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RANDOM EQUATIONS AND APPLICATIONS TO GENERAL RANDOM FIXED POINT THEOREMS

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Abstract. In this paper, random operator equations are considered. Some general random fixed point theorems are obtained or extended. The concept of best random proximity points which is an extension of the notion of random fixed points is also proposed.

1. Introduction and Preliminaries

Random fixed point theory is a stochastic generalization of classical fixed point theory for deterministic mappings and has received much attention in recent years (see, e.g., [**3**], [**5**], [**9**], [**12**], [**15**] and references therein). Some authors (see, e.g., [**5**], [**12**], [**15**]) have shown that under some assumptions the existence of a deterministic fixed point is equivalent to the existence of a random fixed point. In this case, every deterministic fixed point theorem produces a random fixed point theorem.

In this paper, by considering random operator equations we gain some general random fixed point theorems as particular cases. Some results on random fixed points and random coincidence points in the literature (e.g., [**1**], [**5**], [**11**], [**12**], [**13**] and [**14**]) are obtained or extended. We also propose the concept of best random proximity point which is a randomization of the concept of proximity point in deterministic analysis and is an extension of the notion of random fixed point.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and X, Y metric spaces. We denote by $\mathcal{B}(X)$ the Borel σ-algebra of X, by 2^X the family of all nonempty subsets of X, by $C(X)$ the family of all nonempty closed subsets of X. The σ-algebra on $\Omega \times X$ is denoted by $\mathcal{F} \otimes \mathcal{B}(X)$. Hausdorff metric induced by d on $C(X)$ is given by

$$H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

for $A, B \in C(X)$, where $d(a, B) = \inf_{b\in B} d(a, b)$ is the distance from a point $a \in X$ to a subset $B \subset X$. We use $d(A, B)$ to denote the usual distance between two sets A and B

$$d(A, B) = \inf\{d(a, b)|a \in A, b \in B\}.$$

A mapping $\xi : \Omega \to X$ is said to be measurable (or X-valued random variable) if

$$\xi^{-1}(B) = \{\omega \in \Omega|\xi(\omega) \in B\} \in \mathcal{F}$$

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for any $B \in \mathcal{B}(X)$. A set-valued mapping $F : \Omega \to 2^X$ is said to be measurable if

$$F^{-1}(B) = \{\omega \in \Omega | F(\omega) \cap B \neq \emptyset\} \in \mathcal{F}\}$$

for each open subset B of X (Note that in Himmelberg [**7**] this is called weakly measurable). The graph of F is defined by

$$Gr(F) = \{(\omega, x) | \omega \in \Omega, x \in F(\omega).$$

We recall the concept of random operators.

Definition 1.1. (1) *A mapping $f : \Omega \times X \to Y$ is said to be a random operator if for each $x \in X$, the mapping $f(., x)$ is measurable, where $f(., x)$ denotes the mapping $\omega \mapsto f(\omega, x)$.*

(2) *A mapping $T : \Omega \times X \to 2^Y$ is said to be a (multivalued) random operator if for each $x \in X$, the mapping $T(., x)$ is measurable, where $T(., x)$ denotes the mapping $\omega \mapsto T(\omega, x)$.*

(3) *The random operator $f : \Omega \times X \to Y$ is said to be continuous if for each ω the mapping $f(\omega, .)$ is continuous.*

(4) *The random operator $T : \Omega \times X \to C(Y)$ is said to be continuous if for each ω the mapping $T(\omega, .)$ is continuous with Hausdorff distance on $C(Y)$.*

(5) *The random operator $f : \Omega \times X \to Y$ is said to be measurable if the mapping $(\omega, x) \mapsto f(\omega, x)$ is $(\mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(Y))$-measurable.*

(6) *The random operator $T : \Omega \times X \to 2^Y$ is said to be measurable if the mapping $(\omega, x) \mapsto T(\omega, x)$ is $(\mathcal{F} \otimes \mathcal{B}(X), \mathcal{B}(Y))$-measurable.*

Let X be a metric space. We will call X: a Polish space if X is separable and complete; a Suslin space if X is a continuous image of a Polish space.

For later convenience, we list the following four results.

