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THE CURRENT GALILEI ALGEBRA AND ASSOCIATED RANDOM VARIABLES

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Abstract. We compute the Fock kernel for the one-mode (i.e. restricted to a single interval) and current (i.e. extended to simple functions) Galilei algebra. We also compute the characteristic function of a family of random variables naturally associated with the Galilei algebra.

1. The Galilei and RHPWN Lie Algebras

Definition 1. *The (one–mode) Galilei algebra $\mathcal{G}$ is the Lie algebra with generators $\xi_1, \xi_2, \xi_3, \xi_4$ and commutation relations*

$$[\xi_4, \xi_1] = \xi_2 \qquad ; \qquad [\xi_4, \xi_2] = \xi_3.$$

All other commutators among generators are equal to zero.

If $a^\dagger$ and a are a Boson pair, i.e.

$$[a, a^\dagger] = 1 \; ; \; (a)^* = a^\dagger$$

then the Lie algebra generated by $\{1, p, q, q^2\}$, where

$$q = i\,(a - a^\dagger) \; ; \; p = a^\dagger + a$$

is a Boson form of $\mathcal{G}$ since

$$[q, p] = i \; ; \; [q^2, p] = 2\,i\,q \; ; \; [q^2, q] = 0$$

and, in the notation of Definition 1, we may take

$$\xi_1 = \frac{1}{2}\,q^2 \,;\, \xi_2 = -i\,q \,;\, \xi_3 = -1 \,;\, \xi_4 = p.$$

Notice that

$$(q)^* = q \; ; \; (q^2)^* = q^2 \; ; \; (p)^* = p.$$

In order to consider the smeared field form of $\{1, p, q, q^2\}$, i.e. the current Galilei algebra, we recall some basic facts about the $*$-Lie algebra of the Renormalized Higher Powers of White Noise (see [**3**]-[**7**]).

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The quantum white noise functionals $a_t^\dagger$ (creation density) and a_t (annihilation density) satisfy the Boson commutation relations

$$[a_t, a_s^\dagger] = \delta(t-s) \ ; \ [a_t^\dagger, a_s^\dagger] = [a_t, a_s] = 0,$$

where $t, s \in \mathbb{R}$ and δ is the Dirac delta function, as well as the duality relation

$$(a_s)^* = a_s^\dagger.$$

For a test function f and $n, k \in \{0, 1, 2, ...\}$, the sesquilinear forms

$$B_k^n(f) = \int_{\mathbb{R}} f(t)\, {a_t^\dagger}^n\, a_t^k\, dt,$$

where dt denotes integration with respect to Lebesgue measure μ, with involution

$$(B_k^n(f))^* = B_n^k(\bar{f})$$

were defined in [**9**]. In [**3**] and [**4**] we introduced the convolution type renormalization

$$\delta^l(t-s) = \delta(s)\, \delta(t-s) \quad ; \quad l = 2, 3, ... \tag{1.1}$$

of the higher powers of the Dirac delta function and, by restricting to test functions $f(t)$ such that $f(0) = 0$, we obtained the Renormalized Higher Powers of White Noise ($RHPWN$) $*$–Lie algebra commutation relations

$$[B_k^n(f), B_K^N(g)] = (k\, N - K\, n)\ B_{k+K-1}^{n+N-1}(f\, g).$$

2. The RHPWN Form of the Current Galilei Algebra

Lemma 1. *Let $a_t^\dagger$ and a_t be as in Section 1. Define*

$$q_t = i\,(a_t - a_t^\dagger) \ ; \ p_t = a_t^\dagger + a_t.$$

Then

$$[q_t, p_s] = i\, \delta(t-s) \ ; \ [q_t^2, p_s] = 2\, i\, q_t\, \delta(t-s)$$

$$[q_t, q_s] = [p_t, p_s] = [q_t^2, q_s^2] = [q_t^2, q_s] = 0$$

and

$$(q_s)^* = q_s \ ; \ (q_s^2)^* = q_s^2 \ ; \ (p_s)^* = p_s.$$

Proof. The proof follows easily from the commutation and duality relations satisfied by the quantum white noise functionals $a_t^\dagger$ and a_t.

$\square$

Proposition 1. *For step functions f, g vanishing at zero let*

$$\begin{aligned} q(f) &= i\,(B_1^0(f) - B_0^1(f)) \\ p(f) &= B_1^0(f) + B_0^1(f) \\ q^2(f) &= 2\,B_1^1(f) - B_2^0(f) - B_0^2(f). \end{aligned}$$

Then

$$[q(f), p(g)] = 2\,i \int_{\mathbb{R}} f\,g\,dt\ ;\ [q^2(f), p(g)] = 4\,i\,q(f\,g)$$

$$[q(f), q(g)] = [p(f), p(g)] = [q^2(f), q^2(g)] = [q^2(f), q(g)] = 0$$

and

$$(q(f))^* = q(\bar{f})\ ;\ (q^2(f))^* = q^2(\bar{f})\ ;\ (p(f))^* = p(\bar{f}).$$

