NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 42 (2012), 27-36

ON THE GROWTH OF SOLUTIONS OF HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

Nan Wu and Ye-Zhou Li

(Received 26 February, 2011)

Abstract. In this paper, we discuss the growth order of the meromorphic solution of the differential equation $A_n(z)f^{(n)}(z)+\cdots+A_1(z)f'(z)+A_0f(z)=F(z)$ with meromorphic coefficients $A_0(z), A_1(z), \cdots, A_n(z)$ in some angular regions and generalize a result in the complex plane $\mathbb{C}$.

1. Introduction and Main Results

Let $f(z)$ be a meromorphic function on the whole complex plane. We suppose the readers know the standard notation of the Nevanlinna theory of meromorphic functions, such as $T(r,f), N(r,f), m(r,f), \delta(a,f)$. For the details, see [**5, 6, 7**]. The order and lower order of f are defined as follows:

$$\sigma(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}$$

and

$$\mu(f)=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

In this article, $\overline{\Omega}(\alpha,\beta)=\{z:\alpha\le\arg z\le\beta\}$, $\Omega(\alpha,\beta)=\{z:\alpha<\arg z<\beta\}$, Ω denotes an angular region.

In 2003, Belaidi and Hamani [**2**] investigated the growth of solutions to the differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0, \tag{1.1}$$

where $A_0,\cdots,A_{k-1}$ are entire functions with $A_0\not\equiv 0$ in $\mathbb{C}$. They obtained a theorem as follows:

Theorem A. *Let $A_0,\cdots,A_{k-1}$ be entire functions that satisfy*

$$\max_{0\le j\le k-1}\{\sigma(A_j)\}<\sigma(A_1).$$

Then every solution $f\not\equiv 0$ to (1.1) *of finite order satisfies $\sigma(f)\ge\sigma(A_1)$.*

Let $f(z)$ be a meromorphic function in an angular region $\overline{\Omega}(\alpha,\beta)$. Recall the definition of Ahlfors-Shimizu characteristic in an angular region (see [**6**]). Set $\Omega(r)=\Omega(\alpha,\beta)\cap\{z:1<|z|<r\}=\{z:\alpha<\arg z<\beta, 1<|z|<r\}$. Define

$$\mathcal{S}(r,\Omega,f)=\frac{1}{\pi}\iint_{\Omega(r)}\left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma$$

2010 *Mathematics Subject Classification* Primary 30D10; Secondary 30D20, 30B10, 34M05.
Key words and phrases: Differential equations, Meromorphic solutions, Order, Angular region.

and

$$\mathcal{T}(r,\Omega,f)=\int_1^r\frac{\mathcal{S}(t,\Omega,f)}{t}dt.$$

The order and lower order of f on Ω are defined as follows

$$\sigma_{\alpha,\beta}(f)=\limsup_{r\to\infty}\frac{\log\mathcal{T}(r,\Omega,f)}{\log r}$$

and

$$\mu_{\alpha,\beta}(f)=\liminf_{r\to\infty}\frac{\log\mathcal{T}(r,\Omega,f)}{\log r}.$$

Remark. We remark that the order $\sigma_{\alpha,\beta}(f)$ of a meormorphic function f on an angular region here we give is reasonable, because $\mathcal{T}(r,\mathbb{C},f)=T(r,f)+O(1)$.

Suppose $f\not\equiv 0$ is a function analytic on for fixed r. We define the maximum modulus $M(r,\overline{\Omega}(\alpha,\beta),f)=\max_{\alpha\le\theta\le\beta}|f(re^{i\theta})|$ of $f(z)$ on $\overline{\Omega}(r,\alpha,\beta)$ and the order $\rho_{\alpha,\beta}(f)$ on $\overline{\Omega}(\alpha,\beta)$ by

$$\rho_{\alpha,\beta}(f)=\limsup_{r\to\infty}\frac{\log\log M(r,\overline{\Omega}(\alpha,\beta),f)}{\log r}.$$

In 1994, Wu [**9**] studied the growth of solutions to the second order linear differential equation in the angle and gave Theorem B as follows.

