $\sqrt{H} = I(+)M$ for some ideal I of R. So

$$\sqrt{H} = \sqrt{\sqrt{H}} = \sqrt{I(+)M} = \sqrt{I}(+)M \neq R(M),$$

and this shows that $R(M)$ is a u-ring. □

A ring R is called a *multiplication ring* if every ideal of R is multiplication, [21]. It is called a *herditary* ring if every ideal of R is projective, [18]. A ring R is said to be flat if every finitely generated (and hence every ideal of R) is flat. It is called von Neumann regular ring if every ideal of R is pure. For properties of flat rings and von Neumann regular rings, see [18]. Finally a ring R is called a *Prüfer* ring if every finitely generated regular ideal of R is invertible, [20] and [21]. We close our work by giving a result showing the important role of homogeneous ideals to study some properties of $R(M)$.

Theorem 11. *Let R be a ring and M an R-module.*

(1) If every homogeneous ideal of $R(M)$ is finitely generated (resp. multiplication) then $R(M)$ is a Noetherian (resp. multiplication) ring, and hence R is Noetherian (resp. multiplication).

(2) If every homogeneous ideal of $R(M)$ is projective (resp. finitely generated flat, pure) then $R(M) \cong R$ is a herditary (resp. flat, von Neumann regular) ring.

(3) If every homogeneous ideal of $R(M)$ is principal then $R(M)$ is a PIR, and hence R is a PIR.

(4) Let M be divisible. If every finitely generated regular homogeneous ideal of $R(M)$ is invertible then $R(M)$ is a Prüfer ring, and hence R is a Prüfer ring.

Proof. (1) Let H be an ideal of $R(M)$. Then $H + 0(+)M$ and $H \cap 0(+)M$ are homogeneous ideals of $R(M)$. So $H + 0(+)M$ and $H \cap 0(+)M$ are finitely generated (resp. multiplication). It follows by [22, Ex. 23, p. 13] and [28, Theorem 8] that H is a finitely generated (resp. multiplication) ideal of $R(M)$. Hence $R(M)$ is Noetherian (resp. multiplication). Next, $0(+)M$ is a finitely generated (resp. multiplication) ideal of $R(M)$. It follows by [12, Theorem 3.1] that M is finitely generated (resp. multiplication). The fact that R is Noetherian (resp. multiplication) follows by [13, Theorem 4.8] and [4, Theorem 11].

(2) Let H be an ideal of $R(M)$. Then $H + 0(+)M$ and $H \cap 0(+)M$ are homogeneous ideals of $R(M)$. Assume $H \cap 0(+)M = 0(+)N$ for some submodule N of M. It is shown, [4, Proposition 4] that if $0(+)N$ is projective (resp. finitely generated flat, pure) then $N = 0$. This implies that $H \cap 0(+)M = 0$, and hence $H \oplus 0(+)M$ is projective (resp. finitely generated flat, pure). It follows that H is projective (resp. finitely generated flat, pure), and hence $R(M)$ is a herditary (resp. flat, von Neumann regular) ring. Next, since $0(+)M$ is projective (resp. finitely generated flat, pure), $M = 0$ and hence $R(M) \cong R$.

(3) Suppose that H is an ideal of $R(M)$. Then $H + 0(+)M$ and $H \cap 0(+)M$ are principal ideals of $R(M)$ since they are homogeneous. Since $H + 0(+)M$ is principal (hence multiplication), we infer from [9, Corollary 2.2] that

$$H(0(+)M) = (H + 0(+)M)(H \cap 0(+)M),$$

and hence $H(0(+)M)$ is principal. But $0(+)M$ is homogeneous, and hence is principal. So H is principal and hence $R(M)$ is PIR. The fact that R is a PIR follows from [4, Theorem 11].

(4) Let H be a finitely generated regular ideal of $R(M)$. Since M is divisible, it follows by [13, Theorem 3.9] that H is homogeneous. So H is invertible and hence $R(M)$ is a Prüfer ring. It follows by [3, Theorem 15] that R is a Prüfer ring. $\square$

References

[1] T. Albu and P.F. Smith, *Primality, irreducibility, and complete irreducibility in modules over commutative rings*, Rev. Roumaine Math. Pures Appl., **54** (2009) 275-286.