Proof. For $t, s \in \mathbb{R}$

$$q_t\,q_s\,\delta(t-s) = -(a_t - a_t^\dagger)(a_s - a_s^\dagger)\,\delta(t-s)$$
$$= -a_t^\dagger\,a_s^\dagger\,\delta(t-s) + a_t^\dagger\,a_s\,\delta(t-s) + a_t\,a_s^\dagger\,\delta(t-s) - a_t\,a_s\,\delta(t-s)$$
$$= -a_t^\dagger\,a_s^\dagger\,\delta(t-s) + a_t^\dagger\,a_s\,\delta(t-s) + a_s^\dagger\,a_t\,\delta(t-s) + \delta^2(t-s) - a_t\,a_s\,\delta(t-s)$$

Taking $\int_{\mathbb{R}}\int_{\mathbb{R}} f(s) \ldots ds\,dt$ of both sides and using the renormalization (1.1) we have that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} f(s)\,\delta^2(t-s)\,ds\,dt = \int_{\mathbb{R}}\int_{\mathbb{R}} f(s)\,\delta(s)\,\delta(t-s)\,ds\,dt = f(0) = 0$$

and so

$$q^2(f) = \int_{\mathbb{R}} f(t)\,q_t^2\,dt = 2\,B_1^1(f) - B_2^0(f) - B_0^2(f).$$

For $p(f)$ and $q(f)$ we directly find that

$$q(f) = \int_{\mathbb{R}} f(t)\,q_t\,dt = \int_{\mathbb{R}} f(t)\,i\,(a_t - a_t^\dagger)\,dt = i\,(B_1^0(f) - B_0^1(f))$$

and

$$p(f) = \int_{\mathbb{R}} f(t)\,p_t\,dt = \int_{\mathbb{R}} f(t)\,(a_t + a_t^\dagger)\,dt = B_1^0(f) + B_0^1(f).$$

Moreover

$$\begin{aligned} [q(f), p(g)] &= i\,[B_1^0(f) - B_0^1(f), B_1^0(g) + B_0^1(g)] \\ &= i\,[B_1^0(f), B_0^1(g)] - i\,[B_0^1(f), B_1^0(g)] \\ &= 2\,i\,[B_1^0(f), B_0^1(g)] \\ &= 2\,i\,B_0^0(f\,g) \\ &= 2\,i \int_{\mathbb{R}} f\,g\,dt \end{aligned}$$

and

$$\begin{aligned}
[q^2(f), p(g)] &= [2\,B_1^1(f) - B_2^0(f) - B_0^2(f), B_1^0(g) + B_0^1(g)] \\
&= 2\,[B_1^1(f), B_1^0(g)] + 2\,[B_1^1(f), B_0^1(g)] - [B_2^0(f), B_0^1(g) - [B_0^2(f), B_1^0(g) \\
&= -2\,B_1^0(f\,g) + 2\,B_0^1(f\,g) - 2\,B_1^0(f\,g) + 2\,B_0^1(f\,g) \\
&= 4\,\left(B_0^1(f\,g) - B_1^0(f\,g)\right) \\
&= 4\,i\,q(f\,g).
\end{aligned}$$

Finally

$$\begin{aligned}
\left(q^2(f)\right)^* &= \left(2\,B_1^1(f) - B_2^0(f) - B_0^2(f)\right)^* \\
&= 2\,B_1^1(\bar{f}) - B_0^2(\bar{f}) - B_2^0(\bar{f}) \\
&= q^2(\bar{f})
\end{aligned}$$

and, similarly,

$$(q(f))^* = q(\bar{f}) \ ; \ (p(f))^* = p(\bar{f}).$$

$\square$

Definition 2. *The current Galilei algebra is the $*$–Lie algebra generated by*

$$\{q(f), p(f), q^2(f), 1 \ ; \ f \in \mathcal{S}_0\}$$

where $\mathcal{S}_0$ is the set of step functions vanishing at zero.

3. The Fock Kernel for the One-Mode Galilei Algebra

Definition 3. *Let $I \subset \mathbb{R}$ with $\mu(I) > 0$ and $a, b, c \in \mathbb{C}$. The exponential vector $\psi_{a,b,c}(I)$ is defined by*

$$\begin{gathered}
\psi_{a,b,c}(I) = e^{a\,q^2(\chi_I)}\,e^{b\,q(\chi_I)}\,e^{c\,p(\chi_I)}\,\Phi \\
= e^{a\,\left(2\,B_1^1(\chi_I) - B_2^0(\chi_I) - B_0^2(\chi_I)\right)}\,e^{i\,b\,\left(B_1^0(\chi_I) - B_0^1(\chi_I)\right)}\,e^{c\,\left(B_1^0(\chi_I) + B_0^1(\chi_I)\right)}\,\Phi
\end{gathered}$$

where the Fock vacuum vector Φ is such that

$$B_1^0(\chi_I)\,\Phi = B_2^0(\chi_I)\,\Phi = 0 \ ; \ B_1^1(\chi_I)\,\Phi = \frac{\mu(I)}{2}\,\Phi$$

and χ_I denotes the characteristic function of the set I.

