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RATIONAL HOMOTOPY STABILITY FOR THE SPACES OF ALGEBRAIC MAPS

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Abstract. Let X be a path connected nilpotent (e.g. simply connected) complex algebraic variety with $\pi_2(X)$ a free abelian group of rank r. For a based algebraic map $f : (\mathbb{CP}^1, \infty) \to (X, x_0)$, we can assign it a multiple degree $\mathbf{n} = f_*(1)$ under the induced homomorphism $f_* : \pi_2(\mathbb{CP}^1) \to \pi_2(X)$. Let $\mathrm{Alg}^{\mathbf{n}}_{x_0}(\mathbb{CP}^1, X)$ be the space of based algebraic maps of degree $\mathbf{n}$ from $\mathbb{CP}^1$ into X. Under some assumption we prove that the map $\mathrm{Alg}^{\mathbf{n}}_{x_0}(\mathbb{CP}^1, X) \longrightarrow \mathrm{Alg}^{d\mathbf{n}}_{x_0}(\mathbb{CP}^1, X)$ obtained by compositing $f \in \mathrm{Alg}^{\mathbf{n}}_{x_0}(\mathbb{CP}^1, X)$ with $g(z) = z^d, z \in \mathbb{CP}^1$ induces rational homotopy equivalence up to some dimension, which tends to infinity as the degree $\mathbf{n}$ grows.

1. Introduction

Rational curves play an essential role in quantum cohomology (e.g. Manin [15]), higher dimensional algebraic geometry (e.g. Kollár [11]) and other related areas. Every rational curve on a complex algebraic variety X can be viewed as the image of an algebraic map from $\mathbb{CP}^1$ to X. To investigate rational curves on X, people often consider the space of algebraic maps from $\mathbb{CP}^1$ to X. A lot of work has been done on the space of algebraic maps in several branches of mathematics.

Let $\mathrm{Alg}^n_{x_0}(\mathbb{CP}^1, X)$ and $F^n_{x_0}(\mathbb{CP}^1, X)$ be the space of based algebraic maps and the space of based continuous maps of degree n from $(\mathbb{CP}^1, \infty)$ into a complex algebraic variety (X, x_0), respectively. In 1979, Segal [16] proved that the inclusion map $\mathrm{Alg}^n_{x_0}(\mathbb{CP}^1, \mathbb{CP}^m) \subset F^n_{x_0}(\mathbb{CP}^1, \mathbb{CP}^m)$ is a homotopy equivalence up to dimension $n(2m-1)$. His work was motivated by the observation of Atiyah that the only critical points of an "energy" functional on the space of continuous maps is the space of algebraic maps (where the functional achieves an absolute minimum) and by the extrapolation of finite dimensional Morse theory to the infinite dimensional case. Segal's theorem was then generalized to the case X a complex Grassmannian by Kirwan [10], and certain $SL(n, \mathbb{C})$ flags by Guest [5]. Later Mann and Milgram [13] increased the range of the isomorphisms obtained by Kirwan for Grassmannian and treated all $SL(n, \mathbb{C})$ flag manifolds in [14]. Similar stability theorems were proved for any generalized flag manifold G/P by Boyer, Mann, Hurtubise and Milgram [2] and Hurtubise [8], following on a stable result of Gravesen [4] and for toric varieties by Guest [6]. We refer the interested reader to [9] for a survey.

More recently, Boyer, Hurtubise and Milgram [1] questioned what is the most general complex target space X which admits stability theorem, which, loosely

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speaking, says that the homotopy (homology) groups of the space of algebraic maps is isomorphic to the homotopy (homology) groups of the entire continuous map space through a range that grows with degree, as degree moves to infinity. They noticed that there is sort of link between the stability and some rationality property of the manifold. The stability theorem should hold for certain subclass of rationally connected varieties (a variety is called rationally connected if any pairs of its points can be connected by a rational curve, see Kollár [12] for an elementary introduction to rationally connected varieties). Boyer, Hurtubise and Milgram [1] proved that the stability theorems hold for principal almost solv-varieties, a subclass of rationally connected varieties on which a complex solvable linear group acts with a free dense open orbit.

In Symplectic setting, Cohen, Jones and Segal [3] began to investigate what condition on a closed, connected, integral symplectic manifold (X, ω) ensures that the stability theorems hold. They consider the limit $\varinjlim \mathrm{Alg}_{x_0}(\mathbb{CP}^1, X)$, which is viewed as a kind of stabilization of $\mathrm{Alg}_{x_0}(\mathbb{CP}^1, X)$ under certain gluing operations. They conjecture that the stability theorem hold if and only if the evaluation map $E : \varinjlim \mathrm{Alg}_{x_0}(\mathbb{CP}^1, X) \longrightarrow X$ is a quasifibration.

