

## LITTLE HANKEL OPERATORS BETWEEN BERGMAN SPACES OF THE RIGHT HALF PLANE

NAMITA DAS

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**Abstract.** In this paper we consider a class of weighted integral operators on  $L^2(0, \infty)$  and show that they are unitarily equivalent to little Hankel operators between weighted Bergman spaces of the right half plane. We use two parameters  $\alpha, \beta \in (-1, \infty)$  and involve two weights to define Bergman spaces of the domain and range of the little Hankel operators. We obtained conditions for the Hankel integral operator to be Hilbert-Schmidt, nuclear, finite rank and compact, expressed in terms of the kernel of the integral operator. For certain class of weights, these operators are shown to be unitarily equivalent to little Hankel operators between weighted Bergman spaces of the disk, and the symbol correspondence is given. In view of the strong link between Hankel operators and best approximation, some asymptotic results on the singular values of Hankel integral operators are also provided.

### 1. Introduction

Little Hankel operators between Bergman spaces of the disk have been considered by several authors (e.g. see [8], [13], [12], [15] and [3]). These operators on the Bergman space behave more like Hankel operators on the Hardy space. The main characteristic of these operators is that the operator only depends on the conjugate analytic part of the symbol. In this paper we shall show that there are many equivalent ways of regarding little Hankel operators between weighted Bergman spaces and shall concentrate particularly on their representation as integral operators. We shall discuss about the size estimates of these operators: boundedness, compactness and membership in the Schatten classes.

The organization of this paper is as follows. We begin with general weights on Bergman spaces and the associated Hankel integral operators on  $(0, \infty)$ . Then we specialize to a particular class of weights studied by Rochberg [13] where the operators can be related to little Hankel operators between weighted Bergman spaces of the disk in a natural manner. Finally, we discuss the asymptotic behavior of the singular values from a point of view similar to that given in [5] and [6] for classical Hankel operators. These relate to problems of approximation theory and systems theory.

Hankel and Toeplitz operators on the Bergman space of the right half plane and the unit disk may be defined in several equivalent ways and we begin by summarizing some of these. Let  $\mathbb{C}_+ = \{F \in \mathbb{C} : \operatorname{Re} F > \kappa\}$  be the right half plane: then  $L^2(\mathbb{C}_+, \tilde{\mathbb{A}})$  is the space of complex-valued, absolutely square-integrable, measurable functions on  $\mathbb{C}_+$  with respect to the area measure  $d\tilde{A} = dx dy$  and  $L_a^2(\mathbb{C}_+)$  is the closed subspace consisting of those functions in  $L^2(\mathbb{C}_+, \tilde{\mathbb{A}})$  that are analytic. Let  $P_+$  denote the orthogonal projection of  $L^2(\mathbb{C}_+, \tilde{\mathbb{A}})$  onto  $L_a^2(\mathbb{C}_+)$ . Let  $L^\infty(\mathbb{C}_+)$  be

the space of complex-valued, essentially bounded, Lebesgue measurable functions on  $\mathbb{C}_+$  and  $H^\infty(\mathbb{C}_+)$  be the subspace consisting of those functions that are analytic in  $L^\infty(\mathbb{C}_+)$ .

For  $\phi \in L^\infty(\mathbb{C}_+)$ , we define the Toeplitz operator  $\widetilde{T}_\phi$  from  $L_a^2(\mathbb{C}_+)$  into  $L_a^2(\mathbb{C}_+)$  by  $\widetilde{T}_\phi f = P_+(\phi f)$ . The little Hankel operator  $\widetilde{h}_\phi$  is a mapping from  $L_a^2(\mathbb{C}_+)$  into  $\overline{L_a^2(\mathbb{C}_+)}$  (the space consisting of conjugates of functions in  $L_a^2(\mathbb{C}_+)$ ) defined by  $\widetilde{h}_\phi f = \overline{P_+}(\phi f)$  where  $\overline{P_+}$  is the projection operator from  $L^2(\mathbb{C}_+, \tilde{\mathbb{A}})$  onto  $\overline{L_a^2(\mathbb{C}_+)}$ .

Let  $\widetilde{S}_\phi$  be the mapping from  $L_a^2(\mathbb{C}_+)$  into  $L_a^2(\mathbb{C}_+)$  defined by  $\widetilde{S}_\phi f = P_+(\tilde{J}(\phi f))$  where  $\tilde{J}$  is the mapping from  $L^2(\mathbb{C}_+, \tilde{\mathbb{A}})$  into  $L^2(\mathbb{C}_+, \tilde{\mathbb{A}})$  such that  $\tilde{J}f(s) = f(\bar{s})$ . Note that  $\tilde{J}$  is unitary and  $\tilde{J}S_\phi f = \tilde{J}(P_+(\tilde{J}(\phi f))) = \tilde{J}P_+(\tilde{J}(\phi f)) = \overline{P_+}(\phi f) = \widetilde{h}_\phi f$  for all  $f \in L_a^2(\mathbb{C}_+)$ .

Let  $\widetilde{\Gamma}_\phi$  be the mapping from  $L_a^2(\mathbb{C}_+)$  into  $L_a^2(\mathbb{C}_+)$  defined by  $\widetilde{\Gamma}_\phi f = P_+(\widetilde{M}_\phi \tilde{J}f)$  where  $\widetilde{M}_\phi$  is the mapping from  $L^2(\mathbb{C}_+, \tilde{\mathbb{A}})$  into  $L^2(\mathbb{C}_+, \tilde{\mathbb{A}})$  defined by  $\widetilde{M}_\phi f = \phi f$ . Thus  $(\widetilde{\Gamma}_\phi f)(z) = (P_+(\widetilde{M}_\phi \tilde{J}f))(z) = P_+(\phi(z)f(\bar{z})) = P_+(\tilde{J}(\phi(\bar{z})f(z))) = P_+(\tilde{J}(\tilde{J}\phi(z)f(z))) = P_+(\tilde{J}(\tilde{J}\phi)f(z)) = (\tilde{S}_{\tilde{J}\phi}f)(z)$  for all  $f \in L_a^2(\mathbb{C}_+)$ , hence  $\widetilde{\Gamma}_\phi = \widetilde{S}_{\tilde{J}\phi}$ .

Let  $\mathbb{D} = \{F \in \mathbb{C} : |F| < 1\}$  be the open unit disk in the complex plane  $\mathbb{C}$ . Let  $L^2(\mathbb{D}, \mathbb{A})$  be the space of complex-valued, absolutely square-integrable, measurable functions on  $\mathbb{D}$  with respect to the normalized area measure  $dA = \frac{1}{\pi} dx dy$ . Let  $L_a^2(\mathbb{D})$  be the closed subspace consisting of those functions in  $L^2(\mathbb{D}, \mathbb{A})$  that are analytic and let  $P$  denote the orthogonal projection of  $L^2(\mathbb{D}, \mathbb{A})$  onto  $L_a^2(\mathbb{D})$ . The functions  $\{e_n(z)\} = \{\sqrt{n+1}z^n\}_{n=0}^\infty$  form an orthonormal basis for  $L_a^2(\mathbb{D})$ .

We can define similarly operators  $T_\phi, M_\phi, h_\phi, S_\phi, \Gamma_\phi, J$  for  $\phi \in L^\infty(\mathbb{D})$ , replacing  $L_a^2(\mathbb{C}_+)$  by  $L_a^2(\mathbb{D})$ ,  $\overline{L_a^2(\mathbb{C}_+)}$  by  $\overline{L_a^2(\mathbb{D})}$ , and  $L^2(\mathbb{C}_+, \tilde{\mathbb{A}})$  by  $L^2(\mathbb{D}, \mathbb{A})$  in the same way as we defined  $\widetilde{T}_\phi, \widetilde{M}_\phi, \widetilde{h}_\phi, \widetilde{S}_\phi, \widetilde{\Gamma}_\phi, \tilde{J}$  for  $\phi \in L^\infty(\mathbb{C}_+)$ . Since  $\widetilde{h}_\phi$  is unitarily equivalent to some  $\widetilde{S}_\psi$  and  $\widetilde{\Gamma}_\phi$  is unitarily equivalent to some  $\widetilde{S}_\theta$ , we shall refer all these operators in the sequel as the little Hankel operators on  $L_a^2(\mathbb{C}_+)$  and  $h_\phi, S_\phi, \Gamma_\phi$  as the little Hankel operators on  $L_a^2(\mathbb{D})$ .

Let  $dA(z)$  be the area measure on  $\mathbb{D}$  normalised so that the area of  $\mathbb{D}$  is 1. For  $-1 < \alpha < \infty$ , let  $dA_\alpha$  be the probability measure on  $\mathbb{D}$  defined by

$$dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z).$$

Let  $L^2(dA_\alpha)$  be the space of all measurable functions on the unit disk  $\mathbb{D}$  for which the norm

$$\|f\|_\alpha^2 = (\alpha + 1) \int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty.$$

The weighted Bergman space  $L_a^2(dA_\alpha)$  is the subspace of functions in  $L^2(dA_\alpha)$  that are analytic and  $L_a^2(dA_\alpha)$  is a closed subspace [15] of  $L^2(dA_\alpha)$ . Let  $P_\alpha$  be the orthogonal projection from the Hilbert space  $L^2(dA_\alpha)$  onto the closed subspace  $L_a^2(dA_\alpha)$ , given by

$$P_\alpha f(z) = \int_{\mathbb{D}} K^\alpha(z, w) f(w) dA_\alpha(w)$$

where  $K^\alpha(z, w) = K(z, w)^{\frac{1+\alpha}{2}} = \frac{1}{(1-z\bar{w})^{\alpha+2}}$ ,  $z, w \in \mathbb{D}$  is the reproducing kernel of  $L_a^2(dA_\alpha)$ . Let  $\phi$  be a measurable function on  $\mathbb{D}$ . The little Hankel operator with symbol  $\phi$  denoted by  $h_\phi$  is defined by  $h_\phi f = \overline{P_\alpha}(\phi f)$ ,  $f \in L_a^2(dA_\alpha)$  where  $\overline{P_\alpha}$  is the orthogonal projection from the Hilbert space  $L^2(dA_\alpha)$  onto  $\overline{L_a^2(dA_\alpha)}$ , conjugates

of functions in  $L_a^2(dA_\alpha)$ . Let  $L^\infty(dA_\alpha)$  be the space of complex-valued, essentially bounded, measurable functions on  $\mathbb{D}$  and  $H^\infty(dA_\alpha)$  be the subspace consisting of those functions that are analytic in  $L^\infty(dA_\alpha)$ . Here we shall consider only those symbols  $\phi$  that are bounded in  $H^\infty + \overline{H^\infty}$ , where  $\overline{H^\infty(dA_\alpha)}$  constitutes the conjugates of functions in  $H^\infty(dA_\alpha)$ . If  $\phi \in H^\infty$ ,  $h_\phi = 0$ . Let  $\Gamma_\phi$  be a map from  $L_a^2(dA_\alpha)$  into  $L_a^2(dA_\alpha)$  such that  $\Gamma_\phi f = P_\alpha(\phi Jf)$  for all  $f \in L_a^2(dA_\alpha)$  where  $J$  is the mapping from  $L^2(dA_\alpha)$  onto  $L^2(dA_\alpha)$  such that  $Jf(z) = f(\bar{z})$ . Note that  $J$  is unitary. It can easily be checked that the operators  $\Gamma_\phi$  is unitarily equivalent to an operator  $h_\psi$ , for some  $\psi \in L^\infty(dA_\alpha)$ .