Theorem B. *Let $A(z)$ and $B(z)$ be analytic on $\overline{\Omega}(\alpha,\beta)$. If for any $K>0$, the measure of*

$$\left\{\theta:\alpha<\theta<\beta,\liminf_{r\to\infty}\frac{(|A(re^{i\theta})|+1)r^K}{|B(re^{i\theta})|}=0\right\}$$

is bigger than zero, then any solution $f\not\equiv 0$ to the equation

$$f''+A(z)f'+B(z)f=0 \tag{1.2}$$

has $\rho_{\alpha,\beta}(f)=+\infty$.

In 1992, Wu [**8**] also obtained a growth theorem in the angular region as follows.

Theorem C. *Let A and B be meromorphic in $\mathbb{C}$ with $\sigma(A)<\sigma(B)$ and $\delta(\infty,B)>0$. Then every nontrivial meromorphic solution f to (1.2) has infinite order. Furthermore, if $\sigma(B)\le 1/2$ and $\delta(\infty,B)=1$, then $\sigma_{\alpha,\beta}(f)=+\infty$ for every angular region $\Omega(\alpha,\beta)$.*

In this paper, we consider the following differential equation

$$A_n(z)f^{(n)}(z)+A_{n-1}(z)f^{(n-1)}(z)+\cdots+A_0(z)f(z)=F(z), \tag{1.3}$$

where $A_0(z),\cdots,A_n(z),F(z)$ are meromorphic functions in $\mathbb{C}$. We investigate the higher order and meromorphic coefficient case and give estimations of the growth of the solution in some angular regions.

Theorem 1.1. *Let $A_0(z)$ be a meromorphic function in $\mathbb{C}$ of finite lower order $\mu<\infty$ and nonzero order $0<\lambda\le\infty$ and $\delta(\infty,A_0)>0$. For any positive and finite τ with $\mu\le\tau\le\lambda$, consider the angular region $\Omega(\alpha,\beta)$ with*

$$\beta-\alpha>2\pi-\frac{4}{\tau}\arcsin\sqrt{\frac{\delta(\infty,A_0)}{2}}.$$

If $A_j(z)(j=1,2,\cdots,n)$ are meromorphic functions in $\mathbb{C}$ with $T(r,A_j)=o(T(r,A_0))$, then every solution $f\not\equiv 0$ to the equation

$$A_nf^{(n)}+A_{n-1}f^{(n-1)}+\cdots+A_0f=0$$

has the order $\sigma_{\alpha,\beta}(f) = +\infty$ *in* $\Omega(\alpha, \beta)$.

It is easy to see that Theorem C is a consequence of Theorem 1.1.

Theorem 1.2. *Let* $A_1, \cdots, A_n, F$ *be meromorphic functions in the complex plane* $\mathbb{C}$, $f \not\equiv 0$ *be a meromorphic solution to equation* (1.3) *of finite order. If the angular region* $\Omega(\alpha, \beta)$ *satisfies*

$$\beta - \alpha > \left\{ \frac{\pi}{\sigma_{\alpha,\beta}(A_j)}, \frac{\pi}{\sigma_{\alpha,\beta}(F)}, \frac{\pi}{\sigma_{\alpha,\beta}(f)}; j = 1, 2, \cdots, n \right\}$$

and $\max\limits_{j \neq k}\{\sigma_{\alpha,\beta}(A_j), \sigma_{\alpha,\beta}(F)\} < \sigma_{\alpha+\delta,\beta-\delta}(A_k)$, *where* $\delta > 0$ *is small enough, then* $\sigma_{\alpha,\beta}(f) \geq \sigma_{\alpha+\delta,\beta-\delta}(A_k)$.

Finally, we give Theorem 1.3, which is the complex plane case of Theorem 1.2 and is a generalization of Theorem A.

Theorem 1.3. *Let* $A_1, \cdots, A_n, F$ *be meromorphic functions in the complex plane* $\mathbb{C}$, $f \not\equiv 0$ *be a meromorphic solution of equation* (1.3) *of finite order. If*

$$\max_{0 \leq j \leq n, j \neq k}\{\sigma(A_j), \sigma(F)\} < \sigma(A_k),$$

then $\sigma(f) \geq \sigma(A_k)$.

2. Basic Knowledge and Some Lemmas

In order to prove the theorems, we give some lemmas. The following result is from [**11, 12, 13**].