Theorem 1.2 ([**6**, Theorem III.22])**.** *Let X be a Suslin space and $F : \Omega \to 2^X$ a multivalued mapping which has measurable graph. Then there exists a sequence of measurable selections (ξ_n) of F such that $(\xi_n(\omega))$ is dense in $F(\omega)$ for every $\omega \in \Omega$.*

Theorem 1.3 ([**7**, Theorem 6.1])**.** *Let X be a separable metric space, Y a metric space and $f : \Omega \times X \to Y$ a continuous random operator. Then f is a measurable random operator.*

Theorem 1.4 ([**7**, Theorem 3.5])**.** *Let X be a Suslin space and $F : \Omega \to C(X)$ a multivalued mapping. Then four following statements are equivalent*

a) *F is measurable;*
b) *F is $\mathcal{B}$-measurable, i.e. $F^{-1}(B) \in \mathcal{F}$ for each $B \in \mathcal{B}(X)$;*
c) *For each $x \in X$, the function $\omega \mapsto d(x, F(\omega))$ is measurable;*
d) *$Gr(F)$ is measurable;*

Lemma 1.5 ([**8**, Lemma 2.4])**.** *Let X be a separable metric space and $\xi : \Omega \to X$, $F : \Omega \to C(X)$ two measurable mappings. Then the mapping $\omega \mapsto d(\xi(\omega), F(\omega))$ is measurable.*

In the rest of this paper, we assume that X, Y are Polish spaces and $(\Omega, \mathcal{F}, P)$ is a complete probability space.

2. Random Equations and Random Fixed Points

Definition 2.1. *Let $f, g : \Omega \times X \to Y$ be random operators. Consider the random equation of the form*

$$f(\omega, x) = g(\omega, x). \tag{1}$$

We say that the equation (1) has a random solution if there exists an X-valued random variable $\xi : \Omega \to X$ such that, for every ω,

$$f(\omega, \xi(\omega)) = g(\omega, \xi(\omega)).$$

We call ξ a random solution of the equation (1).

Clearly, if the equation (1) has a random solution then it has a deterministic solution for each ω. However, the following simple example shows that the converse is not true.

Example 2. *Let $\Omega = [0; 1]$ and $\mathcal{F}$ be the family of subsets $A \subset \Omega$ with the property that either A is countable or the complement A^c is countable. Define a probability measure P on $\mathcal{F}$ by*

$$P(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{otherwise.} \end{cases}$$

Let $X = [0; 1]$. Define two mappings $f, g : \Omega \times X \to X$ by

$$f(\omega, x) = \begin{cases} x & \text{if } \omega = x \\ 1 & \text{otherwise} \end{cases}$$

$$g(\omega, x) = \begin{cases} x & \text{if } \omega = x \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that $(\Omega, \mathcal{F}, P)$ forms a complete probability space and f, g are random operators. For each $\omega \in \Omega$, $u(\omega) = \omega$ is a unique solution of the equation (1). Suppose that ξ is a random solution of the equation (1). Then $\xi(\omega) = \omega$. Hence, the mapping $u : \Omega \to X$ defined by $u(\omega) = \omega$ must be measurable. For $B = [0; 1/2) \in \mathcal{B}(X)$ we have

$$u^{-1}(B) = B = [0; 1/2) \notin \mathcal{F}$$

showing that u is not measurable and we get a contradiction.

The following theorem gives a sufficient condition ensuring that the existence of a deterministic solution for each ω implies the existence of a random solution for a general random equation.

Theorem 2.3. *Let $f, g : \Omega \times X \to Y$ be measurable random operators and $F : \Omega \to C(X)$ a measurable mapping. If for each ω, the random equation $f(\omega, x) = g(\omega, x)$ has a deterministic solution in $F(\omega)$ then it has a random solution in $F(\omega)$.*

Proof. Define a multivalued mapping $D : \Omega \to 2^X$ by

$$D(\omega) = \{x \in F(\omega) | f(\omega, x) = g(\omega, x)\}.$$

We will point out that D has measurable graph. Indeed,

$$\begin{aligned}Gr(D) &= \{(\omega,x)|\omega\in\Omega, x\in F(\omega), f(\omega,x)=g(\omega,x)\}\\ &= \{(\omega,x)|\omega\in\Omega, x\in F(\omega)\}\cap\{(\omega,x)|\omega\in\Omega, x\in X, f(\omega,x)=g(\omega,x)\}\\ &= \{(\omega,x)|\omega\in\Omega, x\in F(\omega)\}\cap\{(\omega,x)|\omega\in\Omega, x\in X, d(f(\omega,x),g(\omega,x))=0\}.\end{aligned}$$