Let  $z = \frac{1-s}{1+s}$ . Hence  $2\operatorname{Re} s = \frac{2(1-|z|^2)}{|1+z|^2}$ . For convenience we shall write  $L^2(dA_\alpha) = L^{2,\alpha}(\mathbb{D})$  and  $L_a^2(dA_\alpha) = L_a^{2,\alpha}(\mathbb{D})$ . We say an analytic function  $F \in L_a^{2,\alpha}(\mathbb{C}_+)$  if and only if  $\int_{\mathbb{C}_+} |F(s)|^2 x^\alpha d\tilde{A}(s) < \infty$ . Let  $f(z) = F(\frac{1-z}{1+z})$ ,  $s = \frac{1-z}{1+z}$ . Thus  $F \in L_a^{2,\alpha}(\mathbb{C}_+)$  if and only if

$$\int_{\mathbb{D}} |f(z)|^2 \frac{(1-|z|^2)^\alpha}{|1+z|^{2\alpha}} \frac{4}{|1+z|^4} dA(z) < \infty.$$

This is possible if and only if  $\int_{\mathbb{D}} |\frac{2f(z)}{|1+z|^{\alpha+2}}|^2 (1-|z|^2)^\alpha dA(z) < \infty$ . Hence  $F \in L_a^{2,\alpha}(\mathbb{C}_+)$  if and only if  $\frac{2f(z)}{(1+z)^{\alpha+2}} \in L_a^{2,\alpha}(\mathbb{D})$ . Therefore  $f \in L_a^{2,\alpha}(\mathbb{D})$  if and only if  $\frac{2^{\alpha+1}}{(1+s)^{\alpha+2}} F(s) \in L_a^{2,\alpha}(\mathbb{C}_+)$ .

For  $h(t) \in L^2((0, \infty), dt)$  define the Laplace transform  $H(s) = (\mathcal{L}h)(s) = \int_0^\infty e^{-st} h(t) dt$ . Then  $(\mathcal{L}^{-1}H)(t) = \frac{1}{2\pi i} \int_\Omega H(s) e^{st} ds$ , where  $\Omega$  is the contour  $\{\operatorname{Re} s = \gamma\}$  for any  $\gamma > 0$ .

Let  $w_\alpha(t), t \geq 0$  be a positive increasing function (a weight function) related to a nonnegative real function  $\Omega_\alpha(s), s \geq 0$  by  $w_\alpha(t)^{-1} = (\mathcal{L}\Omega_\alpha)(2t)$ , where  $\alpha > -1$ .

Examples include  $\Omega_\alpha(x) = x^\alpha e^{-ax}$ , with  $a \geq 0$ , corresponding to  $w_\alpha(t) = (t + \frac{a}{2})^{\alpha+1}$ . Further we shall assume that

$$\begin{aligned} w_\alpha(t)w_\beta(\tau) &\leq w_{\alpha+\beta+1}(t+\tau) \\ w_\alpha(t)w_\beta(t) &= w_{\alpha+\beta+1}(t), \end{aligned} \quad (*)$$

for all  $\alpha, \beta > -1$  and  $t, \tau \geq 0$ .

Let  $L_a^2(\mathbb{C}_+, \not\leq_\alpha(\curvearrowright)\tilde{A}(\sim))$  be the space of complex analytic functions  $F$  on  $\mathbb{C}_+$  such that  $\int_{\mathbb{C}_+} |F(s)|^2 \Omega_\alpha(x) d\tilde{A}(s) < \infty$  where  $s = x + iy$ . We shall define Toeplitz and little Hankel operators on this space using the same notation for these operators on this space as we used for  $L_a^2(\mathbb{C}_+)$ . It will be clear from the context on which space we are considering these operators. Finally, let  $L^2((0, \infty), \frac{dt}{w_\alpha(t)})$  be the space of complex-valued, absolutely square-integrable, measurable functions on  $(0, \infty)$  with respect to the measure  $\frac{dt}{w_\alpha(t)}$  where  $w_\alpha(t)$  is as above.

We shall show that the Laplace transform  $\mathcal{L}$  determines a linear bijection between  $L^2((0, \infty), \frac{dt}{w_\alpha(t)})$  and  $L_a^2(\mathbb{C}_+, \not\leq_\alpha(\curvearrowright)\tilde{A}(\sim))$  such that

$$\|\mathcal{L}h\|_{L_a^2(\mathbb{C}_+, \not\leq_\alpha(\curvearrowright)\tilde{A}(\sim))} = \sqrt{2\pi} \|h\|_{L^2((0, \infty), \frac{dt}{w_\alpha(t)})}$$

where  $w_\alpha$  is a positive weight function as described above and  $w_\alpha(t)^{-1} = (\mathcal{L}\Omega_\alpha)(2t)$ .

## 2. Little Hankel Operators Between Weighted Bergman Spaces

In this section we are going to define a bounded integral operator and show how this operator is unitarily equivalent to a little Hankel operator between weighted Bergman spaces. R. Rochberg [13] has referred to some special classes of these integral operators and here we provide a more detailed account of these operators and their spectral properties.

We shall consider the integral operator  $K_h$  from  $L^2((0, \infty), dt)$  into itself defined by

$$(K_h u)(t) = \int_0^\infty \frac{\sqrt{w_\beta(t)}\sqrt{w_\alpha(\tau)}}{\sqrt{w_{\alpha+\beta+1}(t+\tau)}} h(t+\tau)u(\tau)d\tau,$$

where  $w_\alpha(t)$  and  $w_\beta(t)$  are the positive increasing weight functions as above.

**Theorem 2.1** If  $h(t) \in L^1((0, \infty), dt) \cap L^2((0, \infty), dt)$  then the Hankel integral operator  $K_h$  is well-defined and bounded with  $\|K_h\| \leq \|h\|_1$ .

**Proof** Let  $f, g \in L^2((0, \infty), dt)$  be such that  $\|f\|_{L^2} \leq 1$  and  $\|g\|_{L^2} \leq 1$ . Then,

$$\left| \int_0^\infty \overline{(K_h f)(t)} g(t) dt \right| = \left| \int_0^\infty \int_0^\infty \frac{\sqrt{w_\beta(t)}\sqrt{w_\alpha(\tau)}}{\sqrt{w_{\alpha+\beta+1}(t+\tau)}} \overline{h(t+\tau)} f(\tau) g(t) dt d\tau \right|.$$

The result follows as in [4] since the modulus of  $\frac{\sqrt{w_\beta(t)}\sqrt{w_\alpha(\tau)}}{\sqrt{w_{\alpha+\beta+1}(t+\tau)}}$  will not exceed 1.  $\square$

We now show the existence of a unitary map between  $L_a^2(\mathbb{C}_+, \not\prec_\alpha(\curvearrowright)\tilde{\mathbb{A}}(\sim))$  and  $L^2((0, \infty), \frac{dt}{w_\alpha(t)})$  in order to establish that  $K_h$  is unitarily equivalent to a little Hankel operator.

**Theorem 2.2** If  $\Omega_\alpha$  is a positive function of  $x$  alone, then there exists a unitary map  $W$  from  $L_a^2(\mathbb{C}_+, \not\prec_\alpha(\curvearrowright)\tilde{\mathbb{A}}(\sim))$  onto  $L^2((0, \infty), \frac{dt}{w_\alpha(t)})$  where  $w_\alpha(t)^{-1} = (\mathcal{L}\Omega_\alpha)(2t)$ , and  $s = x + iy$ .

**Proof** Let  $F \in L_a^2(\mathbb{C}_+, \not\prec_\alpha(\curvearrowright)\tilde{\mathbb{A}}(\sim))$ . Then

$$\int_{\mathbb{C}_+} |F(x + iy)|^2 \Omega_\alpha(x) dx dy < \infty.$$

Let  $\mathcal{L}(f(t)e^{-xt})(iy) = F(x + iy)$ . Then by Fubini's theorem and Plancherel's theorem [14],

$$\begin{aligned} \|F\|_{L_a^2(\mathbb{C}_+, \not\prec_\alpha(\curvearrowright)\tilde{\mathbb{A}}(\sim))}^2 &= \int_{\mathbb{C}_+} |F(x + iy)|^2 \Omega_\alpha(x) dx dy \\ &= \int_{y=-\infty}^\infty \int_{x=0}^\infty |F(x + iy)|^2 \Omega_\alpha(x) dx dy \\ &= \int_{y=-\infty}^\infty \int_{x=0}^\infty |\mathcal{L}(f(t)e^{-xt})(iy)|^2 \Omega_\alpha(x) dx dy \\ &= \int_{x=0}^\infty \left[ \int_{y=-\infty}^\infty |\mathcal{L}(f(t)e^{-xt})(iy)|^2 dy \right] \Omega_\alpha(x) dx \\ &= \int_{x=0}^\infty [2\pi \int_{t=0}^\infty |f(t)|^2 e^{-2tx} dt] \Omega_\alpha(x) dx \\ &= 2\pi \int_{t=0}^\infty |f(t)|^2 \left[ \int_{x=0}^\infty e^{-2tx} \Omega_\alpha(x) dx \right] dt < \infty. \end{aligned}$$

With  $w_\alpha(t)^{-1} = \int_{x=0}^\infty e^{-2tx} \Omega_\alpha(x) dx = (\mathcal{L}\Omega_\alpha)(2t)$ , we have

$$2\pi \int_{t=0}^\infty |f(t)|^2 \frac{dt}{w_\alpha(t)} < \infty.$$

Thus  $f \in L^2((0, \infty), \frac{dt}{w_\alpha(t)})$  and the map  $W$  is given by  $WF = \sqrt{2\pi}f$ .  $\square$

**Theorem 2.3** For  $\beta > \alpha > -1$ , the little Hankel operator  $\widetilde{\Gamma}_G$  from  $L_a^2(\mathbb{C}_+, \not\leq_\alpha(\curvearrowright)\tilde{\mathbb{A}}(\sim))$  into  $L_a^2(\mathbb{C}_+, \not\leq_\beta(\curvearrowright)\tilde{\mathbb{A}}(\sim))$  with symbol  $G \in H^\infty(\mathbb{C}_+)$  is unitarily equivalent to the integral operator  $K_g$  defined above where  $G = \mathcal{L}(\sqrt{w_{\beta-\alpha-1}}(t)g(t))$ .