All the above stability theorems have considered pairs of spaces at the same degree. A natural question is to ask that what is the connection among the spaces with varying degrees. The motivation to ask such a question is that we attempt to define certain stability property on algebraic varieties over algebraic closed field k other than $\mathbb{C}$. The space $\mathrm{Alg}^n_{x_0}(\mathbb{P}^1, X)$ is $\mathrm{Hom}^n_{x_0}(\mathbb{P}^1, X)$ in algebraic geometry. Because Zariski topology on an algebraic variety is not Huasdorff, in general the space $F^n_{x_0}(\mathbb{P}^1, X)$ is not very useful. To define stability properties on an algebraic variety over an arbitrary algebraic closed field k, we had better only use the spaces $\mathrm{Alg}^*_{x_0}(\mathbb{P}^1, X)$. However, we have not figured out a way to define stability properties without using homotopy (homology) yet. In this paper we still work over the complex number field $\mathbb{C}$.

Segal [16] constructed a gluing map between $\mathrm{Alg}^n_{x_0}(\mathbb{CP}^1, \mathbb{CP}^1)$ and $\mathrm{Alg}^{n+1}_{x_0}(\mathbb{CP}^1, \mathbb{CP}^1)$ and showed that this gluing map induced isomorphisms among the lower homology groups, i.e. he showed that $H_*(\mathrm{Alg}^n_{x_0}(\mathbb{CP}^1, \mathbb{CP}^1))$ is independent of n when n is large.

In general it is not easy to define gluing maps between $\mathrm{Alg}^*_{x_0}(\mathbb{CP}^1, X)$. However we find that the finite covering maps from $\mathbb{CP}^1$ to itself provide a natural connection among $\mathrm{Alg}^*_{x_0}(\mathbb{CP}^1, X)$, specifically we can define an injective map $\Phi_{n,dn} : \mathrm{Alg}^n_{x_0}(\mathbb{CP}^1, X) \longrightarrow \mathrm{Alg}^{dn}_{x_0}(\mathbb{CP}^1, X)$ which sends any $f \in \mathrm{Alg}^n_{x_0}(\mathbb{CP}^1, X)$ to $\Phi_{n,dn}(f)(z) = f(z^d), z \in \mathbb{CP}^1$.

Now we can define stability property for a variety.

Definition 1.1. We say that X is a variety with stability property if and only if the injective map

$$\Phi_{n,dn} : \mathrm{Alg}^n_{x_0}(\mathbb{P}^1, X) \longrightarrow \mathrm{Alg}^{dn}_{x_0}(\mathbb{P}^1, X)$$

is a rational homotopy equivalence up to dimension k_n for any $x_0 \in X$, where k_n is a natural number dependent on n, x_0 and X, and $\lim_{n \longrightarrow \infty} k_n = \infty$.

Remark 1.2. Unfortunately the above injection does not induce homotopy equivalence among $\mathrm{Alg}^*_{x_0}(\mathbb{P}^1, X)$ even in the case $X = \mathbb{CP}^1$, so rational homotopy enters into the picture.

We wonder whether the following is true.

(1) Is the stability property birationally invariant? That is if X is birational to Y and X is a variety with stability property, so is Y?
(2) Is it true that all rationally connected varieties have the stability property defined in Definition 1.1?

It is well known that any rationally connected variety is simply connected. In this paper, we will investigate stability property on slightly general spaces, the so called nilpotent spaces. Recall that a spaces X is nilpotent if its fundamental group $\pi_1(X)$ is a nilpotent group and the action of $\pi_1(X)$ on the higher homotopy groups of X is a nilpotent action.