By Lemma 1.5 and the measurability of f and g, it follows that $\varphi:\Omega\times X\to\mathbb{R}$ defined by $\varphi(\omega,x)=d(f(\omega,x),g(\omega,x))$ is measurable. Thus,

$$\{(\omega,x)|\omega\in\Omega, x\in X, d(f(\omega,x),g(\omega,x))=0\}=\varphi^{-1}(\{0\})$$

is a measurable set. By Theorem 1.4, $Gr(F)=\{(\omega,x)|\omega\in\Omega, x\in F(\omega)\}$ is measurable. Hence

$$Gr(D)=Gr(F)\cap\varphi^{-1}(\{0\})$$

is measurable.
By Theorem 1.2, D has a measurable selection denoted by ξ. Hence, we have $f(\omega,\xi(\omega))=g(\omega,\xi(\omega))$ and $\xi(\omega)\in F(\omega)$ for every ω, i.e. the random equation $f(\omega,x)=g(\omega,x)$ has a random solution in $F(\omega)$. □

Corollary 2.4. *Let $f,g:\Omega\times X\to Y$ be continuous random operators and $F:\Omega\to C(X)$ a measurable mapping. If for each ω the random equation $f(\omega,x)=g(\omega,x)$ has a deterministic solution in $F(\omega)$ then it has a random solution in $F(\omega)$.*

Proof. By Theorem 1.3, f and g are measurable random operators. Thus the conclusion follows from Theorem 2.3. □

Corollary 2.5. *Let $g:\Omega\times X\to Y$ be a measurable random operator and $F:\Omega\to C(X)$ a measurable mapping. Then for every measurable mapping $h:\Omega\to Y$ satisfing, for every ω,*

$$h(\omega)\in g(\omega,F(\omega))$$

there exists a measurable selection $\xi(\omega)\in F(\omega)$ such that $h(\omega)=g(\omega,\xi(\omega))$ for every ω.

Proof. Define $f:\Omega\times X\to Y$ by $f(\omega,x)=h(\omega)$ for any $x\in X,\omega\in\Omega$. Then f is a measurable random operator and the random equation $f(\omega,x)=g(\omega,x)$ has a deterministic solution in $F(\omega)$ for each $\omega\in\Omega$. By Theorem 2.3, the random equation $f(\omega,x)=g(\omega,x)$ has a random solution in $F(\omega)$, i.e. there exists a measurable selection $\xi(\omega)\in F(\omega)$ such that $h(\omega)=g(\omega,\xi(\omega))$ for every ω. □

Remark 6. (1) *Theorem 2.3 extends* [**14**, Lemma 3.1]*, which plays a crucial role in the proof of its main results, where random equation is of the form $f(\omega,x)=0$ and it is assumed that f is a continuous random operator defined in a weakly compact subset of a separable Banach space and $f(\omega,.)$ is demiclosed at zero for each ω.*
(2) *Corollary 2.5 is an extension of Filippov's theorem* [**1**, Theorem 8.2.10]*, where it is assumed that g is a continuous random operator.*

Let X be a separable metric space, S a nonempty complete subset of X, $f:\Omega\times S\to X$ a random operator and $T:\Omega\times S\to 2^X$ a multivalued random operator. Recall that

(1) An X-valued random variable ξ is said to be a random fixed point of f if $f(\omega,\xi(\omega))=\xi(\omega)$ for every ω.

(2) An X-valued random variable ξ is said to be a random fixed point of T if $\xi(\omega) \in T(\omega, \xi(\omega))$ for every ω.
(3) An X-valued random variable ξ is called a random coincidence point of f and T if $f(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$ for every ω.