**Proof** For  $\alpha > -1$ , let  $w_\alpha(t)^{-1} = (\mathcal{L}\Omega_\alpha)(2t)$ . Let  $S : L^2((0, \infty), dt) \longrightarrow L^2\left((0, \infty), \frac{dt}{w_\alpha(t)}\right)$  be such that

$$(Sf)(t) = \sqrt{w_\alpha(t)}f(t).$$

Let  $T : L^2\left((0, \infty), \frac{dt}{w_\beta(t)}\right) \longrightarrow L^2((0, \infty), dt)$  be such that

$$(Tf)(t) = \frac{1}{\sqrt{w_\beta(t)}}f(t).$$

It can easily be checked that  $S$  and  $T$  are unitary maps. Let  $\widetilde{K}_h$  be the operator unitarily equivalent to  $K_h$  by the relation

$$\widetilde{K}_h = T^{-1}K_hS^{-1}.$$

Then the operator

$$\widetilde{K}_h : L^2\left((0, \infty), \frac{dt}{w_\alpha(t)}\right) \longrightarrow L^2\left((0, \infty), \frac{dt}{w_\beta(t)}\right)$$

satisfies

$$\begin{aligned} (\widetilde{K}_h u)(s) &= (T^{-1}K_hS^{-1}u)(s) \\ &= \int_0^\infty \frac{w_\beta(s)}{\sqrt{w_{\alpha+\beta+1}(s+t)}} h(s+t)u(t)dt. \end{aligned}$$

For  $G \in H^\infty(\mathbb{C}_+)$ , the little Hankel operator

$$\widetilde{\Gamma}_G : L_a^2(\mathbb{C}_+, \not\leq_\alpha(\curvearrowright)\tilde{\mathbb{A}}(\sim)) \longrightarrow \mathbb{L}_\mathbb{C}^\#(\mathbb{C}_+, \not\leq_\beta(\curvearrowright)\tilde{\mathbb{A}}(\sim))$$

is given by

$$(\widetilde{\Gamma}_G U)(s) = P_{\alpha\beta}(G(s)U(\bar{s}))$$

for  $U \in L_a^2(\mathbb{C}_+, \not\leq_\alpha(\curvearrowright)\tilde{\mathbb{A}}(\sim))$  where  $P_{\alpha\beta}$  is the orthogonal projection of  $L^2(\mathbb{C}_+, \not\leq_\alpha(\curvearrowright)\tilde{\mathbb{A}}(\sim))$  onto  $L_a^2(\mathbb{C}_+, \not\leq_\beta(\curvearrowright)\tilde{\mathbb{A}}(\sim))$ . The operator  $\widetilde{\Gamma}_G$  is bounded (for proof see [8]).

Let  $G(s) = \mathcal{L}(\sqrt{w_{\beta-\alpha-1}}(t)g(t))$ ,  $U(s) = \mathcal{L}(\sqrt{w_\alpha(t)}u(t))$  and  $(\widetilde{\Gamma}_G U)(s) = P_{\alpha\beta}(G(s)U(\bar{s})) = R(s)$ . Then

$$\langle G(s)U(\bar{s}), F(s) \rangle = \langle R(s), F(s) \rangle$$

for all  $F \in L_a^2(\mathbb{C}_+, \not\leq_\beta(\curvearrowright)\tilde{\mathbb{A}}(\sim))$ . Thus

$$\langle G(s), \overline{U(\bar{s})}F(s) \rangle = \langle R(s), F(s) \rangle$$

for all  $F \in L_a^2(\mathbb{C}_+, \not\leq_\beta(\curvearrowright)\tilde{\mathbb{A}}(\sim))$ . Also  $\overline{U(\bar{s})} = \mathcal{L}(\sqrt{w_\alpha(t)}\bar{u}(t))$ . Thus

$$\begin{aligned} &\int_0^\infty \sqrt{w_{\beta-\alpha-1}(t)g(t)} \overline{(\sqrt{w_\alpha(t)}\bar{u}(t)) * (\sqrt{w_\beta(t)}f(t))} \frac{dt}{w_\beta(t)} \\ &= \int_0^\infty \sqrt{w_\beta(t)r(t)} \sqrt{w_\beta(t)f(t)} \frac{dt}{w_\beta(t)} \end{aligned}$$

where  $*$  denotes convolution,  $\sqrt{w_\beta(t)}f(t) = \mathcal{L}^{-1}\{F(s)\}$ ,  $\sqrt{w_\beta(t)}r(t) = \mathcal{L}^{-1}\{R(s)\}$  and

$$\begin{aligned} & \overline{(\sqrt{w_\alpha(t)}\bar{u}(t)) * (\sqrt{w_\beta(t)}f(t))} \\ &= \int_0^t \sqrt{w_\alpha(\tau)}\bar{u}(\tau)\sqrt{w_\beta(t-\tau)}f(t-\tau)d\tau \\ &= \int_0^t \sqrt{w_\alpha(\tau)}u(\tau)\sqrt{w_\beta(t-\tau)}\overline{f(t-\tau)}d\tau. \end{aligned}$$

$$\begin{aligned} & \text{Hence } \int_0^\infty \sqrt{w_{\beta-\alpha-1}(t)}g(t)((\sqrt{w_\alpha(t)}\bar{u}(t)) * (\sqrt{w_\beta(t)}f(t)))\frac{dt}{w_\beta(t)} \\ &= \int_0^\infty \sqrt{w_{\beta-\alpha-1}(t)}g(t)(\int_0^t \sqrt{w_\alpha(\tau)}u(\tau)\sqrt{w_\beta(t-\tau)}\overline{f(t-\tau)}d\tau)\frac{dt}{w_\beta(t)} \\ &= \int_{x=0}^\infty \int_{\tau=0}^\infty \sqrt{w_{\beta-\alpha-1}(x+\tau)}g(x+\tau)\sqrt{w_\alpha(\tau)}u(\tau)\sqrt{w_\beta(x)}\overline{f(x)}\frac{d\tau}{w_\beta(x+\tau)}dx \\ &= \int_{x=0}^\infty [\int_{\tau=0}^\infty \frac{\sqrt{w_{\beta-\alpha-1}(x+\tau)}}{w_\beta(x+\tau)}g(x+\tau)\sqrt{w_\alpha(\tau)}u(\tau)d\tau]\sqrt{w_\beta(x)}\overline{f(x)}dx \\ &= \int_{x=0}^\infty \frac{1}{w_\beta(x)}(\widetilde{K_g(\sqrt{w_\alpha}u)})(x)\sqrt{w_\beta(x)}\overline{f(x)}dx \\ &= \int_{x=0}^\infty (\widetilde{K_g(\sqrt{w_\alpha}u)})(x)\sqrt{w_\beta(x)}\overline{f(x)}\frac{dx}{w_\beta(x)} \\ &= \langle (\widetilde{K_g(\sqrt{w_\alpha}u)})(x), \sqrt{w_\beta(x)}\overline{f(x)} \rangle_{L^2((0,\infty), \frac{dx}{w_\beta(x)})}. \end{aligned}$$

$$\text{Thus } \langle \widetilde{K_g(\sqrt{w_\alpha}u)}(x), \sqrt{w_\beta(x)}f(x) \rangle_{L^2((0,\infty), \frac{dx}{w_\beta(x)})}$$

$$= \langle \sqrt{w_\beta(x)}r(x), \sqrt{w_\beta(x)}f(x) \rangle_{L^2((0,\infty), \frac{dx}{w_\beta(x)})}.$$

Hence  $(\widetilde{K_g(\sqrt{w_\alpha}u)})(x) = \sqrt{w_\beta(x)}r(x) = \mathcal{L}^{-1}\{R(s)\}$ , and  $\mathcal{L}(\widetilde{K_g(\sqrt{w_\alpha}u)})(s) = R(s) = (\Gamma_G U)(s)$ .  $\square$

We are now in a position to discuss various properties that these integral operators may possess, expressed in terms of the integral kernels.

**Theorem 2.4** If  $h(x) \in L^1(0, \infty) \cap L^2(0, \infty)$ , then the integral operator  $K_h$  is finite rank if and only if  $\frac{h(t)}{\sqrt{w_{\alpha+\beta+1}(t)}}$  is of the form  $\sum_{n=1}^N P_n(t)e^{-\lambda_n t}$ , where  $P_n$  is a polynomial and  $\lambda_n \in \mathbb{C}$  are distinct with  $\text{Re } \lambda_n > 0$ . Its rank is  $\sum_{n=1}^N \deg(P_n)$ .

**Proof** The operator  $\widetilde{K}_h$  defined in the theorem 2.3 is unitarily equivalent to  $K_h$ . Consider the operator  $(Au)(s) = \int_0^\infty h(s+t)u(t)dt$ . The operator  $A$  is a mapping from  $L^2(0, \infty)$  into  $L^2(0, \infty)$  and  $A$  is finite rank if and only if  $h$  is a linear combination of  $P_n(t)e^{-\lambda_n t}$  where  $P_n$  is a polynomial and  $\lambda_n \in \mathbb{C}$ . Thus  $\widetilde{K}_h$  is finite rank [12] if and only if  $\frac{h(t)}{\sqrt{w_{\alpha+\beta+1}(t)}}$  is a linear combination of  $P_n(t)e^{-\lambda_n t}$ .  $\square$

**Theorem 2.5** The operator  $K_h$  defined above is Hilbert-Schmidt if and only if  $\sqrt{\frac{c(t)}{w_{\alpha+\beta+1}(t)}}h(t)$  is in  $L^2(0, \infty)$  and  $\|K_h\|_{HS} = \|\sqrt{\frac{c(t)}{w_{\alpha+\beta+1}(t)}}h\|_{L^2}$  where  $c(t) = \int_{x=0}^t w_\beta(x)w_\alpha(t-x)dx$ .