Let X be a path connected nilpotent complex algebraic variety with $\pi_2(X)$ a free abelian group of rank r. For a based algebraic map $f : (\mathbb{CP}^1, \infty) \to (X, x_0)$, we can assign it a multiple degree $\mathbf{n} = f_*(1)$ under the induced homomorphism $f_* : \pi_2(\mathbb{CP}^1) \to \pi_2(X)$. We can define $\Phi_{\mathbf{n},d\mathbf{n}} : \mathrm{Alg}^{\mathbf{n}}_{x_0}(\mathbb{CP}^1, X) \longrightarrow \mathrm{Alg}^{d\mathbf{n}}_{x_0}(\mathbb{CP}^1, X)$ in the same way as we did for $\Phi_{n,dn}$. We conjecture the following

Conjecture 1.3. *For any path connected nilpotent complex algebraic variety X with $\pi_2(X)$ a free abelian group of rank r, the stability property defined in Definition* 1.1 *holds true, that is, the injective map $\Phi_{\mathbf{n},d\mathbf{n}}$ induces rational homotopy equivalence up to some dimension which grows with $\mathbf{n}$, as $\mathbf{n}$ moves to infinity in a suitable positive cones.*

We do not know how to prove Conjecture 1.3 directly by only using tools in algebraic geometry. In this paper we transfer the problem on the space of algebraic maps to that on the space of continuous maps and show the following:

Theorem 1.4. *Let X be a path connected nilpotent complex algebraic variety with $\pi_2(X)$ a free abelian group of rank r. Assume that the inclusion map $\mathrm{Alg}^{\mathbf{n}}_{x_0}(\mathbb{CP}^1, X) \subset F^{\mathbf{n}}_{x_0}(\mathbb{CP}^1, X)$ induces rational homotopy equivalence up to dimension $k_{\mathbf{n}}$, where $k_{\mathbf{n}}$ grows with $\mathbf{n}$, as $\mathbf{n}$ moves to infinity in a suitable positive cones. Then $g_{\mathbf{n},d\mathbf{n}}$ induces rational homotopy equivalence up to dimension $\min\{k_{\mathbf{n}}, k_{d\mathbf{n}}\}$.*

The assumption in Theorem 1.4 is true for Grassmannians, flag manifolds, toric varieties, hence stability property holds true on these varieties.

Observe that Cohen, Jones and Segal [3] defined some kind of stabilization of $\mathrm{Alg}_{x_0}(\mathbb{CP}^1, X)$ under certain gluing operations. We can also define a stabilization of $\mathrm{Alg}_{x_0}(\mathbb{CP}^1, X)$ by using the maps $\Phi_{\mathbf{n},d\mathbf{n}}$. The set $\{\mathbf{n}\}$ forms a partial order set. There is a morphism from $\mathbf{n}$ to $\mathbf{m}$ if and only if $\mathbf{m} = d\mathbf{n}$. We can define the colimit $\varinjlim \mathrm{Alg}_{x_0}(\mathbb{P}^1, X)$ over the directed system $\{\mathbf{n}\}$. It is worthwhile to investigate the relations between these two stabilizations of $\mathrm{Alg}_{x_0}(\mathbb{CP}^1, X)$.

The idea of the proof of Theorem 1.4 is very simple. Look at the following commutative diagram,

$$\begin{array}{ccc} \mathrm{Alg}^{\mathbf{n}}_{x_0}(\mathbb{CP}^1, X) & \xrightarrow{\quad i \quad} & F^{\mathbf{n}}_{x_0}(\mathbb{CP}^1, X) \\ \Big\downarrow{\scriptstyle \Phi_{\mathbf{n},d\mathbf{n}}} & & \Big\downarrow{\scriptstyle \Phi_{\mathbf{n},d\mathbf{n}}} \\ \mathrm{Alg}^{d\mathbf{n}}_{x_0}(\mathbb{CP}^1, X) & \xrightarrow{\quad i \quad} & F^{d\mathbf{n}}_{x_0}(\mathbb{CP}^1, X) \end{array}$$

The horizontal arrows induce rational homotopy equivalence up to dimension $k_{\mathbf{n}}$ and $k_{d\mathbf{n}}$ respectively. If we can prove that the right arrow induces rational homotopy equivalence, then the left arrow induces rational homotopy equivalence up to dimension $\min\{k_{\mathbf{n}}, k_{d\mathbf{n}}\}$.

We first show that the map $\Phi_{\mathbf{n},d\mathbf{n}} : F^{\mathbf{n}}(\mathbb{CP}^1, X) \longrightarrow F^{d\mathbf{n}}(\mathbb{CP}^1, X)$ induces rational homotopy equivalence for free maps by using the Sullivan-Haefliger model in rational homotopy setting. The base map space $F^*_{x_0}(\mathbb{CP}^1, X)$ is exactly the fiber of the evaluation map $E : F^*(\mathbb{CP}^1, X) \longrightarrow X$ which sends f to $f(\infty) = x_0 \in X$ so that we can use the long exact sequence of the fibration to pass the rational homotopy equivalence from the space of free maps to the space of based maps.