Notice that

$$\begin{aligned}
q^2(\chi_I)\,\psi_{a,b,c}(I) &= \frac{\partial}{\partial\,a}\,\psi_{a,b,c} = \frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,\psi_{a+\epsilon,b,c}(I) \\
q(\chi_I)\,\psi_{a,b,c}(I) &= \frac{\partial}{\partial\,b}\,\psi_{a,b,c} = \frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,\psi_{a,b+\epsilon,c}(I) \\
p(\chi_I)\,\psi_{a,b,c}(I) &= \frac{\partial}{\partial\,c}\,\psi_{a,b,c} = \frac{\partial}{\partial\,\epsilon}\,|_{\epsilon=0}\,\psi_{a,b,c+\epsilon}(I)
\end{aligned}$$

Lemma 2. *For* $|a|<1$

$$(1-a)^{-B_1^1(\chi_I)}\,\Phi = (1-a)^{-\frac{\mu(I)}{2}}\,\Phi.$$

Proof.

$$\begin{aligned}
(1-a)^{-B_1^1(\chi_I)}\,\Phi &= e^{-\ln(1-a)\,B_1^1(\chi_I)}\,\Phi \\
&= \sum_{n=0}^{\infty}\frac{(-\ln(1-a))^n}{n!}\,\left(B_1^1(\chi_I)\right)^n\,\Phi \\
&= \sum_{n=0}^{\infty}\frac{(-\ln(1-a))^n}{n!}\,\left(\frac{\mu(I)}{2}\right)^n\,\Phi \\
&= e^{-\ln(1-a)\,\frac{\mu(I)}{2}}\,\Phi \\
&= (1-a)^{-\frac{\mu(I)}{2}}\,\Phi.
\end{aligned}$$

□

Lemma 3. *(i) If x, d, N and h satisfy the oscillator algebra commutation relations*

$$[d,x]=h\ ;\ [d,h]=[x,h]=0\ ;\ [N,x]=x\ ;\ [d,N]=d \tag{3.1}$$

then for all $a,b,t,s\in\mathbb{C}$

$$e^{t\,d}\,e^{a\,x}=e^{a\,x}\,e^{t\,d}\,e^{a\,t\,h} \tag{3.2}$$

$$e^{s\,(x+a\,d+b\,h)}=e^{s\,x}\,e^{s\,a\,d}\,e^{(s\,b+s^2\,a/2)h} \tag{3.3}$$

and

$$e^{a\,N}\,e^{b\,x}=e^{e^a\,b\,x}\,e^{a\,N}. \tag{3.4}$$

(ii) If Δ, R and ρ satisfy the $sl(2)$ algebra commutation relations

$$[\Delta,R]=\rho\ ;\ [\rho,R]=2\,R\ ;\ [\Delta,\rho]=2\,\Delta \tag{3.5}$$

then

$$e^{t\,\Delta}\,e^{a\,R}=e^{\frac{a}{1-a\,t}\,R}\,(1-a\,t)^{-\rho}\,e^{\frac{t}{1-a\,t}\,\Delta} \tag{3.6}$$

and

$$e^{a\,(\rho-R-\Delta)}=e^{\frac{a}{a-1}\,R}\,(1-a)^{-\rho}\,e^{\frac{a}{a-1}\,\Delta}. \tag{3.7}$$

Proof. (3.2), (3.3), (3.4) and (3.6) are, respectively, Propositions 2.2.1, 4.1.1 2.4.2 (for $N=x\,D$) and 3.3.2 of [**10**]. For (3.7) we notice that by Proposition 4.3.1 of [**10**], for all $A,B,s\in\mathbb{C}$

$$e^{s\,(R+A\,\rho+B\,\Delta)}=e^{V(s)\,R}\,e^{H(s)\,\rho}\,e^{U(s)\,\Delta}$$

where

$$\begin{aligned} V'(s) &= 1 + 2\,A\,V(s) + B\,V(s)^2 \;;\; V(0)=0 \text{ (Riccati ODE)} \\ H'(s) &= A + B\,V(s) \;;\; H(0)=0 \\ U'(s) &= B\,e^{2\,H(s)} \;;\; U(0)=0. \end{aligned}$$

For $A=-1$ and $B=1$ we obtain

$$\begin{aligned} V'(s) &= 1 - 2\,V(s) + V(s)^2 \;;\; V(0)=0 \\ H'(s) &= -1 + V(s) \;;\; H(0)=0 \\ U'(s) &= e^{2\,H(s)} \;;\; U(0)=0 \end{aligned}$$

which imply that

$$V(s) = U(s) = \frac{s}{s+1} \;;\; H(s) = -\ln(s+1)$$

i.e.

$$e^{s\,(R-\,\rho+\Delta)} = e^{\frac{s}{s+1}\,R}\,e^{-\ln(s+1)\,\rho}\,e^{\frac{s}{s+1}\,\Delta}$$

from which (3.7) follows by letting $s=-a$.