**Proof** The square of the Hilbert-Schmidt norm of  $K_h$  is given by

$$\begin{aligned} & \int_{x=0}^\infty \int_{\tau=0}^\infty \frac{w_\beta(x)w_\alpha(\tau)}{w_{\alpha+\beta+1}(x+\tau)} |h(x+\tau)|^2 d\tau dx \\ &= \int_{y=0}^\infty \int_{x=0}^y \frac{w_\beta(x)w_\alpha(y-x)}{w_{\alpha+\beta+1}(y)} |h(y)|^2 dx dy, \end{aligned}$$

where  $y = x + \tau$ . Suppose  $c(y) = \int_{x=0}^y w_\beta(x)w_\alpha(y-x)dx$ . Then the last integral is equal to

$$\int_{y=0}^{\infty} \frac{c(y)}{w_{\alpha+\beta+1}(y)} |h(y)|^2 dy = \left\| \sqrt{\frac{c(y)}{w_{\alpha+\beta+1}(y)}} h \right\|_{L^2}^2.$$

Thus  $K_h$  is Hilbert-Schmidt if and only if  $\sqrt{\frac{c(t)}{w_{\alpha+\beta+1}(t)}} h(t) \in L^2(0, \infty)$ .  $\square$

**Theorem 2.6** If  $h \in L^1(0, \infty)$ , then  $K_h$  is compact.

**Proof** Without loss of generality, we may assume that  $h \in L^1(0, \infty)$  is real and positive. Let  $M > 0$ ,  $h_M = h \wedge M \left( \sqrt{\frac{w_{\alpha+\beta+1}(t)}{c(t)}} \right) (t+1)^{-1}$ , so that  $h_M$  increases to  $h$  as  $M \rightarrow \infty$ . Hence  $\|h_M - h\|_{L^1} \rightarrow 0$ . Also  $h_M$  induces a Hilbert-Schmidt operator since  $(t+1)^{-1} \in L^2(0, \infty)$  and  $\sqrt{\frac{c(t)}{w_{\alpha+\beta+1}(t)}} h_M \leq \frac{M}{t+1} \in L^2(0, \infty)$ . Now  $\|K_{h_M} - K_h\| \leq \|h_M - h\|_{L^1} \rightarrow 0$  as  $M \rightarrow \infty$ . Therefore given  $\epsilon > 0$ , there exists  $M$  such that  $\|K_{h_M} - K_h\| < \epsilon$  and each  $K_{h_M}$  is Hilbert-Schmidt. Thus  $K_h$  is compact.  $\square$

**Theorem 2.7** If  $w_\alpha$  is a positive increasing weight function described before such that  $\left| \frac{\sqrt{w_\beta(t)}\sqrt{w_\alpha(t)}}{\sqrt{w_{\alpha+\beta+1}(2t)}} \right| \leq m < 1$  for all  $t \geq 0$  and for some positive constant  $m$  and  $w_{\beta+1}(t) \leq w_{\alpha+1}(t)$  then the integral operator  $K_h$  on  $L^2(0, \infty)$  with  $h \in L^1$  defined by

$$(K_h u)(t) = \int_0^\infty \frac{\sqrt{w_\beta(t)}\sqrt{w_\alpha(\tau)}}{\sqrt{w_{\alpha+\beta+1}(t+\tau)}} h(t+\tau)u(\tau)d\tau$$

satisfies the following inequalities:

$$\|K_h\| \leq \|\mathcal{L}(\sqrt{w_{\beta-\alpha-1}(t)} h)\|_\infty \leq \|h\|_{L^1} \leq \frac{2}{m} \|K_h\|_N.$$

where  $\|K_h\|_N = \text{tr}|K_h|$  and  $|K_h| = (K_h^* K_h)^{\frac{1}{2}}$  is the modulus of  $K_h$ .

**Proof** The first inequality follows from theorem 2.3 and the second inequality from the definition of Laplace transform and the condition that  $w_{\beta+1}(t) \leq w_{\alpha+1}(t)$ . Only the final inequality requires detailed proof. Suppose first that  $h$  is continuous and of compact support  $[0, M]$ , say. For  $\delta > 0$  and  $n = 0, 1, 2, \dots$  define the functions

$$e_n^\delta(t) = \delta^{-\frac{1}{2}} \chi_{(n\delta, (n+1)\delta)}(t).$$

Then  $e_0^\delta, e_1^\delta, e_2^\delta, \dots$  is an orthonormal sequence in  $L^2(0, \infty)$  and

$$\langle K_h e_n^\delta, e_n^\delta \rangle = \frac{1}{\delta} \int_{n\delta}^{(n+1)\delta} \int_{n\delta}^{(n+1)\delta} \frac{\sqrt{w_\beta(t)}\sqrt{w_\alpha(\tau)}}{\sqrt{w_{\alpha+\beta+1}(t+\tau)}} h(t+\tau) dt d\tau. \quad (1)$$

By the uniform continuity of  $h$  we have given  $\epsilon > 0$  and for sufficiently small  $\delta$ ,  $|h(v_1) - h(v_2)| < \frac{\epsilon}{2M}$  if  $|v_1 - v_2| \leq 2\delta$ . Thus

$$\begin{aligned} & \left| \langle K_h e_n^\delta, e_n^\delta \rangle - \delta \frac{\sqrt{w_\beta(n\delta)}\sqrt{w_\alpha(n\delta)}}{\sqrt{w_{\alpha+\beta+1}(2n\delta)}} |h(2n\delta)| \right| \\ &= \left| \frac{1}{\delta} \int_{n\delta}^{(n+1)\delta} \int_{n\delta}^{(n+1)\delta} \frac{\sqrt{w_\beta(t)}\sqrt{w_\alpha(\tau)}}{\sqrt{w_{\alpha+\beta+1}(t+\tau)}} h(t+\tau) dt d\tau \right| \end{aligned}$$

$$\begin{aligned}
& \left| -\frac{1}{\delta} \int_{n\delta}^{(n+1)\delta} \int_{n\delta}^{(n+1)\delta} \frac{\sqrt{w_\beta(n\delta)}\sqrt{w_\alpha(n\delta)}}{\sqrt{w_{\alpha+\beta+1}(2n\delta)}} |h(2n\delta)| dt d\tau \right| \\
& \leq \frac{1}{\delta} \int_{n\delta}^{(n+1)\delta} \int_{n\delta}^{(n+1)\delta} |h(t+\tau) - h(2n\delta)| dt d\tau \leq \frac{1}{\delta} \frac{\epsilon}{2M} \delta^2 = \frac{\delta\epsilon}{2M}
\end{aligned}$$

since  $w_{\alpha+\beta+1}(t)$  is an increasing function and satisfies the condition (\*). Also

$$\begin{aligned}
& \left| \frac{1}{2} \frac{\sqrt{w_\beta(n\delta)}\sqrt{w_\alpha(n\delta)}}{\sqrt{w_{\alpha+\beta+1}(2n\delta)}} \int_{2n\delta}^{2(n+1)\delta} |h(\tau)| d\tau \right. \\
& \quad \left. - \delta \frac{\sqrt{w_\beta(n\delta)}\sqrt{w_\alpha(n\delta)}}{\sqrt{w_{\alpha+\beta+1}(2n\delta)}} |h(2n\delta)| \right| \\
& = \left| \frac{1}{2} \frac{\sqrt{w_\beta(n\delta)}\sqrt{w_\alpha(n\delta)}}{\sqrt{w_{\alpha+\beta+1}(2n\delta)}} \int_{2n\delta}^{2(n+1)\delta} |h(\tau)| d\tau \right. \\
& \quad \left. - \frac{1}{2} \int_{2n\delta}^{2(n+1)\delta} \frac{\sqrt{w_\beta(n\delta)}\sqrt{w_\alpha(n\delta)}}{\sqrt{w_{\alpha+\beta+1}(2n\delta)}} |h(2n\delta)| d\tau \right| \\
& \leq \frac{1}{2} \int_{2n\delta}^{2(n+1)\delta} | |h(\tau)| - |h(2n\delta)| | d\tau \leq \frac{1}{2} \frac{\epsilon}{2M} 2\delta = \frac{\delta\epsilon}{2M}.
\end{aligned}$$

Hence

$$\begin{aligned}
& \left| |\langle K_h e_n^\delta, e_n^\delta \rangle| - \frac{1}{2} \frac{\sqrt{w_\beta(n\delta)}\sqrt{w_\alpha(n\delta)}}{\sqrt{w_{\alpha+\beta+1}(2n\delta)}} \int_{2n\delta}^{2(n+1)\delta} |h(\tau)| d\tau \right| \\
& \leq \frac{\delta\epsilon}{2M} + \frac{\delta\epsilon}{2M} = \frac{\delta\epsilon}{M}
\end{aligned}$$

and so since  $h$  has support in  $[0, M]$  we obtain

$$\begin{aligned}
& \left| \sum_{n=0}^{\infty} |\langle K_h e_n^\delta, e_n^\delta \rangle| - \frac{m}{2} \|h\|_1 \right| \leq \left| \sum_{n=0}^{\infty} |\langle K_h e_n^\delta, e_n^\delta \rangle| - \frac{1}{2} \frac{\sqrt{w_\beta(n\delta)}\sqrt{w_\alpha(n\delta)}}{\sqrt{w_{\alpha+\beta+1}(2n\delta)}} \|h\|_1 \right| \\
& \leq \frac{\delta\epsilon}{M} \left( 1 + \frac{M}{\delta} \right)
\end{aligned}$$

so that  $\sum_{n=0}^{\infty} |\langle K_h e_n^\delta, e_n^\delta \rangle| \rightarrow \frac{m}{2} \|h\|_1$  as  $\delta \rightarrow 0$ . But for any  $h_0 \in L^1$ , corresponding to the operator  $K_{h_0}$ , and  $\delta > 0$  we have

$$|\langle K_{h_0} e_n^\delta, e_n^\delta \rangle| \leq \int_{2n\delta}^{2(n+1)\delta} |h_0(\tau)| d\tau,$$

since for each  $\tau$  we obtain an obvious upper bound in (1). Thus given  $\epsilon > 0$  and  $h \in L^1$ , let  $h = h_1 + h_2$  where  $h_1$  is continuous with compact support in  $E$  and  $\|h_2\|_1 < \frac{\epsilon}{4}$  and  $h_2(x) = 0, x \in E$ . Then

$$\begin{aligned}
& \left| \sum_{n=0}^{\infty} |\langle K_h e_n^\delta, e_n^\delta \rangle| - \frac{m}{2} \|h\|_1 \right| \\
& \leq \sum_{n=0}^{\infty} |\langle K_{h_2} e_n^\delta, e_n^\delta \rangle| + \left| \sum_{n=0}^{\infty} |\langle K_{h_1} e_n^\delta, e_n^\delta \rangle| - \frac{m}{2} \|h\|_1 \right|
\end{aligned}$$

$$\leq \|h_2\|_1 + \left| \sum_{n=0}^{\infty} |\langle K_{h_1} e_n^\delta, e_n^\delta \rangle| - \frac{m}{2} \|h_1\|_1 \right| + \frac{1}{2} \|h_2\|_1 < \epsilon$$

if  $\delta$  is sufficiently small. Since  $\sum_{n=0}^{\infty} |\langle K_h e_n^\delta, e_n^\delta \rangle| \leq \|K_h\|_N$ , thus

$$\|K_h\|_N \geq \frac{m}{2} \|h\|_1.$$

Therefore,  $\|h\|_1 \leq \frac{2}{m} \|K_h\|_N$ .  $\square$

### 3. Hankel Operators Between Weighted Bergman Spaces of the Disk

As an important special case of the results in the previous section we consider the integral operator  $K_h : L^2((0, \infty), dt) \longrightarrow L^2((0, \infty), dt)$  defined by

$$(K_h u)(t) = \int_0^\infty \frac{t^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}} h(t+\tau) u(\tau) d\tau,$$

with  $\alpha, \beta > -1$ . These operators are unitarily equivalent to little Hankel operators between weighted Bergman spaces of the disk.