Lemma 2.1. *Let* $T(r)$ *be a non-negative and non-decreasing real function in* $(0, \infty)$ *with lower order*

$$\mu(T) = \liminf_{r \to \infty} \frac{\log T(r)}{\log r} < \infty$$

and order

$$0 < \lambda(T) = \limsup_{r \to \infty} \frac{\log T(r)}{\log r} \leq \infty.$$

Then for any positive number $\mu \leq \sigma \leq \lambda$ *and any set* E *with finite measure, there exists a sequence* $\{r_n\}$ *such that*

(1) $r_n \notin E$, $\lim\limits_{n \to \infty} \frac{r_n}{n} = \infty$;

(2) $\liminf\limits_{n \to \infty} \frac{\log T(r_n)}{\log r_n} \geq \sigma$;

(3) $T(t) < (1 + o(1))(\frac{2t}{r_n})^\sigma T(r_n/2), t \in [r_n/n, nr_n]$;

(4) $T(t)/t^{\sigma - \varepsilon_n} \leq 2^{\sigma+1} T(r_n)/r_n^{\sigma - \varepsilon_n}, 1 \leq t \leq nr_n, \varepsilon_n = [\log n]^{-2}$.

The sequence $\{r_n\}$ is called the Pólya peaks of order σ outside E. Given a positive function $\Lambda(r)$ satisfying $\lim_{r \to \infty} \Lambda(r) = 0$, for $r > 0$ and $a \in \mathbb{C}$, we define

$$D_\Lambda(r, a) = \left\{ \theta \in [-\pi, \pi) : \log^+ \frac{1}{|f(re^{i\theta}) - a|} > \Lambda(r) T(r, f) \right\},$$

and

$$D_\Lambda(r, \infty) = \{\theta \in [-\pi, \pi) : \log^+ |f(re^{i\theta})| > \Lambda(r) T(r, f)\}.$$

The following result is called the spread relation, which was conjectured by Edrei [**3**] and proved by Baernstein [**1**].

Lemma 2.2. *Let $f(z)$ be transcendental and meromorphic in $\mathbb{C}$ of the finite lower order $\mu < \infty$ and the positive order $0 < \lambda \leq \infty$ and have one deficient value $a \in \widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. Then for any sequence of Pólya peaks $\{r_n\}$ of order $\sigma > 0, \mu \leq \sigma \leq \lambda$ and any positive function $\Lambda(r) \to 0$ as $r \to +\infty$, we have*

$$\liminf_{n\to\infty} \operatorname{meas} D_\Lambda(r_n, a) \geq \min\left\{2\pi, \frac{4}{\sigma} \arcsin\sqrt{\frac{\delta(a,f)}{2}}\right\},$$

where meas *denotes the Lebesgue measure of the set $D_\Lambda(r_n, a)$.*

Nevanlinna theory on the angular domain plays an important role in this paper. Let us recall the following terms (see [**4**]):

$$A_{\alpha,\beta}(r,f) = \frac{\omega}{\pi}\int_1^r \left(\frac{1}{t^\omega} - \frac{t^\omega}{r^{2\omega}}\right)[\log^+|f(te^{i\alpha})| + \log^+|f(te^{i\beta})|]\frac{dt}{t},$$
$$B_{\alpha,\beta}(r,f) = \frac{2\omega}{\pi r^\omega}\int_\alpha^\beta \log^+|f(re^{i\theta})|\sin\omega(\theta-\alpha)d\theta,$$
$$C_{\alpha,\beta}(r,f) = 2\sum_{1<|b_n|<r}\left(\frac{1}{|b_n|^\omega} - \frac{|b_n|^\omega}{r^{2\omega}}\right)\sin\omega(\theta_n - \alpha),$$

where $\omega = \pi/(\beta - \alpha)$, and $b_n = |b_n|e^{i\theta_n}$ is a pole of $f(z)$ in the angular domain $\Omega(\alpha, \beta)$, appears according to its multiplicity. The Nevanlinna's angular characteristic is defined as follows:

$$S_{\alpha,\beta}(r,f) = A_{\alpha,\beta}(r,f) + B_{\alpha,\beta}(r,f) + C_{\alpha,\beta}(r,f).$$

The following lemma (see [**4**]) is the Nevanlinna first and second fundamental theorem on the angular domains.