□

Corollary 1. *For $I \subset \mathbb{R}$ and $a,b,A,t,T,s \in \mathbb{C}$*

$$e^{t\,B_1^0(\chi_I)}\,e^{a\,B_0^1(\chi_I)} = e^{a\,B_0^1(\chi_I)}\,e^{t\,B_1^0(\chi_I)}\,e^{a\,t\,\mu(I)}$$

$$e^{s\,(B_0^1(\chi_I)+a\,B_1^0(\chi_I)+b\,\mu(I))} = e^{s\,B_0^1(\chi_I)}\,e^{s\,a\,B_1^0(\chi_I)}\,e^{(s\,b+s^2\,a/2)\,\mu(I)}$$

$$e^{a\,B_1^1(\chi_I)}\,e^{b\,B_0^1(\chi_I)} = e^{e^a\,b\,B_0^1(\chi_I)}\,e^{a\,B_1^1(\chi_I)} \tag{3.8}$$

$$e^{T\,B_2^0(\chi_I)}\,e^{A\,B_0^2(\chi_I)} = e^{\frac{A}{1-4\,A\,T}\,B_0^2(\chi_I)}\,(1-4\,A\,T)^{-B_1^1(\chi_I)}\,e^{\frac{T}{1-4\,A\,T}\,B_2^0(\chi_I)}$$

$$e^{A\,(2\,B_1^1(\chi_I)-B_0^2(\chi_I)-B_2^0(\chi_I))} = e^{\frac{A}{2\,A-1}\,B_0^2(\chi_I)}\,(1-2\,A)^{-B_1^1(\chi_I)}\,e^{\frac{A}{2\,A-1}\,B_2^0(\chi_I)}.$$

In particular (3.8) implies that, for $|a|<1$,

$$(1-a)^{-B_1^1(\chi_I)}\,e^{b\,B_0^1(\chi_I)} = e^{\frac{b}{1-a}\,B_0^1(\chi_I)}\,(1-a)^{-B_1^1(\chi_I)}$$

Proof. The proof follows from Lemma 3 by noticing that for a fixed $I \subset \mathbb{R}$, the operators x, D, N, h defined by $B_0^1(\chi_I)=x$, $B_1^0(\chi_I)=D$ and $B_1^1(\chi_I)=N$ and $\mu(I)\,1=h$, satisfy the oscillator algebra commutation relations (3.1) while the operators R,Δ,ρ defined by $B_0^2(\chi_I)=2\,R$, $B_2^0(\chi_I)=2\,\Delta$ and $B_1^1(\chi_I)=\rho$, satisfy the $sl(2)$ algebra commutation relations (3.5) and by letting $\frac{a}{2}=A$ and $\frac{t}{2}=T$ in (3.6) and (3.7).

□

Lemma 4. *For $I \subset \mathbb{R}$ and $a,b \in \mathbb{C}$*

$$e^{a\,B_2^0(\chi_I)}\,e^{b\,B_0^1(\chi_I)}\,\Phi = e^{a\,b^2\,\mu(I)}\,e^{b\,B_0^1(\chi_I)}\,\Phi.$$

Proof. In [**1**] we showed that

$$B_2^0(\chi_I)\, e^{b\, B_0^1(\chi_I)}\, \Phi = b^2\, \mu(I)\, e^{b\, B_0^1(\chi_I)}\, \Phi.$$

Iterating we find that, for $n \geq 0$,

$$\left(B_2^0(\chi_I)\right)^n\, e^{b\, B_0^1(\chi_I)}\, \Phi = \left(b^2\, \mu(I)\right)^n\, e^{b\, B_0^1(\chi_I)}\, \Phi.$$

Therefore

$$\begin{aligned}
e^{a\, B_2^0(\chi_I)}\, e^{b\, B_0^1(\chi_I)}\, \Phi &= \sum_{n=0}^{\infty} \frac{a^n}{n!}\, \left(B_2^0(\chi_I)\right)^n\, e^{b\, B_0^1(\chi_I)}\, \Phi \\
&= \sum_{n=0}^{\infty} \frac{a^n}{n!}\, \left(b^2\, \mu(I)\right)^n\, e^{b\, B_0^1(\chi_I)}\, \Phi \\
&= e^{a\, b^2\, \mu(I)}\, e^{b\, B_0^1(\chi_I)}\, \Phi.
\end{aligned}$$

□

Lemma 5. *For all $a, b, c \in \mathbb{C}$ and $I \subset \mathbb{R}$ with $\mu(I) > 0$ and $|a| < \frac{1}{2}$*

$$\begin{aligned}
\psi_{a,b,c}(I) &= (1-2\,a)^{-\frac{\mu(I)}{2}}\, \exp\left(\frac{b^2 + c^2 - 4\,a\,c^2 + 2\,i\,b\,c}{2-4\,a}\, \mu(I)\right) \\
&\quad \exp\left(\frac{a}{2\,a-1}\, B_0^2(\chi_I)\right)\, \exp\left(\frac{c - i\,b}{1-2\,a}\, B_0^1(\chi_I)\right)\, \Phi
\end{aligned}$$