**Theorem 3.1** If  $h(x) \in L^1(0, \infty) \cap L^2(0, \infty)$ , then the integral operator  $K_h : L^2(0, \infty) \longrightarrow L^2(0, \infty)$  given by  $(K_h u)(t) = \int_0^\infty \frac{t^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}} h(t+\tau) u(\tau) d\tau$ , with  $\alpha, \beta > -1$  is well-defined and bounded with  $\|K_h\| \leq \|h\|_1$ .

**Proof** The proof follows by substituting  $w_\alpha(t) = t^{\alpha+1}$  in theorem 2.1.  $\square$

**Theorem 3.2** Let  $G(s) \in L^\infty(\mathbb{C}_+)$ . Then the little Hankel operator  $\widetilde{\Gamma}_G$  defined from  $L_a^{2,\alpha}(\mathbb{C}_+)$  into  $L_a^{2,\beta}(\mathbb{C}_+)$  by  $G$  is equivalent to the little Hankel operator  $\Gamma_\phi$  from  $L_a^{2,\alpha}(\mathbb{D})$  into  $L_a^{2,\beta}(\mathbb{D})$  determined by the function  $\phi(z) = (\frac{1+\bar{z}}{1+z})^{\alpha+2} G(Mz)$ .

**Proof** Let  $W : L_a^{2,\alpha}(\mathbb{D}) \longrightarrow \mathbb{L}_\mathbb{D}^{\mathcal{K},\alpha}(\mathbb{C}_+)$  be defined by

$$(Wg)(s) = \frac{2^{\frac{\alpha}{2}+1}}{\sqrt{\pi}} g(Ms) \frac{1}{(1+s)^{\alpha+2}}$$

where  $Ms = \frac{1-s}{1+s}$ . The inverse map  $W^{-1} : L_a^{2,\alpha}(\mathbb{C}_+) \longrightarrow \mathbb{L}_\mathbb{D}^{\mathcal{K},\alpha}(\mathbb{D})$  satisfies  $(W^{-1}G)(z) = 2^{\frac{\alpha}{2}+1} \sqrt{\pi} G(Mz) \frac{1}{(1+z)^{\alpha+2}}$  where  $Mz = \frac{1-z}{1+z}$ . Further, we shall define  $V : L_a^{2,\beta}(\mathbb{C}_+) \longrightarrow \mathbb{L}_\mathbb{D}^{\mathcal{K},\beta}(\mathbb{D})$  by  $(VG)(z) = 2^{\frac{\beta}{2}+1} \sqrt{\pi} G(Mz) \frac{1}{(1+z)^{\beta+2}}$  where  $Mz = \frac{1-z}{1+z}$ . The inverse map  $V^{-1} : L_a^{2,\beta}(\mathbb{D}) \longrightarrow \mathbb{L}_\mathbb{D}^{\mathcal{K},\beta}(\mathbb{C}_+)$  satisfies  $(V^{-1}g)(s) = \frac{2^{\frac{\beta}{2}+1}}{\sqrt{\pi}} g(Ms) \frac{1}{(1+s)^{\beta+2}}$ . It can easily be checked that  $V$  and  $W$  are unitary maps. Note  $W$  can also be defined from  $L^{2,\alpha}(\mathbb{D})$  into  $L^{2,\alpha}(\mathbb{C}_+)$  and similarly  $V$  can be defined from  $L^{2,\beta}(\mathbb{C}_+)$  into  $L^{2,\beta}(\mathbb{D})$  and are also unitary on these spaces. Then  $\nu_{n,\alpha}^2 = \|z^n\|_\alpha^2 = (\alpha+1) \int_\mathbb{D} |z|^{2n} (1-|z|^2)^\alpha dA(z) = (\alpha+1) \int_0^1 x^n (1-x)^\alpha dx = (\alpha+1) \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} \sim (n+1)^{-\alpha-1}$ . Hence  $\nu_{n,\alpha} \sim n^{-\frac{\alpha+1}{2}}$ ,  $n \geq 1$  and  $\{\frac{z^n}{\nu_{n,\alpha}}\}$  is an orthonormal basis for  $L_a^{2,\alpha}(\mathbb{D})$ .

Let  $\widetilde{P}_{\alpha\beta}$  be the orthogonal projection of  $L_a^{2,\alpha}(\mathbb{C}_+)$  onto  $L_a^{2,\beta}(\mathbb{C}_+)$  and  $P_{\alpha\beta}$  be the orthogonal projection of  $L_a^{2,\alpha}(\mathbb{D})$  onto  $L_a^{2,\beta}(\mathbb{D})$ . Define the map  $\widetilde{J} : L^{2,\alpha}(\mathbb{C}_+) \longrightarrow \mathbb{L}^{\mathcal{K},\alpha}(\mathbb{C}_+)$  such that  $\widetilde{J}f(s) = f(\bar{s})$ . In what follows we shall show that  $V\widetilde{\Gamma}_G W(\frac{z^n}{\nu_{n,\alpha}}) =$

$\Gamma_\phi(\frac{z^n}{\nu_{n,\alpha}})$ . That is,  $\tilde{\Gamma}_G W(\frac{z^n}{\nu_{n,\alpha}}) = V^{-1} \Gamma_\phi(\frac{z^n}{\nu_{n,\alpha}})$ . Note

$$\begin{aligned}
\tilde{\Gamma}_G W(\frac{z^n}{\nu_{n,\alpha}}) &= \tilde{P}_{\alpha\beta} G \tilde{J}(W(\frac{z^n}{\nu_{n,\alpha}})) \\
&= \tilde{P}_{\alpha\beta} G \tilde{J}(\frac{2^{\frac{\alpha}{2}+1}}{\sqrt{\pi}} \frac{1}{\nu_{n,\alpha}} (Ms)^n \frac{1}{(1+s)^{\alpha+2}}) \\
&= \tilde{P}_{\alpha\beta} G \tilde{J}(\frac{2^{\frac{\alpha}{2}+1}}{\sqrt{\pi}} \frac{1}{\nu_{n,\alpha}} \frac{(1-s)}{(1+s)} \frac{1}{(1+s)^{\alpha+2}}) \\
&= \tilde{P}_{\alpha\beta} G(\frac{2^{\frac{\alpha}{2}+1}}{\sqrt{\pi}} \frac{1}{\nu_{n,\alpha}} \frac{(1-\bar{s})}{(1+\bar{s})} \frac{1}{(1+\bar{s})^{\alpha+2}}) \\
&= V^{-1} P_{\alpha\beta} W^{-1}(G(s) \frac{2^{\frac{\alpha}{2}+1}}{\sqrt{\pi}} \frac{1}{\nu_{n,\alpha}} \frac{(1-\bar{s})}{(1+\bar{s})} \frac{1}{(1+\bar{s})^{\alpha+2}}) \\
&= V^{-1} P_{\alpha\beta} (\frac{2^{\frac{\alpha}{2}+1}}{\sqrt{\pi}} \frac{1}{\nu_{n,\alpha}} 2^{\frac{\alpha}{2}+1} \sqrt{\pi} (\frac{1-\frac{1+\bar{z}}{1+\frac{1-\bar{z}}{1+\bar{z}}}}{1+\frac{1-\bar{z}}{1+\bar{z}}})^n \frac{1}{(1+\frac{1-\bar{z}}{1+\bar{z}})^{\alpha+2}} G(Mz) \frac{1}{(1+z)^{\alpha+2}}) \\
&= V^{-1} P_{\alpha\beta} (2^{\alpha+2} \frac{1}{\nu_{n,\alpha}} \bar{z}^n (\frac{1+\bar{z}}{2})^{\alpha+2} G(Mz) \frac{1}{(1+z)^{\alpha+2}}) \\
&= V^{-1} P_{\alpha\beta} (G(Mz) (\frac{1+\bar{z}}{1+z})^{\alpha+2} J(\frac{z^n}{\nu_{n,\alpha}})).
\end{aligned}$$

Let  $\phi(z) = G(Mz) (\frac{1+\bar{z}}{1+z})^{\alpha+2}$ . Then

$$\begin{aligned}
\tilde{\Gamma}_G W(\frac{z^n}{\nu_{n,\alpha}}) &= V^{-1} P_{\alpha\beta} (\phi J(\frac{z^n}{\nu_{n,\alpha}})) \\
&= V^{-1} \Gamma_\phi(\frac{z^n}{\nu_{n,\alpha}}).
\end{aligned}$$

Thus  $V \tilde{\Gamma}_G W(\frac{z^n}{\nu_{n,\alpha}}) = \Gamma_\phi(\frac{z^n}{\nu_{n,\alpha}})$  and  $\tilde{\Gamma}_G$  is unitarily equivalent to  $\Gamma_\phi$ .  $\square$

In the following two theorems we shall assume that  $\alpha = \beta$ .

**Theorem 3.3** A Hankel operator  $\tilde{\Gamma}_G$  from  $L_a^{2,\alpha}(\mathbb{C}_+)$  into  $L_a^{2,\alpha}(\mathbb{C}_+)$  is nuclear if and only if it has a symbol of the form

$$G(s) = \sum_{k=1}^{\infty} \lambda_k \left( \frac{2\operatorname{Re} a_k}{s - a_k} \right)^{\alpha+2}$$

with  $\sum_{k=1}^{\infty} |\lambda_k| < \infty$  and  $a_k \in \mathbb{C}_-$  (the left half plane), the series converging in  $H^\infty$ . Moreover,

$$\inf \left\{ \sum_{k=1}^{\infty} |\lambda_k| : G(s) \text{ can be written as above} \right\} \leq c_\alpha \|\tilde{\Gamma}_G\|_N$$

where  $c_\alpha$  is a constant depending on  $\alpha$ .