2. Sullivan-Haefliger model and Proof of Theorem 1.4

Naturally we identify $F(\mathbb{CP}^1, X, f)$ with $\Gamma_{f'}$, the space of sections of the trivial fibration $p_1 : \mathbb{CP}^1 \times X \longrightarrow \mathbb{CP}^1$ homotopic to the section $f' = (id, f) : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \times X$. To prove that the injection $F(\mathbb{CP}^1, X, f) \longrightarrow F(\mathbb{CP}^1, X, f \circ g)$ induces rational homotopy equivalence is equivalent to show that so does for $\Gamma_{f'} \longrightarrow \Gamma_{(f\circ g)'}$, where $(f \circ g)' = (id, f \circ g)$ and $g(z) = z^d, z \in \mathbb{CP}^1 = S^2, d \in \mathbb{N}$.

Denote (A, d_A) a minimal model for $\mathbb{CP}^1$ and $(\wedge V, d)$ a minimal model for X. Let f' be the section $f' = (id, f) : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \times X$. It gives a morphism $\sigma_{f'} : A \otimes \wedge V \longrightarrow A$ which is the identity on A. Let ψ be the A-algebra automorphism of $A \otimes \wedge V$ mapping $1 \otimes v$ to $1 \otimes v - \sigma_{f'}(v)$; define on $A \otimes \wedge V$ a new differential d' such that $(d_A \otimes d) \circ \psi = \psi \circ d'$. Then ψ is a DG-automorphism and $\sigma_{f'} \circ \psi$ maps $\wedge V$ to zero. Denote the graded dual vector space of A by $A^\vee$:

$$(A^\vee)_n = \mathrm{Hom}(A^n, \mathbb{Q}).$$

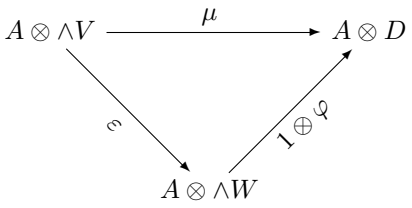
Let $(a_i)_{i\in I}$ be a basis of A. Then $A^\vee$ is naturally equipped with the dual basis a_i^* such that $< a_i^*; a_j >= \delta_{ij}$.

We now look at the map of algebras defined by:

$$\varepsilon : A \otimes \wedge V \longrightarrow A \otimes (A^\vee \otimes \wedge V) : \varepsilon(v) = \sum_{i\in I} a_i \otimes (a_i^* \otimes v); \varepsilon(a) = a, a \in A$$

In [7], Haefliger shows how to put a uniquely defined differential $d_A \otimes \delta$ on $A \otimes (A^\vee \otimes \wedge V)$ in such a way that ε becomes a morphism of commutative differential graded algebras. Let W be the quotient of $A^\vee \otimes \wedge V$ by the subspace of elements of degree < 0, and by the subspace formed by the δ-cocycles in degree 0. Haefliger showed that $(\wedge W, \delta)$ is a model (the so-called Sullivan-Haefliger model) of the space $\Gamma_{f'}$. It has the following universal property. Let D be a DG-algebra such that $D^0 = \mathbb{Q}$, and let $\mu : A \otimes \wedge V \longrightarrow A \otimes D$ be a morphism of augmented DG-algebras

over A. Then there is a unique $\varphi : \wedge W \longrightarrow D$ such that the diagram



commutes.

Let $(A \otimes \wedge V, d_i), i = 1, 2$ be two differential graded commutative algebras. Follow the construction in Haefliger [7], there are $(\wedge W_i, \delta_i), i = 1, 2$ such that $\varepsilon_i, i = 1, 2$ become morphisms of commutative differential graded algebras.

We have the following theorem:

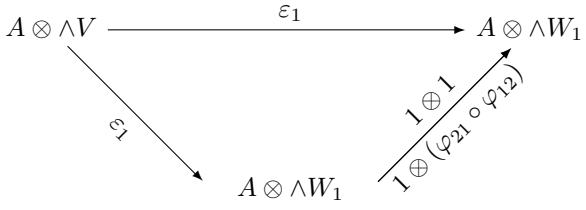
Theorem 2.1. *Notations as above. If $h : (A \otimes \wedge V, d_1) \longrightarrow (A \otimes \wedge V, d_2)$ is an isomorphism which restricts to an automorphism on A, then $(\wedge W_1, \delta_1)$ is isomorphic to $(\wedge W_2, \delta_2)$.*