Lemma 2.3. *Let f be a nonconstant meromorphic function on the angular domain $\Omega(\alpha, \beta)$. Then for any complex number a,*

$$S_{\alpha,\beta}(r,f) = S_{\alpha,\beta}\left(r, \frac{1}{f-a}\right) + O(1), \quad r \to \infty;$$

and for any $q(\geq 3)$ distinct points $a_j \in \widehat{\mathbb{C}}$ $(j = 1, 2, \ldots, q)$,

$$(q-2)S_{\alpha,\beta}(r,f) \leq \sum_{j=1}^q C_{\alpha,\beta}\left(r, \frac{1}{f-a_j}\right) + Q_{\alpha,\beta}(r,f),$$

where

$$Q_{\alpha,\beta}(r,f) = (A+B)_{\alpha,\beta}\left(r, \frac{f'}{f}\right) + \sum_{j=1}^q (A+B)_{\alpha,\beta}\left(r, \frac{f'}{f-a_j}\right) + O(1).$$

The key point is the estimation of the error term $Q_{\alpha,\beta}(r, f)$, which can be obtained for our purpose of this paper as follows. And the following is true (see [**4**]). Write

$$Q_{\alpha,\beta}(r,f) = A_{\alpha,\beta}\left(r, \frac{f^{(p)}}{f}\right) + B_{\alpha,\beta}\left(r, \frac{f^{(p)}}{f}\right).$$

Then

(1)$Q_{\alpha,\beta}(r, f) = O(\log r)$ as $r \to \infty$, when $\sigma(f) < \infty$;

(2)$Q_{\alpha,\beta}(r,f) = O(\log r + \log T(r,f))$ as $r \to \infty$ and $r \notin E$ when $\sigma(f) = \infty$, where E is a set with finite linear measure.

Zheng [**13**] compared the relationship between $T(r,f)$ and $T(r,f^{(k)})$ and obtained

$$\begin{aligned} T(r,f^{(k)}) &\leq N(r,f^{(k)}) + m(r,f) + m(r,\frac{f^{(k)}}{f}) \\ &= k\overline{N}(r,f) + T(r,f) + m(r,\frac{f^{(k)}}{f}) \\ &\leq (k+1)T(r,f) + m(r,\frac{f^{(k)}}{f}). \end{aligned}$$

The following result is useful for our study, the proof of which is similar to the case of the characteristic function $T(r,f)$ and $T(r,f^{(k)})$ on the whole complex plane, see [**12**].

Lemma 2.4. *Let $f(z)$ be a meromorphic function on the whole complex plane. Then for any angular domain $\Omega(\alpha,\beta)$, we have*

$$S_{\alpha,\beta}(r,f^{(p)}) \leq (p+1)S_{\alpha,\beta}(r,f) + Q_{\alpha,\beta}(r,f).$$

Wu [**8**] established the following which is estimates for the logarithmic derivative of a meromorphic function in an angular region.

Lemma 2.5. *Let f be a meromorphic function on the angular region $\overline{\Omega}(\alpha,\beta)$ with finite order ρ, let $\Gamma = \{(n_1,m_1),(n_2,m_2),\cdots,(n_j,m_j)\}$ denote a finite set of distinct pairs of integers satisfying $n_i > m_i \geq 0$ for $i = 1,2,\cdots,j$, and let $\varepsilon > 0$ and $\delta > 0$ be given constants. Then there exists $K > 0$ depending only on f,ε,δ such that*

$$\left|\frac{f^{(n)}(z)}{f^{(m)}(z)}\right| < K|z|^{(n-m)(k_\delta+2\rho+1+\varepsilon)}(\sin k_\delta(\varphi-\alpha-\delta))^{-2^{n-m}}, \tag{2.1}$$

for all $(n,m) \in \Gamma$ and all $z = re^{i\varphi} \in \Omega(\alpha+\delta,\beta-\delta)$ except for an $\mathbb{R}-$set, that is, a countable union of discs whose total radii have finite sum, where $k_\delta = \pi/(\beta-\alpha-2\delta)$.

In order to make it clearly, we give the definition of an $\mathbb{R}-$set.

Definition 2.1. Let $B(z_n,r_n) = \{z : |z-z_n| < r_n\}$ be an open disk on the complex plane. If $\sum_{n=1}^{\infty} r_n < \infty$. We call $\cup_{n=1}^{\infty} B(z_n,r_n)$ an $\mathbb{R}-$set.