[2] M.M. Ali, *Multiplication modules and homogeneous idealization III*, Beitrage zur Algebra und Geom., **49** (2008) 449-479.

[3] M.M. Ali, *Idealization and theorems of D. D. Anderson II*, Commun. Algebra, **35** (2007) 2767-2792.

[4] M.M. Ali, *Idealization and theorems of D. D. Anderson*, Commun. Algebra, **34** (2006) 4479-4501.

[5] M.M. Ali, : *Some remarks on multiplication and flat modules*, J. Commutative Algebra, **4** (2012) 1-27.

[6] M.M. Ali, *Residual submodules of multiplication modules*, Beitrage zur Algebra und Geom., **46** (2005) 405-422.

[7] M.M. Ali, *Invertibility of multiplication modules*, New Zealand J. Math., **35** (2006) 17-29.

[8] M.M. Ali and D.J. Smith, *Some remarks on multiplication and projective modules*, Commun. Algebra, **32** (2004) 3897-3909.

[9] M.M. Ali and D.J. Smith, *Finite and infinite collections of multiplication modules*, Beitrage zur Algebra und Geom., **42** (2001) 557-573.

[10] M.M. Ali and D.J. Smith, *Pure submodules of multiplication modules*, Beitrage zur Algebra und Geom., **44** (2004) 61-74.

[11] M.M. Ali and D.J. Smith, *Locally cyclic projective modules*, New Zealand J. Math., **35** (2006) 17-29.

[12] D.D. Anderson, *Cancellation modules and related modules*, Lecture Notes Pure Appl. Math., **220** (2001) 13-25. Dekker, New York.

[13] D.D. Anderson and M. Winders, *Idealization of a module*, J. Commutative Algebra, 1**(2009)** 1-56.

[14] H. Ansari-Toroghy and F. Farshadifar, *The dual notion of multiplication modules*, Taiwanese J. Math, **11** (2007) 1189-1201.

[15] H. Ansari-Toroghy and F. Farshadifar, *On the dual notion of multiplication modules,* Arab. J. Sci. Eng., **36** (2011) 925–932.

[16] A. Barnard, *Multiplication modules*, J. Algebra, **71** (1981) 174-178.

[17] Z.A. El-Bast, P. F. Smith, *Multiplication modules*, Commun. Algebra, **16** (1988) 755-799.

[18] C. Faith, *Algebra I: Rings, Modules and Categories*, Springer-Verlag, 1981.

[19] R. Gilmer, *Multiplicative Ideal Theory,* Queen's Paper in Pure and Applied Mathematics 90, Kingston, 1992.

[20] J.A. Huckab, *Commutative Rings with Zero Divisors*, Marcel Dekker, New York, 1988.

[21] M.D. Larsen and P. J. McCarthy, *Multiplicative Theory of Ideals*, Pure and Applied Mathematics 43, New York-London, 1971.

[22] H. Matsumura, *Commutative Ring Theory,* Cambridge University Press, 1986.

[23] M. Nagata, *Local Rings*, Interscience Publisher, New York-London-Sydney, 1962.

[24] A.G. Naoum, *Flat modules and multiplication modules*, Periodica Math. Hungar., **21** (1990) 309-317.

[25] A.G. Naoum and F.H. Al-Alwan, *Dedekind modules*, Commun. Algebra, **24** (1996) 397-412.

[26] A.G. Naoum and M.A.K. Hasan, *The residual of finitely generated multiplication modules*, Arch. Math., **46** (1986) 225-230.

[27] A.G. Naoum and M.A.K. Hasan, *On finitely generated projective and flat ideals*, Period. Math. Hungar., **16** (1985) 251-260.

[28] P.F. Smith, *Some remarks on multiplication modules*, Arch. Math., **50** (1988) 223-235.

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