Proof. For brevity, in what follows, we drop the use of χ_I. Using Lemmas 2 and 4, Corollary 1, and the fact that, for all $\lambda \in \mathbb{C}$, $e^{\lambda\, B_1^0}\, \Phi = e^{\lambda\, B_2^0}\, \Phi = \Phi$ we have

$$\begin{aligned}
\psi_{a,b,c}(I) &= e^{a\left(2\, B_1^1(\chi_I) - B_2^0(\chi_I) - B_0^2(\chi_I)\right)}\, e^{i\,b\left(B_1^0(\chi_I) - B_0^1(\chi_I)\right)}\, e^{c\left(B_1^0(\chi_I) + B_0^1(\chi_I)\right)}\, \Phi \\
&= e^{\frac{b^2+c^2}{2}\, \mu(I)}\, e^{\frac{a}{2\,a-1}\, B_0^2}\, (1-2\,a)^{-B_1^1}\, e^{\frac{a}{2\,a-1}\, B_2^0}\, e^{-i\,b\, B_0^1}\, e^{i\,b\, B_1^0}\, e^{c\, B_0^1}\, e^{c\, B_1^0}\, \Phi \\
&= e^{\left(\frac{b^2+c^2}{2} + i\,b\,c\right)\mu(I)}\, e^{\frac{a}{2\,a-1}\, B_0^2}\, (1-2\,a)^{-B_1^1}\, e^{\frac{a}{2\,a-1}\, B_2^0}\, e^{-i\,b\, B_0^1}\, e^{c\, B_0^1}\, e^{i\,b\, B_1^0}\, \Phi \\
&= e^{\left(\frac{b^2+c^2}{2} + i\,b\,c\right)\mu(I)}\, e^{\frac{a}{2\,a-1}\, B_0^2}\, (1-2\,a)^{-B_1^1}\, e^{\frac{a}{2\,a-1}\, B_2^0}\, e^{c - i\,b\, B_0^1}\, \Phi \\
&= e^{\left(\frac{b^2+c^2}{2} + i\,b\,c + \frac{a\,(c-i\,b)^2}{2\,a-1}\right)\mu(I)}\, e^{\frac{a}{2\,a-1}\, B_0^2}\, (1-2\,a)^{-B_1^1}\, e^{c - i\,b\, B_0^1}\, \Phi \\
&= e^{\frac{b^2+c^2-4\,a\,c^2+2\,i\,b\,c}{2-4\,a}\, \mu(I)}\, e^{\frac{a}{2\,a-1}\, B_0^2}\, e^{\frac{c-i\,b}{1-2\,a}\, B_0^1} (1-2\,a)^{-B_1^1}\, \Phi \\
&= (1-2\,a)^{-\frac{\mu(I)}{2}}\, e^{\frac{b^2+c^2-4\,a\,c^2+2\,i\,b\,c}{2-4\,a}\, \mu(I)}\, e^{\frac{a}{2\,a-1}\, B_0^2(\chi_I)}\, e^{\frac{c-i\,b}{1-2\,a}\, B_0^1(\chi_I)}\, \Phi.
\end{aligned}$$

□

Proposition 2. *(i) Let $I \subset \mathbb{R}$ with $\mu(I) > 0$ and $a, b, c, A, B, C \in \mathbb{C}$ with $|a| < \frac{1}{2}$ and $|A| < \frac{1}{2}$. Then, the Galilei Fock space inner product of the exponential vectors $\psi_{A,B,C}(I)$ and $\psi_{a,b,c}(I)$ is given by*

$$\langle \psi_{A,B,C}(I), \psi_{a,b,c}(I)\rangle \tag{3.9}$$
$$= (1-2\,a)^{-\frac{\mu(I)}{2}}\,(1-2\,\bar{A})^{-\frac{\mu(I)}{2}}\left(1-\frac{\bar{a}\,A}{4\,(2\,\bar{a}-1)\,(2\,A-1)}\right)^{-\frac{\mu(I)}{2}}$$
$$\exp\left(\frac{b^2+c^2-4\,a\,c^2+2\,i\,b\,c}{2-4\,a}\,\mu(I)\right)\exp\left(\frac{\bar{B}^2+\bar{C}^2-4\,\bar{A}\,\bar{C}^2-2\,i\,\bar{B}\,\bar{C}}{2-4\,\bar{A}}\,\mu(I)\right)$$
$$\exp\left(\frac{\mu(I)}{4}\,\frac{\frac{\bar{a}\,(C-i\,B)^2}{(2\,\bar{a}-1)\,(2\,A-1)^2}+4\,\frac{(\bar{c}+i\,\bar{b})\,(C-i\,B)}{(2\,\bar{a}-1)\,(2\,A-1)}+\frac{A\,(\bar{c}+i\,\bar{b})^2}{(2\,\bar{a}-1)^2\,(2\,A-1)}}{4-\frac{\bar{a}\,A}{(2\,\bar{a}-1)\,(2\,A-1)}}\right).$$