The following result was firstly established by Zheng [12, Theorem 2.4.7], which gives the relation between $S_{\alpha,\beta}(r,f)$ and $\mathcal{T}(r,\Omega,f)$, and it is crucial for our study.

Lemma 2.6. *Let $f(z)$ be a function meromorphic on $\Omega = \Omega(\alpha,\beta)$. Then*

$$S_{\alpha,\beta}(r,f) \leq 2\omega^2\frac{\mathcal{T}(r,\Omega,f)}{r^\omega} + \omega^3\int_1^r \frac{\mathcal{T}(t,\Omega,f)}{t^{\omega+1}}dt + O(1),$$

and for $\varepsilon > 0$ small enough, putting $\Omega_\varepsilon = \{z : \alpha+\varepsilon < \arg z < \beta-\varepsilon\}$,

$$S_{\alpha,\beta}(r,f) \geq \omega\sin(\omega\varepsilon)\frac{\mathcal{T}(r,\Omega_\varepsilon,f)}{r^\omega} + \omega^2\sin(\omega\varepsilon)\int_1^r \frac{\mathcal{T}(t,\Omega_\varepsilon,f)}{t^{\omega+1}}dt + O(1),\ \ \omega = \frac{\pi}{\beta-\alpha}.$$

3. Proof of Theorem 1.1

Proof. We suppose that there exists a nontrivial meromorphic solution f such that $\sigma_{\alpha,\beta}(f) < +\infty$. In view of Lemma 2.5, there exists a constant $M > 0$ not depending on z such that

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| < |z|^M \text{ for } \ j = 1, 2, \cdots, n$$

for all $z \in \Omega(\alpha + \varepsilon, \beta - \varepsilon)$ except for a $\mathbb{R}-$set D. For D, we can define a set $E = \{r > 0 | \exists z \in D, s.t. |z| = r\}$ thus

$$\operatorname{meas} E < \infty.$$

Since $\mu(A_0) < \infty$, $T(r, A_0)$ satisfies the condition of Lemma 2.1, there exists a sequence of Pólya peaks $\{r_n\}$ of order τ of $T(r, A_0)$ and $r_n \notin E$. We define a real function $\Lambda(r)$ by

$$\Lambda(r)^2 = \max\left\{\frac{T(r_n, A_j)}{T(r_n, A_0)}, \frac{\log r_n}{T(r_n, A_0)}; j = 1, 2, \cdots, n\right\}, \text{ for } r_n \leq r < r_{n+1}.$$

Obviously $\lim_{r\to\infty} \Lambda(r) = 0$. In light of Lemma 2.2, for all sufficiently large n and small $\varepsilon > 0$ with $\beta - \alpha > 2\pi - \frac{4}{\tau} \arcsin \sqrt{\frac{\delta(\infty, A_0)}{2}} + 4\varepsilon$, we have

$$\operatorname{meas} D_\Lambda(r_n) \geq \min\left\{2\pi, \frac{4}{\tau} \arcsin \sqrt{\frac{\delta(\infty, A_0)}{2}}\right\} - \varepsilon,$$

where

$$D_\Lambda(r_n) = \{\theta \in [0, 2\pi) : \log^+ |A_0(r_n e^{i\theta})| > \Lambda(r_n)T(r_n, A_0)\}.$$

We easily see that

$$\begin{aligned}\operatorname{meas}(D_\Lambda(r_n) \cap [\alpha + \varepsilon, \beta - \varepsilon]) &\geq \operatorname{meas}(D_\Lambda(r_n)) - \operatorname{meas}([-\pi, \pi] \setminus (\alpha + \varepsilon, \beta - \varepsilon)) \\ &= \operatorname{meas}(D_\Lambda(r_n)) - \operatorname{meas}([-\pi, \alpha + \varepsilon] \cup [\beta - \varepsilon, \pi]) \\ &> \varepsilon\end{aligned}$$

and

$$\begin{aligned}\int_{\alpha+\varepsilon}^{\beta-\varepsilon} \log^+ |A_0(r_n e^{i\theta})| d\theta &\geq \int_{D_\Lambda(r_n)\cap[\alpha+\varepsilon,\beta-\varepsilon]} \log^+ |A_0(r_n e^{i\theta})| d\theta \\ &\geq \varepsilon\Lambda(r_n)T(r_n, A_0).\end{aligned}$$