Proof. We will apply the universal property of the Sullivan-Haefliger model to prove our theorem. Let us look at the following diagram

$$\begin{array}{ccccc} (A \otimes \wedge V, d_1) & \xrightarrow{\varepsilon_1} & (A \otimes \wedge W_1, d_A \otimes \delta_1) & \xrightarrow{h|_A \otimes 1} & (A \otimes \wedge W_1, d_A \otimes \delta_1) \\ \downarrow h & & & & \\ (A \otimes \wedge V, d_2) & \xrightarrow{\varepsilon_2} & (A \otimes \wedge W_2, d_A \otimes \delta_2) & \xrightarrow{{h|_A}^{-1} \otimes 1} & (A \otimes \wedge W_2, d_A \otimes \delta_2) \end{array}$$

Obviously the map $(h|_A \otimes 1) \circ \varepsilon_1 \circ (h^{-1}) : (A \otimes \wedge V, d_2) \longrightarrow (A_1 \otimes \wedge V, d_1) \longrightarrow (A \otimes \wedge W_1, d_A \otimes \delta_1) \longrightarrow (A \otimes \wedge W_1, d_A \otimes \delta_1)$ is a morphism of augmented DG-algebras over A and $\wedge W_1$ is a DG-algebra with $W_1^0 = \mathbb{Q}$. Apply the universal property, there is a unique morphism $\varphi_{21} : \wedge W_2 \longrightarrow \wedge W_1$ such that $(1 \otimes \varphi_{21}) \circ \varepsilon_2 = (h|_A \otimes 1) \circ \varepsilon_1 \circ h^{-1}$. Similarly we can prove that there exist a unique morphism $\varphi_{12} : \wedge W_1 \longrightarrow \wedge W_2$ such that $(1 \otimes \varphi_{12}) \circ \varepsilon_1 = ({h|_A}^{-1} \otimes 1) \circ \varepsilon_2 \circ h : (A \otimes \wedge V, d_1) \longrightarrow (A_1 \otimes \wedge V, d_2) \longrightarrow (A \otimes \wedge W_2, d_A \otimes \delta_2) \longrightarrow (A \otimes \wedge W_2, d_A \otimes \delta_2)$. Now we apply the same trick as above to show that $\varphi_{21} \circ \varphi_{12} = 1_{\wedge W_1}$ and $\varphi_{12} \circ \varphi_{21} = 1_{\wedge W_2}$.

Since $(1 \otimes \varphi_{12}) \circ \varepsilon_1 = ({h|_A}^{-1} \otimes 1) \circ \varepsilon_2 \circ h$, we have $(h|_A \otimes \varphi_{21} \circ \varphi_{12}) \circ \varepsilon_1 = (1 \otimes \varphi_{21}) \circ (h|_A \otimes 1) \circ (1 \otimes \varphi_{12}) \circ \varepsilon_1 = (1 \otimes \varphi_{21}) \circ (h|_A \otimes 1) \circ ({h|_A}^{-1} \otimes 1) \circ \varepsilon_2 \circ h = (1 \otimes \varphi_{21}) \circ \varepsilon_2 \circ h = (h|_A \otimes 1) \circ \varepsilon_1 \circ h^{-1} \circ h = (h|_A \otimes 1) \circ \varepsilon_1$. So $(1 \otimes \varphi_{21} \circ \varphi_{12}) \circ \varepsilon_1 = ({h|_A}^{-1} \otimes 1) \circ (h|_A \otimes \varphi_{21} \circ \varphi_{12}) \circ \varepsilon_1 = ({h|_A}^{-1} \otimes 1) \circ (h|_A \otimes 1) \circ \varepsilon_1 = \varepsilon_1$. Therefore there are two morphisms from $\wedge W_1$ to itself such that the following diagram commutes.



By the uniqueness of such morphisms, we have $\varphi_{21} \circ \varphi_{12} = 1_{\wedge W_1}$. Similarly we can prove $\varphi_{12} \circ \varphi_{21} = 1_{\wedge W_2}$. This completes the proof of Theorem 2.1.