(ii) If $I\cap J=\oslash$ then

$$\langle \psi_{A,B,C}(I), \psi_{a,b,c}(J)\rangle = 0$$

Proof. To prove (i), as in Lemma 5, we drop the use of χ_{I}. By Lemma 5

$$\langle \psi_{A,B,C}, \psi_{a,b,c}\rangle$$
$$= (1-2\,a)^{-\frac{\mu(I)}{2}}\,(1-2\,\bar{A})^{-\frac{\mu(I)}{2}}\,e^{\frac{b^2+c^2-4\,a\,c^2+2\,i\,b\,c}{2-4\,a}\,\mu(I)}\,e^{\frac{\bar{B}^2+\bar{C}^2-4\,\bar{A}\,\bar{C}^2-2\,i\,\bar{B}\,\bar{C}}{2-4\,\bar{A}}\,\mu(I)}$$
$$\langle e^{\frac{a}{2\,a-1}\,B_0^2}\,e^{\frac{c-i\,b}{1-2\,a}\,B_0^1}\,\Phi, e^{\frac{A}{2\,A-1}\,B_0^2}\,e^{\frac{C-i\,B}{1-2\,A}\,B_0^1}\,\Phi\rangle.$$

Using the Feinsilver-Kocik-Schott Fock kernel for the Schrödinger algebra (see [**12**]), in [**2**] we showed that for all $a,b,A,B\in\mathbb{C}$, with $B_1^1(\chi_I)\,\Phi=\frac{\mu(I)}{2}\,\Phi$,

$$\langle e^{a\,B_0^2}\,e^{b\,B_0^1}\,\Phi, e^{A\,B_0^2}\,e^{B\,B_0^1}\,\Phi\rangle = \left(1-\frac{\bar{a}\,A}{4}\right)^{-\frac{\mu(I)}{2}}\,e^{\frac{\mu(I)}{4}\,\frac{\bar{a}\,B^2+4\,\bar{b}\,B+\bar{b}^2\,A}{4-\bar{a}\,A}}.$$

Therefore

$$\langle e^{\frac{a}{2\,a-1}\,B_0^2}\,e^{\frac{c-i\,b}{1-2\,a}\,B_0^1}\,\Phi, e^{\frac{A}{2\,A-1}\,B_0^2}\,e^{\frac{C-i\,B}{1-2\,A}\,B_0^1}\,\Phi\rangle$$
$$= \left(1-\frac{\bar{a}\,A}{4\,(2\,\bar{a}-1)\,(2\,A-1)}\right)^{-\frac{\mu(I)}{2}}\exp\left(\frac{\mu(I)}{4}\,\frac{\frac{\bar{a}\,(C-i\,B)^2}{(2\,\bar{a}-1)\,(2\,A-1)^2}+4\,\frac{(\bar{c}+i\,\bar{b})\,(C-i\,B)}{(2\,\bar{a}-1)\,(2\,A-1)}+\frac{A\,(\bar{c}+i\,\bar{b})^2}{(2\,\bar{a}-1)^2\,(2\,A-1)}}{4-\frac{\bar{a}\,A}{(2\,\bar{a}-1)\,(2\,A-1)}}\right)$$

and so

$$\langle \psi_{A,B,C}, \psi_{a,b,c}\rangle = (1-2\,a)^{-\frac{\mu(I)}{2}}\,(1-2\,\bar{A})^{-\frac{\mu(I)}{2}}$$
$$\exp\left(\frac{b^2+c^2-4\,a\,c^2+2\,i\,b\,c}{2-4\,a}\,\mu(I)\right)\exp\left(\frac{\bar{B}^2+\bar{C}^2-4\,\bar{A}\,\bar{C}^2-2\,i\,\bar{B}\,\bar{C}}{2-4\,\bar{A}}\,\mu(I)\right)$$
$$\left(1-\frac{\bar{a}\,A}{4\,(2\,\bar{a}-1)\,(2\,A-1)}\right)^{-\frac{\mu(I)}{2}}\exp\left(\frac{\mu(I)}{4}\,\frac{\frac{\bar{a}\,(C-i\,B)^2}{(2\,\bar{a}-1)\,(2\,A-1)^2}+4\,\frac{(\bar{c}+i\,\bar{b})\,(C-i\,B)}{(2\,\bar{a}-1)\,(2\,A-1)}+\frac{A\,(\bar{c}+i\,\bar{b})^2}{(2\,\bar{a}-1)^2\,(2\,A-1)}}{4-\frac{\bar{a}\,A}{(2\,\bar{a}-1)\,(2\,A-1)}}\right).$$

The proof of (ii) follows from the fact that B_2^0 and B_1^0 commute with B_0^2 and B_0^1 on disjoint intervals and also $B_2^0\,\Phi=B_1^0\,\Phi=0$.