It is obvious that

$$\operatorname{meas} D'_\Lambda(r_n) = \operatorname{meas}\{\theta : r_n e^{i\theta} \in D\} = 0.$$

Set $D_{n,\varepsilon} = D_\Lambda(r_n) \cap [\alpha + \varepsilon, \beta - \varepsilon] \backslash D'_\Lambda(r_n)$. We deduce that

$$\begin{aligned}\int_{D_{n,\varepsilon}} \log^+ |A_0(r_n e^{i\theta})| d\theta &\leq \int_{D_{n,\varepsilon}} \left(\sum_{j=1}^{n} \log^+ \left|\frac{f^{(j)}(r_n e^{i\theta})}{f(r_n e^{i\theta})}\right| + \sum_{j=1}^{n} \log^+ |A_j(r_n e^{i\theta})|\right) d\theta \\ &\leq \int_{D_{n,\varepsilon}} \sum_{j=1}^{n} \log^+ |A_j(r_n e^{i\theta})| d\theta + O(\log r_n) \\ &\leq \Lambda^2(r_n)T(r_n, A_0).\end{aligned}$$

These inequalities imply that

$$\varepsilon\Lambda(r_n) \leq \Lambda^2(r_n).$$

This contradicts $\Lambda(r_n) \to 0$. □

4. Proof of Theorem 1.2

Proof. Set $q = \max_{j\neq k}\{\sigma_{\alpha,\beta}(A_j), \sigma_{\alpha,\beta}(F)\} < \sigma_{\alpha+\delta,\beta-\delta}(A_k) = p$. Suppose that f is a solution of (1.3) with $\sigma(f) < +\infty$. It follows from (1.3) that

$$-A_k(z) = A_n(z)\frac{f^{(n)}}{f^{(k)}} + \cdots + A_{k+1}(z)\frac{f^{(k+1)}}{f^{(k)}} + A_{k-1}(z)\frac{f^{(k-1)}}{f^{(k)}} + \cdots + A_0(z)\frac{f}{f^{(k)}} - \frac{F(z)}{f^{(k)}}. \tag{4.1}$$

Hence from Nevanlinna theory of value distribution of meromorphic functions in the angular region, we have

$$\begin{aligned} S_{\alpha,\beta}(r, A_k) &\leq \sum_{0\leq j\leq n, j\neq k} \left(S_{\alpha,\beta}(r, A_j) + S_{\alpha,\beta}\left(r, \frac{f^{(j)}}{f^{(k)}}\right)\right) \\ &\qquad\qquad + S_{\alpha,\beta}(r, F) + S_{\alpha,\beta}\left(r, \frac{1}{f^{(k)}}\right) + O(1) \\ &\leq \sum_{0\leq j\leq n, j\neq k} (S_{\alpha,\beta}(r, A_j) + S_{\alpha,\beta}(r, f^{(j)})) + S_{\alpha,\beta}(r, F) \\ &\qquad\qquad + (n+1)S_{\alpha,\beta}\left(r, \frac{1}{f^{(k)}}\right) + O(1) \\ &= \sum_{0\leq j\leq n, j\neq k} (S_{\alpha,\beta}(r, A_j) + S_{\alpha,\beta}(r, f^{(j)})) \\ &\qquad\qquad + S_{\alpha,\beta}(r, F) + (n+1)S_{\alpha,\beta}(r, f^{(k)}) + O(1) \\ &\leq \sum_{0\leq j\leq n, j\neq k} (S_{\alpha,\beta}(r, A_j) + (j+1)S_{\alpha,\beta}(r, f)) + S_{\alpha,\beta}(r, F) \\ &\qquad\qquad + (n+1)(k+1)S_{\alpha,\beta}(r, f) + O(\log r) \end{aligned} \tag{4.2}$$

holds for all r outside a set $E \subset [0, \infty)$ with $\operatorname{meas} E = \delta < +\infty$. It follows from (4.2) that for all $r \notin E$

$$S_{\alpha,\beta}(r, A_k) \leq \sum_{0\leq j\leq n, j\neq k} S_{\alpha,\beta}(r, A_j) + O(S_{\alpha,\beta}(r, f)) + S_{\alpha,\beta}(r, F) + O(\log r). \tag{4.3}$$