**Proof** We shall prove this using the equivalence established in theorem 3.2 with  $\alpha = \beta$ . Let  $a_k \in \mathbb{C}_-$ ,  $\mathbf{1} \in \mathbb{N}$  and  $w_k = \frac{1+a_k}{1-a_k}$ . Then  $w_k \in \mathbb{D}$  and

$$\left( \frac{1+\bar{z}}{1+z} \right)^{\alpha+2} \left( \frac{2\operatorname{Re} a_k}{s - a_k} \right)^{\alpha+2} = (1+\bar{z})^{\alpha+2} \frac{(|w_k|^2 - 1)^{\alpha+2}}{(1+\bar{w}_k)^{\alpha+2}} \frac{1}{(1-w_k z)^{\alpha+2}}.$$

Now since  $P_\alpha \left[ \frac{(1+\bar{z})^{\alpha+2}}{(1-w_k z)^{\alpha+2}} \right] = \frac{(1+w_k)^{\alpha+2}}{(1-w_k z)^{\alpha+2}}$  hence the bounded analytic part of

$(\frac{1+\bar{z}}{1+z})^{\alpha+2} \left( \frac{2\operatorname{Re} a_k}{s - a_k} \right)^{\alpha+2}$  is equal to  $\phi(z) = \frac{(|w_k|^2 - 1)^{\alpha+2}}{(1+\bar{w}_k)^{\alpha+2}} \frac{(1+w_k)^{\alpha+2}}{(1-w_k z)^{\alpha+2}}$ . Since  $|\frac{1+w_k}{1+\bar{w}_k}| = 1$ , the assertion of the theorem follows from the characterization of the nuclear Hankel operators on the weighted Bergman space of the disc [8] and the facts below. From [8], it follows that  $\Gamma_\phi$  is nuclear if and only if  $\phi \in B_1^1$  where  $B_1^1 = \{f : \mathbb{D} \rightarrow \mathbb{C} : \mathcal{U} \text{ is analytic and } (\mathcal{K} - |F|^{\mathcal{K}})^{\geq -\mathcal{K}} \mathcal{U}^{(\geq)}(F) \in \mathbb{L}^{\mathcal{K}}((\mathcal{K} - |F|^{\mathcal{K}})^{-\mathcal{K}} \mathbb{A})\}$  where  $m$  is a nonnegative integer and  $m > 1$ . It is known [8],[13] that if  $N > 0$ , then there exists

a sequence  $\{\xi_i\}_{i=1}^\infty \subset \mathbb{D}$  such that every function  $f$  in  $B_1^1$  can be decomposed as a (countable) sum  $f(z) = \sum_{i=1}^\infty b_i \frac{(1-|\xi_i|^2)^N}{(1-\xi_i z)^N}$  with  $\sum_{i=1}^\infty |b_i| \leq C \|f\|_{B_1^1}$ .

Thus

$$\inf \left\{ \sum_{k=1}^\infty |\lambda_k| : G(s) = \sum_{k=1}^\infty \lambda_k \left( \frac{2\operatorname{Re} a_k}{s - a_k} \right)^{\alpha+2} \right\} \leq c'_\alpha \|\phi\|_{B_1^1} \leq c_\alpha \|\tilde{\Gamma}_G\|_N$$

where  $c'_\alpha, c_\alpha$  are constants depending on  $\alpha$ . For more details see [12], [2].

Next we shall show that if the integral operator  $K_h$  with kernel  $h$  is nuclear then  $h(t)$  must be well-behaved, both as to smoothness and as to rate of decay at  $\infty$ .

**Theorem 3.4** Suppose  $\alpha = \beta$  is a odd positive integer and  $h \in L^1(0, \infty)$  determines a nuclear integral operator  $K_h$  on  $L^2(0, \infty)$  defined by

$$(K_h u)(t) = \int_0^\infty \frac{t^{\frac{\alpha+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\alpha+1}} h(t+\tau) u(\tau) d\tau.$$

Then  $h$  is equal almost everywhere to a function  $h_0$  which is continuous on  $(0, \infty)$ ; moreover  $h_0(t), t \in (0, \infty)$  satisfies

$$|h_0(t)| \leq 2^{\alpha+2} \left( \frac{\alpha+2}{t} \right)^{\alpha+2} e^{-(\alpha+2)} c_\alpha \|\tilde{\Gamma}_G\|_N.$$

**Proof** Let  $\tilde{\Gamma}_G$  be the Hankel operator on  $L_a^{2,\alpha}(\mathbb{C}_+)$  unitarily equivalent to  $K_h$ . Then  $\tilde{\Gamma}_G$  is nuclear. By theorem 3.3,  $G(s) = \sum_{k=1}^\infty \lambda_k \left( \frac{2\operatorname{Re} a_k}{s - a_k} \right)^{\alpha+2}$ ,  $a_k \in \mathbb{C}_-$ . Given  $\epsilon > 0$ , take  $h_\epsilon(t) = \sum \lambda_k (2\operatorname{Re} a_k)^{\alpha+2} t^{\alpha+1} e^{a_k t}$  where  $\sum |\lambda_k| \leq (c_\alpha + \epsilon) \|\tilde{\Gamma}_G\|_N$  and  $a_k \in \mathbb{C}_-$ . The series  $\frac{h_\epsilon(t)}{t^{\alpha+1}}$  converges uniformly on  $[\delta, \infty)$  for any  $\delta > 0$ , since  $\sup\{|x^{\alpha+2} e^{-tx}| : t \in [\delta, \infty)\} = x^{\alpha+2} e^{-\delta x}$  and

$$\sup\{|x^{\alpha+2} e^{-\delta x}| : x \geq 0\} = \left( \frac{\alpha+2}{\delta} \right)^{\alpha+2} e^{-(\alpha+2)}.$$

It follows therefore that  $\frac{h_\epsilon(t)}{t^{\alpha+1}}$  is continuous and that

$$\left| \frac{h_\epsilon(t)}{t^{\alpha+1}} \right| \leq 2^{\alpha+2} \left( \frac{\alpha+2}{t} \right)^{\alpha+2} e^{-(\alpha+2)} (c_\alpha + \epsilon) \|\tilde{\Gamma}_G\|_N.$$

Since  $\frac{h_\epsilon(t)}{t^{\alpha+1}} = \frac{h(t)}{t^{\alpha+1}}$  almost everywhere, the result follows on letting  $\epsilon \rightarrow 0$ .  $\square$

**Remark 3.5** For specific kernel functions  $h$  and suitable  $f \in L^2(0, \infty)$ , one can also analyze the asymptotic behavior of the integral defining  $K_h$ . We need to develop appropriate mathematical techniques for evaluating the asymptotic behavior of these integrals as it enhances further the applicability of these integral operators. For example, let  $h(t) = t^{\frac{\alpha+\beta+2}{2}} e^{-kt} \chi_{[0,A]}(t)$ . Then

$$\begin{aligned} (K_h f)(t) &= \int_0^{A-t} t^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}} e^{-k(t+\tau)} f(\tau) d\tau \\ &= t^{\frac{\beta+1}{2}} e^{-kt} \int_0^{A-t} \tau^{\frac{\alpha+1}{2}} e^{-k\tau} f(\tau) d\tau, \end{aligned}$$

where we assume  $f$  and  $t^{\frac{\alpha+1}{2}}f$  are integrable in  $(0, \infty)$ , and the asymptotic series expansion of the function  $t^{\frac{\alpha+1}{2}}f$  is as follows:

$$t^{\frac{\alpha+1}{2}}f(t) \sim t^c \sum_{n=0}^{\infty} a_n t^{bn}, \quad (2)$$

$t \rightarrow 0^+$ ,  $c > -1$  and  $b > 0$ .

Assume that for  $A < \infty, t > 0, |f(t)| \leq M$  for some constant  $M$ . Further assume that if  $A = \infty, |t^{\frac{\alpha+1}{2}}f(t)| \leq pe^{qt}$ , where  $p$  and  $q$  are constants. Then  $I(k) = (K_h f)(t) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(c+bn+1)}{k^{c+bn+1}}, k \rightarrow \infty$ . The proof is as follows:

We break the integral  $I = \int_0^{A-t} \tau^{\frac{\alpha+1}{2}} e^{-k\tau} f(\tau) d\tau$  into two parts, i.e.,  $I(k) = I_1(k) + I_2(k)$ , where

$$I_1 = \int_0^R \tau^{\frac{\alpha+1}{2}} e^{-k\tau} f(\tau) d\tau$$

and

$$I_2 = \int_R^{A-t} \tau^{\frac{\alpha+1}{2}} e^{-k\tau} f(\tau) d\tau$$

where  $R$  is a positive constant and  $R < A-t$ . The integral  $I_2$  is exponentially small as  $k \rightarrow \infty$ . If  $A < \infty$ , because  $\tau^{\frac{\alpha+1}{2}}f(\tau)$  is bounded for  $\tau > 0$ , there exists a positive constant  $B$  such that  $|\tau^{\frac{\alpha+1}{2}}f(\tau)| \leq B$  for  $\tau \geq R$ . Thus  $|I_2(k)| \leq B \int_R^{A-t} e^{-k\tau} d\tau = \frac{B}{k} [e^{-kR} - e^{-k(A-t)}]$ . Thus  $I_2(k) = O(\frac{e^{-kR}}{k})$  as  $k \rightarrow \infty$ .

Equation(2) implies that for each positive integer  $N$ ,

$$I_1(k) = \int_0^R [\sum_{n=0}^N a_n \tau^{c+bn} + O(\tau^{c+b(N+1)})] e^{-k\tau} d\tau, k \rightarrow \infty.$$

However,  $\int_0^R \tau^{c+bn} e^{-k\tau} d\tau = \int_0^{\infty} \tau^{c+bn} e^{-k\tau} d\tau - \int_R^{\infty} \tau^{c+bn} e^{-k\tau} d\tau = \frac{\Gamma(c+bn+1)}{k^{c+bn+1}} + O(\frac{e^{-kR}}{k}), k \rightarrow \infty$ , where we have used the definition of the gamma function on the first integral and integration by parts to establish the second. Moreover,  $\int_0^R O(\tau^{c+b(N+1)}) e^{-k\tau} d\tau \leq A_N \int_0^R \tau^{c+b(N+1)} e^{-k\tau} d\tau \leq A_N \frac{\Gamma(c+b(N+1)+1)}{k^{c+b(N+1)+1}}$ . Thus  $I(k) = \sum_{n=0}^N a_n \frac{\Gamma(c+bn+1)}{k^{c+bn+1}} + O(\frac{1}{k^{c+b(N+1)+1}}), k \rightarrow \infty$ .