$\square$

4. The Fock Kernel for the Current Galilei Algebra

Using the commutativity of the Galilei algebra generators on disjoint sets, we may extend the kernel (3.9) to exponential vectors of the form

$$\begin{aligned}\Psi(f,g,h) &= e^{q^2(f)}\, e^{q(g)}\, e^{p(h)}\,\Phi\\ &= \prod_i e^{a_i\, q^2(\chi_{I_i})}\, e^{b_i\, q(\chi_{I_i})}\, e^{c_i\, p(\chi_{I_i})}\,\Phi\\ &= \prod_i \psi_{a_i,b_i,c_i}(I_i)\end{aligned}$$

where $f=\sum_i a_i\,\chi_{I_i}$, $g=\sum_i b_i\,\chi_{I_i}$ and $h=\sum_i c_i\,\chi_{I_i}$ with $I_i\cap I_j=\oslash$ for $i\neq j$, are simple functions with $|f|<\frac{1}{2}$ and, using Proposition 2 (ii) we obtain

$$\begin{gathered}\langle\Psi(f_1,g_1,h_1),\Psi(f_2,g_2,h_2)\rangle=\prod_i\langle\psi_{a_{1i},b_{1i},c_{1i}}(I_i),\psi_{a_{2i},b_{2i},c_{2i}}(I_i)\rangle\\ =\exp\left(-\frac{1}{2}\int\ln\left((1-2\,f_1)\,\left(1-2\,\bar f_2\right)\,\left(1-\frac{\bar f_1\,f_2}{4\left(1-2\,\bar f_1\right)(1-2\,f_2)}\right)\right)\,d\mu\right)\\ \exp\left(\int\,\left(\frac{g_1^2+h_1^2-4\,f_1\,h_1^2+2\,i\,g_1\,h_1}{2-4\,f_1}+\frac{\bar g_2^2+\bar h_2^2-4\,\bar f_2\,\bar h_2^2-2\,i\,\bar g_2\,\bar h_2}{2-4\,\bar f_2}\right)\,d\mu\right)\\ \exp\left(\frac{1}{4}\int\,\frac{\frac{\bar f_1\,(h_2-i\,g_2)^2}{(2\,\bar f_1-1)\,(2\,f_2-1)^2}+4\,\frac{(\bar h_1+i\,\bar g_1)\,(h_2-i\,g_2)}{(2\,\bar f_1-1)\,(2\,f_2-1)}+\frac{f_2\,(\bar h_1+i\,\bar g_1)^2}{(2\,\bar f_1-1)^2\,(2\,f_2-1)}}{4-\frac{\bar f_1\,f_2}{(2\,\bar f_1-1)\,(2\,f_2-1)}}\,d\mu\right).\end{gathered}$$

5. Random Variables Associated with the Galilei Algebra

Definition 4. *Let $a^\dagger$ and a be a Boson pair as in Section 1. The Boson-Schrödinger algebra is the Lie algebra generated by $\{1,a,a^\dagger,a^2,{a^\dagger}^2,a^\dagger\,a\}$ with commutation relations given in the table (see also* [**12**]*)*

	a	$a^\dagger$	a^2	${a^\dagger}^2$	$a^\dagger\,a$	1
a	0	1	0	$2\,a^\dagger$	a	0
$a^\dagger$	-1	0	$-2\,a$	0	$-a^\dagger$	0
a^2	0	$2\,a$	0	$2+4\,a^\dagger\,a$	$2\,a^2$	0
${a^\dagger}^2$	$-2\,a^\dagger$	0	$-2-4\,a^\dagger\,a$	0	$-2\,{a^\dagger}^2$	0
$a^\dagger\,a$	$-a$	$a^\dagger$	$-2\,a^2$	$2\,{a^\dagger}^2$	0	0
1	0	0	0	0	0	0

Lemma 6. *Let $a^\dagger$ and a be a Boson pair. Let also $L\in\mathbb{R}$ and $M,N\in\mathbb{C}$. Then for all $s\in\mathbb{R}$*

$$e^{i\,s\,\left(L\,a^2+L\,{a^\dagger}^2-2\,L\,a^\dagger\,a-L+M\,a+N\,a^\dagger\right)}\,\Phi=e^{w_1(s)\,{a^\dagger}^2}\,e^{w_2(s)\,a^\dagger}\,e^{w_3(s)}\,\Phi$$

where

$$\begin{aligned}w_1(s) &= \frac{L\,s}{2\,L\,s-i}\\ w_2(s) &= \frac{i\,L\,(M+N)\,s^2+N\,s}{2\,L\,s-i}\end{aligned}$$

and

$$w_3(s) = \frac{(M+N)^2\,(L^2\,s^4 - 2\,i\,L\,s^3) - 3\,M\,N\,s^2}{6\,(2\,i\,L\,s+1)} - \frac{\ln{(2\,i\,L\,s+1)}}{2}.$$

Proof. The proof can be found in [**2**].