By Lemma 2.6 and (4.3), we have

$$\omega\sin(\omega\varepsilon)\frac{\mathcal{T}(r, \Omega_\varepsilon, A_k)}{r^\omega} + O(1) \leq 2\omega^2\frac{\mathcal{T}(r, \Omega)}{r^\omega} + \omega^3\int_1^r \frac{\mathcal{T}(t, \Omega)}{t^{\omega+1}}dt + O(\log r), \tag{4.4}$$

where $\mathcal{T}(t, \Omega) = \sum_{0\leq j\leq n, j\neq k} \mathcal{T}(r, \Omega, A_j) + O(\mathcal{T}(r, \Omega, f)) + \mathcal{T}(r, \Omega, F)$.
Since $\sigma_{\alpha+\delta,\beta-\delta}(A_k) = p$, by the existence of Pólya peaks, there exists $\{r_n\} \to \infty$ outside E such that

$$\liminf_{r_n\to\infty} \frac{\log \mathcal{T}(r_n, \Omega_\delta, A_k)}{\log r_n} \geq p. \tag{4.5}$$

Therefore, for any given $0 < \varepsilon < (p-q)/2$, and for $0 \le j \le n, j \ne k$

$$\mathcal{T}(r_n, \Omega, A_j) < r_n^{q+\varepsilon},\ \mathcal{T}(r_n, \Omega, A_k) > r_n^{p-\varepsilon},\ \mathcal{T}(r_n, \Omega, F) < r_n^{q+\varepsilon} \tag{4.6}$$

hold for sufficiently large r_n. Moreover, putting $\sigma = \sigma_{\alpha,\beta}(f)$, then

$$\mathcal{T}(t, \Omega, A_j) < t^{q+\varepsilon},\ \mathcal{T}(t, \Omega, F) < t^{q+\varepsilon},\ \ \mathcal{T}(t, \Omega, f) < t^{\sigma+\varepsilon} \text{ as } t \to \infty.$$

Thus, we have

$$\int_1^{r_n} \frac{\mathcal{T}(t, \Omega)}{t^{\omega+1}} dt \le O(r_n^{q+\varepsilon-\omega} + r_n^{\sigma+\varepsilon-\omega}). \tag{4.7}$$

From (4.4), (4.6) and (4.7) we get for sufficiently large r_n,

$$r_n^{p-\varepsilon} \le O(r_n^{q+\varepsilon} + r_n^{\sigma+\varepsilon}) + O(r_n^{\omega} \log r_n). \tag{4.8}$$

In view of $O(r_n^{q+\varepsilon}) < \frac{1}{2} r_n^{p-\varepsilon}$, and $O(r_n^{\omega} \log r_n) < r_n^{\sigma+\varepsilon}$, we obtain

$$\frac{1}{2} r_n^{p-\varepsilon} \le O(r_n^{\sigma+\varepsilon}).$$

Hence, $\sigma_{\alpha,\beta}(f) = \sigma \ge p$. □

5. Proof of Theorem 1.3

Proof. Set $\iota = \max_{j \ne k}\{\sigma(A_j), \sigma(F)\} < \sigma(A_k) = \kappa$. Suppose that f is a solution to (1.3) with $\sigma(f) = \sigma < +\infty$. It follows from (1.3) that

$$-A_k(z) = A_n(z)\frac{f^{(n)}}{f^{(k)}} + \cdots + A_{k+1}(z)\frac{f^{(k+1)}}{f^{(k)}} + A_{k-1}(z)\frac{f^{(k-1)}}{f^{(k)}} + \cdots + A_0(z)\frac{f}{f^{(k)}} - \frac{F(z)}{f^{(k)}}. \tag{5.1}$$