□

Applying Theorem 2.1 we can prove that

Theorem 2.2. *The map* $\Phi_{\mathbf{n},d\mathbf{n}} : F^{\mathbf{n}}(\mathbb{CP}^1, X) \longrightarrow F^{d\mathbf{n}}(\mathbb{CP}^1, X)$ *is a rational homotopy equivalence.*

Proof. Let (A, d_A) be the minimal model for $\mathbb{CP}^1 = S^2$ and $(\wedge V, d)$ the minimal model for X. Let $g(z) = z^d, z \in \mathbb{CP}^1 = S^2, d \in \mathbb{N}$. It induces an automorphism of (A, d_A), denote as σ_g. For the section f' and $(f \circ g)'$, the induced morphisms on $(A \otimes \wedge V, d_A \otimes d)$ are $\sigma_{f'}$ and $\sigma_{(f\circ g)'}$, respectively. Just as in Haefliger [7] we have two differential graded algebra $(A \otimes \wedge V, d_i), i = 1, 2$, where $d_i, i = 1, 2$ are definded by $(d_A \otimes d) \circ \psi_i = \psi_i \circ d_i, i = 1, 2$, and $\psi_i, i = 1, 2$, are A-algebra automorphism of $A \otimes \wedge V$ mapping $a \in A$ to itself, and $\psi_1(1 \otimes v) = 1 \otimes v - \sigma_{f'}(v)$ and $\psi_2(1 \otimes v) = 1 \otimes v - \sigma_{(f\circ g)'}(v)$.

Look at the following diagram,

$$\begin{array}{ccc} (A \otimes \wedge V, d_A \otimes d) & \xrightarrow{\psi_1} & (A \otimes \wedge V, d_1) \\ \Big\downarrow{\scriptstyle \sigma_g \otimes 1} & & \Big\downarrow{\scriptstyle \sigma_g \otimes 1} \\ (A \otimes \wedge V, d_A \otimes d) & \xrightarrow{\psi_2} & (A \otimes \wedge V, d_2) \end{array} \qquad (*)$$

We claim that diagram $(*)$ is commutative.

To prove our claim, let $a \otimes v$ be any element in $A \otimes \wedge V$. Because $\psi_1(a \otimes v) = a \otimes v - a\sigma_{f'}(v)$, we have $(\sigma_g \otimes 1) \circ \psi_1(a \otimes v) = \sigma_g(a) \otimes v - \sigma_g(a)\sigma_g(\sigma_{f'}(v))$. And we also have $\psi_2 \circ (\sigma_g \otimes 1)(a \otimes v) = \psi_2(\sigma_g(a) \otimes v) = \sigma_g(a) \otimes v - \sigma_g(a)\sigma_{(f\circ g)'}(v)$. So if we can prove that $\sigma_g(\sigma_{f'}(v)) = \sigma_{(f\circ g)'}(v)$, our claim follows.

Let $p_2 : \mathbb{CP}^1 \times X \longrightarrow X$ be the second projection. It induce a map $\sigma_{p_2} : \wedge V \longrightarrow A \otimes \wedge V$ which maps v to $1 \otimes v$. Since $f = p_2 \circ f'$, so $\sigma_f = \sigma_{f'} \circ \sigma_{p_2} : \wedge V \longrightarrow A$, hence $\sigma_f(v) = \sigma_{f'}(v)$. Similarly we can prove that $\sigma_{f\circ g}(v) = \sigma_{(f\circ g)'}(v)$. But $\sigma_{f\circ g}(v) = \sigma_g(\sigma_f(v))$. Therefore $\sigma_g(\sigma_{f'}(v)) = \sigma_{(f\circ g)'}(v)$, which implies that the diagram $(*)$ is commutative. The left vertical arrow and the two horizontal arrows in diagram $(*)$ are isomorphisms of differential graded commutative algebras, so does the right vertical arrow. The map $(\sigma_g \otimes 1)|_A$ is an automorphism of A. So apply Theorem 2.1 we have that the Sullivan-Haefliger models for the space $\Gamma_{f'}$ and $\Gamma_{(f\circ g)'}$ are isomorphic. Therefore $\Gamma_{f'}$ and $\Gamma_{(f\circ g)'}$ have the same rational homotopy type. □

Proof of Theorem 1.4:

Theorem 1.4 follows easily from Theorem 2.2 by considering the long exact homotopy sequences between the fibrations $F^{\mathbf{n}}_{x_0}(\mathbb{CP}^1, X) \longrightarrow F^{\mathbf{n}}(\mathbb{CP}^1, X) \longrightarrow X$ and $F^{d\mathbf{n}}_{x_0}(\mathbb{CP}^1, X) \longrightarrow F^{d\mathbf{n}}(\mathbb{CP}^1, X) \longrightarrow X$ and then applying five-lemma.

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