Theorem 2.7. *Let X be a Polish space, $f : \Omega \times S \to X$ a measurable random operator, $T : \Omega \times S \to C(X)$ a measurable multivalued random operator and $F : \Omega \to C(S)$ a measurable mapping.*
(1) *If $f(\omega, .)$ has a deterministic fixed point in $F(\omega)$ for each ω then f has a random fixed point in $F(\omega)$.*
(2) *If $T(\omega, .)$ has a deterministic fixed point in $F(\omega)$ for each ω then T has a random fixed point in $F(\omega)$.*
(3) *If $f(\omega, .)$ and $T(\omega, .)$ has a deterministic coincidence point in $F(\omega)$ for each ω then f and T has a random coincidence point in $F(\omega)$.*

Proof. (1) Using the Theorem 2.3 for the random equation $f(\omega, x) = g(\omega, x)$, where $g(\omega, x) = x$.
(2) Define $\varphi : \Omega \times S \to \mathbb{R}$ by $\varphi(\omega, x) = d(x, T(\omega, x))$. By Lemma 1.5 and the measurability of T, φ is a measurable random operator. Clearly, for each ω, $T(\omega, .)$ has a deterministic fixed point in $F(\omega)$ if and only if the random equation $\varphi(\omega, x) = 0$ has a deterministic solution in $F(\omega)$. By assumption and Theorem 2.3, there exists a random variable $\xi(\omega) \in F(\omega)$ such that $\varphi(\omega, \xi(\omega)) = 0$ for every ω. Thus $\xi(\omega) \in T(\omega, \xi(\omega))$ for every ω, i.e. ξ is a random fixed point of T.
(3) Define $\varphi : \Omega \times S \to \mathbb{R}$ by $\varphi(\omega, x) = d(f(\omega, x), T(\omega, x))$. By Lemma 1.5 and the measurability of f and T, φ is a measurable random operator. Clearly, for each ω, $f(\omega, .)$ and $T(\omega, .)$ have a deterministic coincidence point in $F(\omega)$ if and only if the random equation $\varphi(\omega, x) = 0$ has a deterministic solution in $F(\omega)$. By assumption and Theorem 2.3, there exists a random variable $\xi(\omega) \in F(\omega)$ such that $\varphi(\omega, \xi(\omega)) = 0$ for every ω. Thus $f(\omega, \xi(\omega)) \in T(\omega, \xi(\omega))$ for every ω, i.e. ξ is a random coincidence point of f and T.

□

Remark 8. *From Theorem 2.7, we obtain or extend some random fixed point theorems.*
(1) *The claim 1 removes some assumptions on f in* [**12**, Theorem 3.2] *and extends* [**11**, Lemma 3.1]*, which plays a crucial role in the proof of its main results, where it is assumed that f is a continuous random operator satisfying the so-called condition (A).*
(2) *The claim 2 removes some assumptions on T in* [**5**, Theorem 3.1, Theorem 3.2, Theorem 3.3] *and* [**12**, Theorem 3.1].
(3) *The claim 3 extends and improves* [**13**, Theorem 3.1, Theorem 3.3, Theorem 3.12] *which contain most of the known random fixed point theorems as special cases (see,* [**13**, Remark 3.16]*).*

3. Best Random Proximity Points

Let $f : A \to B$ where A and B are two closed subsets of a Polish space X. In general, we have $\inf_{x \in A} d(x, f(x)) \geq d(A, B)$. If there is an element $x_0 \in A$ such that $d(x_0, f(x_0)) = d(A, B)$ then x_0 is called a best proximity point of the mapping

f (see, [**2**]). Particularly, if $A \cap B \neq \emptyset$ then best proximity point x_0 becomes a fixed point of f. Thus the notion of best proximity point is an extension of the notion of fixed point.

We now propose the concept of best random proximity point of a random operator.

Definition 3.1. *Let A, B be two closed subsets of a Polish space X and $f : \Omega \times A \to B$ a random operator. A measurable mapping $\xi : \Omega \to A$ is called a best random proximity point of f if*

$$d(\xi(\omega), f(\omega, \xi(\omega))) = d(A, B)$$

for any $\omega \in \Omega$.

Similarly to the deterministic case, a best random proximity point of a random operator f becomes a random fixed point of f if $A \cap B \neq \emptyset$. Hence the concept of best random proximity point is an extension of the concept of random fixed point.

In general, if f has a best random proximity point then for each ω the mapping $f(\omega, .)$ has a best proximity point. However, the following example shows that the converse is not true.