Hence from Nevanlinna theory of value distribution of meromorphic functions in the complex plane,

$$\begin{aligned}
T(r, A_k) &\le \sum_{0 \le j \le n, j \ne k} \left(T(r, A_j) + T\left(r, \frac{f^{(j)}}{f^{(k)}}\right)\right) + T(r, F) + T\left(r, \frac{1}{f^{(k)}}\right) + O(1) \\
&\le \sum_{0 \le j \le n, j \ne k} (T(r, A_j) + T(r, f^{(j)})) + T(r, F) + (n+1)T\left(r, \frac{1}{f^{(k)}}\right) + O(1) \\
&= \sum_{0 \le j \le n, j \ne k} (T(r, A_j) + T(r, f^{(j)})) + T(r, F) + (n+1)T(r, f^{(k)}) + O(1) \\
&\le \sum_{0 \le j \le n, j \ne k} (T(r, A_j) + (j+1)T(r, f)) + T(r, F) \\
&\quad + (n+1)(k+1)T(r, f) + O(\log r)
\end{aligned} \tag{5.2}$$

holds for all r outside a set $E \subset [0, \infty)$ with $\operatorname{meas} E = \delta < +\infty$. It follows from (5.2) that for all $r \notin E$

$$T(r, A_k) \le \sum_{0 \le j \le n, j \ne k} T(r, A_j) + O(T(r, f)) + T(r, F) + O(\log r). \tag{5.3}$$

Since $\sigma(A_k) = \kappa$, by the existence of Pólya peaks, there exists $\{r_n\} \to \infty$ outside E such that

$$\liminf_{r_n \to \infty} \frac{\log T(r_n, A_k)}{\log r_n} \ge \kappa.$$

Therefore for any given $0 < \varepsilon < (\kappa - \iota)/2$, and for $0 \le j \le n, j \ne k$

$$T(r_n, A_j) < r_n^{\iota+\varepsilon},\ T(r_n, A_k) > r_n^{\kappa-\varepsilon},\ \ T(r_n, F) < r_n^{\iota+\varepsilon} \tag{5.4}$$

hold for sufficiently large r_n. From (5.3) and (5.4), we get for sufficiently large r_n,

$$r_n^{\kappa-\varepsilon} \le O(r_n^{\iota+\varepsilon} + r_n^{\sigma+\varepsilon}) + O(\log r_n). \tag{5.5}$$

Therefore $\sigma(f) = \sigma \ge \kappa$. □

References

[1] A. Baernstein,*Proof of Edrei's spead conjecture,* Proc. London Math. Soc., **26** (1973), 418-434.

[2] B. Belaidi and K. Hamani, *Order and hyper-order of entire solutions of linear differential equations with entire coefficients,* Electronic J. of Differ. Equations, **17** (2003), 1-12.

[3] A. Edrei, *Sums of deficiencies of meromorphic function*, J. Analyse Math., **14** (1965), pp.79-107

[4] A. A. Goldberg and I. V. Ostrovskii, *The Distribution of Values of Meromorphic Functions* (in Russian), Izdat. Nauk. Moscow, 1970.

[5] W. K. Hayman, *Meromorphic Functions*, Oxford, 1964.

[6] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen Co. LTD Tokyo, 1959.

[7] L. Yang, *Value Distribution and New Research*, Springer-Verlag, Berlin, 1993.

[8] S. J. Wu, *Estimates for the logarithmic derivative of a meromorphic function in an angle, and their application.* Conference Proceedings and Lecture Notes in Analysis I (Zhong Li, Fu Yao Ren, Lo Yang and Shun Yan Zhang eds.), International Press, Cambridge, MA, 1994, 235-240.

[9] S. J. Wu, *On the growth of solutions of second order linear differential equations in an angle,* Complex Variables, **24** (1994), 241-248.

[10] J. F. Xu and H. X. Yi, *Solutions of higher order linear differential equations in an angle*, Applied Mathematics Letters, **22** (4) (2009), 484-489.

[11] L. Yang, *Borel directions of meromorphic functions in an angular domain*, Sci. in China, Math. (**I**) (1979), 149-163.

[12] J. H. Zheng, *Value Distribution of Meromorphic Functions*, Springer-Verlag, Berlin, 2010.

[13] J. H. Zheng, *On transcendental meromorphic functions with radially distributed values*, Sci. in China Ser.A. Math., **47** (3) (2004), 401-416.

Nan Wu
Department of Mathematics,
School of Science,
China University of Mining and Technology (Beijing),
Beijing, 100083,
People's Republic of China.
wunan2007@163.com

Ye-Zhou Li
School of Science,
Beijing University of Posts and Tele Communications,
Beijing, 100876,
People's Republic of China.
yiyexiaoquan@yahoo.com.cn