□

Corollary 2. *Let $I \subset \mathbb{R}$ with $\mu(I) > 0$. Let also $\alpha \in \mathbb{R}$ and $u, v \in \mathbb{C}$. Then for all $s \in \mathbb{R}$*

$$e^{i\,s\,\left(\alpha\,B_2^0(\chi_I)+\alpha\,B_0^2(\chi_I)-2\,\alpha\,B_1^1(\chi_I)+u\,B_1^0(\chi_I)+v\,B_0^1(\chi_I)\right)}\,\Phi$$
$$= e^{w_1(s)\,B_0^2(\chi_I)}\,e^{w_2(s)\,B_0^1(\chi_I)}\,e^{w_3(s)}\,\Phi$$

where

$$w_1(s) = \frac{\alpha\,s}{2\,\alpha\,s - i}$$
$$w_2(s) = \frac{i\,\alpha\,(u+v)\,s^2 + v\,s}{2\,\alpha\,s - i}\,\sqrt{\mu(I)}$$

and

$$w_3(s) = \frac{(u+v)^2\,(\alpha^2\,s^4 - 2\,i\,\alpha\,s^3) - 3\,u\,v\,s^2}{6\,(2\,i\,\alpha\,s+1)}\,\mu(I) - \frac{\ln{(2\,i\,\alpha\,s+1)}}{2}.$$

Proof. Using the correspondence,

$$B_0^1(\chi_I) = \sqrt{\mu(I)}\,a^\dagger\,;\; B_1^0(\chi_I) = \sqrt{\mu(I)}\,a\,;\; B_0^0(\chi_I) = \mu(I)$$
$$B_0^2(\chi_I) = {a^\dagger}^2\,;\; B_2^0(\chi_I) = a^2\,;\; B_1^1(\chi_I) = a^\dagger\,a + \tfrac{1}{2}$$

we see that

$$\alpha\,B_2^0(\chi_I) + \alpha\,B_0^2(\chi_I) - 2\,\alpha\,B_1^1(\chi_I) + u\,B_1^0(\chi_I) + v\,B_0^1(\chi_I)$$
$$= L\,a^2 + L\,{a^\dagger}^2 - 2\,L\,a^\dagger\,a - L + M\,a + N\,a^\dagger$$

where $L = \alpha$, $M = u\,\sqrt{\mu(I)}$, $N = v\,\sqrt{\mu(I)}$ and the proof follows from Lemma 6. □

Proposition 3. *(Characteristic Function) Let $\lambda_i \in \mathbb{R}\,;\, i = 1,2,3$. In the notation of Corollary 2, and in view of Proposition 1, consider the random variable (i.e. self-adjoint operator on the Galilei Fock space)*

$$X = \lambda_1\,q^2(\chi_I) + \lambda_2\,q(\chi_I) + \lambda_3\,p(\chi_I)$$
$$= \alpha\,B_2^0(\chi_I) + \alpha\,B_0^2(\chi_I) - 2\,\alpha\,B_1^1(\chi_I) + u\,B_1^0(\chi_I) + v\,B_0^1(\chi_I)$$

where $\alpha = -\lambda_1$, $u = \lambda_3 + i\,\lambda_2$, $v = \lambda_3 - i\,\lambda_2$.

Then, for all $s \in \mathbb{R}$, the (vacuum) characteristic function of the random variable X is given by

$$\langle \Phi, e^{i\,s\,X}\,\Phi\rangle = (1-2\,i\,\lambda_1\,s)^{-1/2}\,\exp\left(\frac{4\,\lambda_3^2\,(\lambda_1^2\,s^4+2\,i\,\lambda_1\,s^3)-3\,(\lambda_2^2+\lambda_3^2)\,s^2}{6\,(1-2\,i\,\lambda_1\,s)}\,\mu(I)\right).$$

Proof. Using Corollary 2 and the fact that for all $z\in\mathbb{C}$

$$e^{z\,B_2^0(\chi_I)}\,\Phi = e^{z\,B_1^0(\chi_I)}\,\Phi = \Phi$$

we have

$$\begin{aligned}\langle \Phi, e^{i\,s\,X}\,\Phi\rangle &= \langle \Phi, e^{w_1(s)\,B_0^2(\chi_I)}\,e^{w_2(s)\,B_0^1(\chi_I)}\,e^{w_3(s)}\,\Phi\rangle\\ &= \langle e^{\bar{w}_2(s)\,B_1^0(\chi_I)}\,e^{\bar{w}_1(s)\,B_2^0(\chi_I)}\,\Phi, e^{w_3(s)}\,\Phi\rangle\\ &= \langle \Phi, e^{w_3(s)}\,\Phi\rangle\\ &= e^{w_3(s)}\,\langle \Phi,\Phi\rangle\\ &= e^{w_3(s)}.\end{aligned}$$

Using the formula for $w_3(s)$ provided in Corollary 2 we find that

$$\begin{aligned}&\langle \Phi, e^{i\,s\,X}\,\Phi\rangle\\ &= (2\,i\,\alpha\,s+1)^{-1/2}\,\exp\left(\tfrac{(u+v)^2\,(\alpha^2\,s^4-2\,i\,\alpha\,s^3)-3\,u\,v\,s^2}{6\,(2\,i\,\alpha\,s+1)}\,\mu(I)\right)\\ &= (1-2\,i\,\lambda_1\,s)^{-1/2}\,\exp\left(\tfrac{4\,\lambda_3^2\,(\lambda_1^2\,s^4+2\,i\,\lambda_1\,s^3)-3\,(\lambda_2^2+\lambda_3^2)\,s^2}{6\,(1-2\,i\,\lambda_1\,s)}\,\mu(I)\right).\end{aligned}$$

□

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