Example 2. *Let $\Omega = [0; 1]$ and $\mathcal{F}$ be the family of subsets $A \subset \Omega$ with the property that either A is countable or the complement A^c is countable. Define a probability measure P on $\mathcal{F}$ by*

$$P(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{otherwise.} \end{cases}$$

Let $A = B = [0; 1]$. Define a mapping $f : \Omega \times A \to B$ by

$$f(\omega, x) = \begin{cases} x & \text{if } \omega = x \\ 0 & \text{if } \omega \neq x \end{cases}$$

It is easy to check that $(\Omega, \mathcal{F}, P)$ forms a complete probability space and f is a random operator. We have

$$d(x, f(\omega, x)) = \begin{cases} 0 & \text{if } \omega = x \\ x & \text{if } \omega \neq x. \end{cases}$$

Thus, for each fixed ω, $d(x, f(\omega, x)) = d(A, B) = 0$ if and only if $x = \omega$. Thus $f(\omega, .)$ has a unique best proximity point $x = \omega$. However, the mapping $\xi : \Omega \to X$ defined by $\xi(\omega) = \omega$ is not measurable. Because, for $B = [0; 1/2) \in \mathcal{B}(X)$ we have $\xi^{-1}(B) = B = [0; 1/2) \notin \mathcal{F}$. Hence the random operator f doesn't have a best random proximity point.

The following theorem gives a sufficient condition on f ensuring that the existence of a best proximity point of $f(\omega, .)$ for each ω implies the existence of a best random proximity point of f.

Theorem 3.3. *Let A and B be two closed subsets of a Polish space X, $f : \Omega \times A \to B$ a measurable random operator. If $f(\omega, .)$ has a best proximity point for each $\omega \in \Omega$ then f has a best random proximity point.*

Proof. Define $\varphi : \Omega \times A \to \mathbb{R}$ by $\varphi(\omega, x) = d(x, f(\omega, x))$. Then φ is a measurable random operator. Clearly, if $f(\omega, .)$ has a best proximity point then the random equation $\varphi(\omega, x) = d(A, B)$ has a deterministic solution in $F(\omega) = A$ for each ω. By Theorem 2.3, the random equation $\varphi(\omega, x) = d(A, B)$ has a random solution ξ. Thus $d(\xi(\omega), f(\omega, \xi(\omega))) = d(A, B)$ for every ω, i.e. ξ is a best random proximity point of f. $\square$

Corollary 3.4. *Let A and B be two closed subsets of a Polish space X, $f : \Omega \times A \to B$ a continuous random operator. If for each $\omega \in \Omega$ the deterministic operator $f(\omega, .)$ has a best proximity point then f has a best random proximity point.*

Proof. By Theorem 1.3, f is a measurable random operator. Thus the conclusion follows from Theorem 3.3. $\square$

As an illustration for Theorem 3.3, we give a random version of [**2**, Theorem 2.1].

Theorem 3.5. *Let A and B be nonempty compact subsets of a Polish space X. Suppose that the random operators $f : \Omega \times A \to B$ and $g : \Omega \times B \to A$ satisfy the following conditions.*

a) *f and g are contractive, i.e. $f(\omega, .)$ and $g(\omega, .)$ are contractive for each $\omega \in \Omega$.*
b) *$d(f(\omega, x), g(\omega, y)) < d(x, y)$ whenever $d(x, y) > d(A, B)$ for $x \in A$, $y \in B$ and $\omega \in \Omega$.*

Then f and g have best random proximity points. Moreover, for a fixed element $x_0 \in A$, let $x_{2n+1} = f(\omega, x_{2n})$ and $x_{2n} = g(\omega, x_{2n-1})$. Then the sequence (x_{2n}) converges to a best random proximity point of f and the sequence (x_{2n+1}) converges to a best random proximity point of g for each ω.

Proof. For each ω, by [**2**, Theorem 2.1], $f(\omega, .)$ and $g(\omega, .)$ have best proximity points and the sequence (x_{2n}) converges to a best proximity point of $f(\omega, .)$, the sequence (x_{2n+1}) converges to a best proximity point of $g(\omega, .)$. The continuity of f and g follows from assumption a). By Theorem 3.3, f and g have best random proximity points. $\square$

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