We note that the assumptions  $c > -1, b > 0$  are necessary for convergence at  $t = 0$ . Also, if  $A = \infty$ , then it is only necessary that  $|\tau^{\frac{\alpha+1}{2}}f(\tau)| \leq pe^{q\tau}$  for some real constants  $p$  and  $q$ , in order to have convergence at  $\tau \rightarrow +\infty$ . In this case, the estimate of  $I_2$  gives  $I_2(k) \leq \frac{pe^{-(k-q)R}}{k-q} = O(\frac{e^{-kR}}{k})$  as  $k \rightarrow \infty$ .

#### 4. Asymptotic Results on the Singular Values

The singular values of Hankel operators play a crucial role in rational approximation [5],[6] and we give here some new results which apply to the Bergman space. We consider the Hankel integral operator  $K_h$  defined by

$$(K_h u)(t) = \int_0^{\infty} \frac{t^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}} h(t+\tau) u(\tau) d\tau,$$

where  $\alpha > -1, \beta > -1$ . We shall discuss the asymptotic behavior of its singular values  $\sigma_i(K_h)$ . It is convenient to begin with functions  $h$  which are in  $L^1$  and of compact support  $[0, A]$ , say.

**Proposition 4.1** The singular values of  $K_h$  satisfy  $\alpha_i = \sigma_i(K_h) \leq A^{\frac{\alpha+\beta+2}{2}} \sigma_i(\Theta)$  where  $\Theta$  is the Hankel integral operator defined by

$$(\Theta u)(t) = \int_0^\infty \frac{h(t+\tau)}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}} u(\tau) d\tau$$

and  $h(t) = 0$  for  $t > A$ .

**Proof** Let  $M$  be the operator on  $L^2$  which consists of multiplication by the function  $t^{\frac{\alpha+1}{2}} \chi_{[0,A]}(t)$ . Thus  $\|M\| = A^{\frac{\alpha+1}{2}}$ . Let  $N$  be the operator on  $L^2$  which consists of multiplication by the function  $t^{\frac{\beta+1}{2}} \chi_{[0,A]}(t)$ . Thus  $\|N\| = A^{\frac{\beta+1}{2}}$ . Then

$$\begin{aligned} (N\Theta M f)(t) &= N\Theta(t^{\frac{\alpha+1}{2}} \chi_{[0,A]}(t) f(t)) \\ &= N\left(\int_0^A \frac{h(t+\tau)}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}} \tau^{\frac{\alpha+1}{2}} f(\tau) d\tau\right) \\ &= t^{\frac{\beta+1}{2}} \chi_{[0,A]}(t) \int_0^A \frac{h(t+\tau)}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}} \tau^{\frac{\alpha+1}{2}} f(\tau) d\tau \\ &= \int_0^A \frac{t^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}} h(t+\tau) f(\tau) d\tau \\ &= \int_0^\infty \frac{t^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}} h(t+\tau) f(\tau) d\tau = (K_h f)(t). \end{aligned}$$

So  $K_h = N\Theta M$  and therefore  $\sigma_i(K_h) \leq \|N\| \sigma_i(\Theta) \|M\| = A^{\frac{\alpha+\beta+2}{2}} \sigma_i(\Theta)$ .  $\square$

**Lemma 4.2** Let  $h(t) = t^{\alpha+1} \chi_{[0,T]}(t)$  and  $K_h$  be the Hankel integral operator on  $L^2(0, \infty)$  given by

$$(K_h u)(t) = \int_0^\infty \frac{t^{\frac{\alpha+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\alpha+1}} h(t+\tau) u(\tau) d\tau.$$

The singular values of the Hankel integral operator  $K_h$  satisfy  $i\sigma_i(K_h) \rightarrow \frac{T^{\alpha+2}}{2^{\alpha+3}}$  as  $i \rightarrow \infty$ .

**Proof** Let  $h(t) = t^{\alpha+1} \chi_{[0,T]}(t)$  and  $K_h$  be the Hankel integral operator on  $L^2(0, \infty)$  defined by  $(K_h u)(t) = \int_0^\infty \frac{t^{\frac{\alpha+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\alpha+1}} h(t+\tau) u(\tau) d\tau$ . Since the integral kernel is real and symmetric, it follows that  $K_h$  is Hermitian (and compact) and it is necessary to determine its eigenvalues. Suppose that

$$\lambda u(t) = \int_0^{T-t} t^{\frac{\alpha+1}{2}} \tau^{\frac{\alpha+1}{2}} u(\tau) d\tau.$$

Writing  $u(t) = t^{\frac{\alpha+1}{2}} v(t)$ , we obtain  $\lambda v(t) = \int_0^{T-t} \tau^{\alpha+1} v(\tau) d\tau$ . This gives

$$\lambda \dot{v}(t) = -(T-t)^{\alpha+1} v(T-t)$$

so that

$$\begin{aligned} -\lambda \ddot{v}(t) &= \frac{d}{dt}[(T-t)^{\alpha+1} v(T-t)] \\ &= (\alpha+1)(T-t)^\alpha (-1) v(T-t) + \dot{v}(T-t) (-1)(T-t)^{\alpha+1} \\ &= -(\alpha+1)(T-t)^\alpha v(T-t) - \dot{v}(T-t)(T-t)^{\alpha+1} \\ &= -(\alpha+1)(T-t)^\alpha \left( \frac{\lambda \dot{v}(t)}{(T-t)^{\alpha+1}} - \left( \frac{-t^{\alpha+1}}{\lambda} v(t) \right) \right) (T-t)^{\alpha+1} \\ &= (\alpha+1) \lambda \frac{\dot{v}(t)}{T-t} + \frac{t^{\alpha+1}}{\lambda} v(t) (T-t)^{\alpha+1}, \end{aligned}$$

since  $v(T-t) = -\frac{\lambda \dot{v}(t)}{(T-t)^{\alpha+1}}$  and  $\dot{v}(T-t) = -\frac{t^{\alpha+1}}{\lambda}v(t)$ . Thus

$$-\lambda \ddot{v}(t) = (\alpha+1)\lambda \frac{\dot{v}(t)}{T-t} + \frac{t^{\alpha+1}}{\lambda}v(t)(T-t)^{\alpha+1}.$$

Therefore

$$\ddot{v}(t) = -(\alpha+1)\frac{\dot{v}(t)}{T-t} - \frac{1}{\lambda^2}t^{\alpha+1}v(t)(T-t)^{\alpha+1}.$$

That is,

$$\ddot{v}(t) + (\alpha+1)\frac{\dot{v}(t)}{T-t} + \frac{(T-t)^{\alpha+1}}{\lambda^2}t^{\alpha+1}v(t) = 0.$$

Hence

$$\frac{d}{dt} \left( \frac{\dot{v}(t)}{(T-t)^{\alpha+1}} \right) + \frac{t^{\alpha+1}v(t)}{\lambda^2} = 0$$

and  $v(T) = 0 = \dot{v}(0)$ .

Such Sturm-Liouville type equations may be analyzed by established techniques (see, e.g., [10]). Briefly, the substitutions  $z = \left(\frac{t}{T-t}\right)^{\frac{\alpha+1}{4}}v$ ,  $x = \int_0^T t^{\frac{\alpha+1}{2}}(T-t)^{\frac{\alpha+1}{2}}dt$  reduces the equation to the form  $z'' - f(x)z + \frac{z}{\lambda^2} = 0$  over the interval  $0 \leq x \leq X = \int_0^T t^{\frac{\alpha+1}{2}}(T-t)^{\frac{\alpha+1}{2}}dt \sim \frac{T^{\alpha+2}}{2^{\alpha+3}}\pi$ . Its eigenvalues are asymptotic to those of the equation  $z'' + \frac{z}{\lambda^2} = 0$  on the same interval, the boundary conditions become  $z(0) = z(X) = 0$  and hence  $\frac{1}{\lambda_k^2} \sim \frac{2^{2(\alpha+3)}k^2\pi^2}{T^{2(\alpha+2)}\pi^2} = \frac{2^{2(\alpha+3)}k^2}{T^{2(\alpha+2)}}$  or  $k\lambda_k \rightarrow \frac{T^{\alpha+2}}{2^{\alpha+3}}$  as required. The singular values of the Hankel integral operator induced by the function  $h(t)$  satisfy  $k\sigma_k(K_h) \rightarrow \frac{T^{\alpha+2}}{2^{\alpha+3}}$ .  $\square$

Let  $J$  denote the integral operator  $(Jf)(t) = \int_t^A f(\tau)d\tau$ , and let  $M, N$  be the operators consisting of multiplication by the function  $t^{\frac{\alpha+1}{2}}$  and  $t^{\frac{\beta+1}{2}}$  respectively where  $\alpha \geq \beta > -1$ . Although  $N^{-1}$  is unbounded, the operator  $Z = MJN^{-1}$  is bounded. The following result makes this more explicit.

**Lemma 4.3** The operator  $Z = MJN^{-1}$  defined above is Hilbert-Schmidt (hence compact) and its Hilbert-Schmidt norm is  $\frac{A^{\frac{\alpha-\beta}{2}+1}}{\sqrt{\alpha+2}\sqrt{\alpha-\beta+2}}$ . Its singular values satisfy  $n\sigma_n \rightarrow \frac{A}{(\alpha+1)\pi}$ .

**Proof** The operator may be written as

$$(Zf)(t) = \int_0^A k(t, \tau)f(\tau)d\tau$$

with

$$k(t, \tau) = \begin{cases} \tau^{-(\frac{\beta+1}{2})}t^{(\frac{\alpha+1}{2})} & \text{if } t < \tau; \\ 0 & \text{if } t > \tau. \end{cases}$$

Standard results (cf. [7]) and an elementary calculation give that

$$\begin{aligned}
\|Z\|_{HS} &= \left( \int_0^A \int_0^A |k(t, \tau)|^2 dt d\tau \right)^{\frac{1}{2}} \\
&= \left( \int_0^A \tau^{-(\beta+1)} \int_0^\tau t^{(\alpha+1)} dt d\tau \right)^{\frac{1}{2}} \\
&= \frac{A^{\frac{\alpha-\beta}{2}+1}}{\sqrt{\alpha+2}\sqrt{\alpha-\beta+2}}.
\end{aligned}$$

To obtain an asymptotic form for the singular values of  $Z$  requires us to consider the equation  $N^{-1}J^*M^2JN^{-1}g = \sigma^2g$ , or

$$\int_{\tau=0}^t t^{-(\frac{\beta+1}{2})} \tau^{\alpha+1} \int_{r=\tau}^A g(r) r^{-(\frac{\beta+1}{2})} dr d\tau = \sigma^2 g(t). \quad (3)$$

Let  $h(t) = g(t)t^{\frac{\beta+1}{2}}$ . The above equation reduces to

$$\sigma^2 h(t) t^{-(\frac{\beta+1}{2})} = \int_{\tau=0}^t t^{-(\frac{\beta+1}{2})} \tau^{\alpha+1} \int_{r=\tau}^A h(r) r^{-(\frac{\beta+1}{2})} r^{-(\frac{\beta+1}{2})} dr d\tau.$$

Hence  $\sigma^2 h(t) = \int_{\tau=0}^t \tau^{\alpha+1} \int_{r=\tau}^A h(r) r^{-(\beta+1)} dr d\tau$ . Thus

$$\sigma^2 \dot{h}(t) = t^{\alpha+1} \int_{r=t}^A h(r) r^{-(\beta+1)} dr. \quad (4)$$

Therefore,  $\sigma^2 \frac{\dot{h}(t)}{t^{\alpha+1}} = \int_{r=t}^A h(r) r^{-(\beta+1)} dr$ . Hence

$$\frac{d}{dt} \left[ \sigma^2 \frac{\dot{h}(t)}{t^{\alpha+1}} \right] = \frac{d}{dt} \left[ \int_{r=t}^A h(r) r^{-(\beta+1)} dr \right].$$

This implies  $\sigma^2 \frac{d}{dt} \left[ \frac{\dot{h}(t)}{t^{\alpha+1}} \right] = -h(t) t^{-(\beta+1)}$ . Thus

$$\sigma^2 \left[ \frac{\ddot{h}(t) t^{\alpha+1} - (\alpha+1) t^\alpha \dot{h}(t)}{t^{2(\alpha+1)}} \right] = -h(t) t^{-(\beta+1)}.$$

Hence

$$\sigma^2 \left[ \frac{\ddot{h}(t)}{t^{\alpha+1}} - (\alpha+1) \frac{\dot{h}(t)}{t^{\alpha+2}} \right] + \frac{h(t)}{t^{\beta+1}} = 0.$$

Therefore

$$\sigma^2 \left[ t \ddot{h}(t) - (\alpha+1) \dot{h}(t) \right] + t^{\alpha+1-\beta} h(t) = 0$$

and finally we obtain

$$t \ddot{h}(t) - (\alpha+1) \dot{h}(t) + \frac{h(t)}{t^{\beta-\alpha-1}} \frac{1}{\sigma^2} = 0.$$

This implies

$$\frac{d^2}{dt^2} \left( \frac{h(t)}{t^{\alpha+1}} \right) + \frac{\alpha+1}{t} \frac{d}{dt} \left( \frac{h(t)}{t^{\alpha+1}} \right) + \frac{h(t)}{t^{\alpha+1}} \left[ \frac{1}{\sigma^2} \frac{1}{t^{\beta-\alpha}} - \frac{\alpha+1}{t^2} \right] = 0$$

with boundary conditions  $h(0) = \dot{h}(A) = 0$  (follows from (3) and (4)). Let  $w = \frac{h(t)}{t^{\alpha+1}}$ ,  $\lambda^2 = \frac{1}{\sigma^2}$ , with boundary conditions  $h(0) = \dot{h}(A) = 0$ . The above equation reduces to

$$w'' + \frac{\alpha+1}{t}w' + w \left[ \lambda^2 \frac{1}{t^{\beta-\alpha}} - \frac{\alpha+1}{t^2} \right] = 0.$$

Hence

$$w'' + \frac{\alpha+1}{t}w' + w \left[ \frac{\lambda^2 t^2 - (\alpha+1)t^{\beta-\alpha}}{t^{\beta-\alpha+2}} \right] = 0.$$

Therefore

$$t^{\beta-\alpha+2}w'' + (\alpha+1)t^{\beta-\alpha+1}w' + [\lambda^2 t^2 - (\alpha+1)t^{\beta-\alpha}]w = 0.$$

Thus

$$t^{\beta-\alpha} \left[ t^2 w'' + (\alpha+1)tw' + \left[ \frac{\lambda^2 t^2}{t^{\beta-\alpha}} - (\alpha+1) \right] w \right] = 0.$$

Hence

$$t^2 w'' + (\alpha+1)tw' + \left[ \frac{\lambda^2}{t^{\beta-\alpha-2}} - (\alpha+1) \right] w = 0.$$

That is,

$$t^2 w'' + (\alpha+1)tw' + [\lambda^2 t^{2-\beta+\alpha} - (\alpha+1)]w = 0.$$

Hence

$$\left( \frac{t}{\alpha+1} \right)^2 w'' + \left( \frac{t}{\alpha+1} \right) w' + \left( \lambda^2 (\alpha+1)^{\alpha-\beta} \left( \frac{t}{\alpha+1} \right)^{2-\beta+\alpha} - \frac{1}{\alpha+1} \right) w = 0.$$

Let  $s = \frac{t}{\alpha+1}$ . Then we obtain  $s^2 w'' + s w' + [\lambda^2 (\alpha+1)^{\alpha-\beta} s^{2-\beta+\alpha} - \frac{1}{\alpha+1}]w = 0$ .

From [1], this has the solution  $\frac{h(t)}{t^{\alpha+1}} = L J_1\left(\frac{s}{\sigma}\right) = L J_1\left(\frac{t}{(\alpha+1)\sigma}\right)$ , where  $L$  is a constant,  $J$  is the Bessel function and  $\sigma$  is determined by the boundary conditions. Since  $L t^\alpha \left[ (\alpha+1) J_1\left(\frac{t}{(\alpha+1)\sigma}\right) + t \frac{d}{dt} \left( J_1\left(\frac{t}{(\alpha+1)\sigma}\right) \right) \right] = 0$  at  $t = A$ , hence

$$(\alpha+1) J_1\left(\frac{A}{(\alpha+1)\sigma}\right) + A \frac{d}{dt} \left( J_1\left(\frac{A}{(\alpha+1)\sigma}\right) \right) = 0.$$

This implies

$$\frac{J_1\left(\frac{A}{(\alpha+1)\sigma}\right)}{\frac{A}{\alpha+1}} + \frac{d}{dt} \left( J_1\left(\frac{A}{(\alpha+1)\sigma}\right) \right) = 0.$$

The  $s^{\text{th}}$  positive zero of  $J'_\nu(z)$  is denoted by  $J'_{\nu,s}$ . The zeroes of  $J'_\nu(z)$  are all real when  $\nu \geq 0$ . If  $\nu$  is fixed and positive,  $\beta = (s + \frac{1}{2}\nu - \frac{3}{4})\pi$ , then

$$J'_{\nu,s} = \beta - \frac{4\nu^2 + \beta}{8\beta} - \frac{112\nu^4 + 328\nu^2 - 9}{384\beta^3} + O\left(\frac{1}{s^5}\right)$$

(for more details see [11]). It follows easily that  $\frac{A}{(\alpha+1)\sigma_n} = (n - \frac{1}{4})\pi + O(\frac{1}{n})$ . Therefore,  $\sigma_n \approx \frac{A}{(\alpha+1)n\pi}$ . Thus  $n\sigma_n \rightarrow \frac{A}{(\alpha+1)\pi}$ .  $\square$

From lemma 4.3, taking  $\alpha = \beta$  it follows that the operator  $NJN^{-1}$  is Hilbert-Schmidt and its Hilbert-Schmidt norm is  $\frac{A}{\sqrt{2(\beta+2)}}$  and its singular values satisfy

$$n\sigma_n \rightarrow \frac{A}{(\beta+1)\pi}.$$

**Theorem 4.4** Suppose  $h(t) = t^{\frac{\alpha+\beta+2}{2}}g(t)$ ,  $k(t) = t^{\frac{\alpha+\beta+2}{2}}\dot{g}(t)$  where  $h(t) = 0, t \geq A$  and  $\dot{g} \in L^2$ . If  $i, m$  are nonnegative integers,  $i \geq m$  then  $\sigma_i(K_h) \leq \frac{A}{(\alpha+1)\pi m}(1 + o(1))\sigma_{i-m}(K_k)$ .

**Proof**

$$\begin{aligned}
(K_h u)(t) &= \int_0^\infty \frac{t^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(t+\tau)^{\frac{\alpha+\beta+2}{2}}} h(t+\tau) u(\tau) d\tau \\
&= - \int_{\tau=0}^\infty \int_{s=t}^\infty \dot{g}(s+\tau) t^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}} u(\tau) ds d\tau \\
&= - \int_{s=t}^\infty \left( \int_{\tau=0}^\infty \frac{s^{\frac{\beta+1}{2}} \tau^{\frac{\alpha+1}{2}}}{(s+\tau)^{\frac{\alpha+\beta+2}{2}}} (s+\tau)^{\frac{\alpha+\beta+2}{2}} \dot{g}(s+\tau) u(\tau) d\tau \right) \left( \frac{t^{\frac{\beta+1}{2}}}{s^{\frac{\beta+1}{2}}} \right) ds \\
&= - \int_{s=t}^\infty (K_{t^{\frac{\alpha+\beta+2}{2}} \dot{g}(t)} u)(s) \left( \frac{t^{\frac{\beta+1}{2}}}{s^{\frac{\beta+1}{2}}} \right) ds \\
&= -t^{\frac{\beta+1}{2}} \int_{s=t}^\infty (K_k u)(s) \frac{ds}{s^{\frac{\beta+1}{2}}} \\
&= -N J N^{-1} K_k u(t)
\end{aligned}$$

where  $Nf(t) = t^{\frac{\beta+1}{2}}f(t)$  for by Fubini's theorem, we may interchange the order of integration when  $u$  varies over a dense subset of  $L^2(0, \infty)$  and deduce that  $K_h = -N J N^{-1} K_k$  everywhere since  $N J N^{-1}, K_h, K_k$  are bounded. Since for any operators  $P$  and  $Q$  it is true that the singular values satisfy  $\sigma_{m+n+1}(PQ) \leq \sigma_{m+1}(P)\sigma_{n+1}(Q)$  (cf. [12]) for all  $m, n \geq 0$ , the result follows from lemma 4.3 on writing  $i = m + n + 1, P = N J N^{-1}$  and  $Q = K_k$ .  $\square$

**Corollary 4.5** Suppose  $h(t) = t^{\frac{\alpha+\beta+2}{2}}g(t)$ ,  $k(t) = t^{\frac{\alpha+\beta+2}{2}}\dot{g}(t)$  where  $h(t) = 0, t \geq A$ ,  $t^{-\frac{1}{2}}h(t) \in L^2$  and  $\dot{g} \in L^2$ , then  $\sigma_{2m}(K_h) \leq \frac{A}{\pi(\alpha+1)}cm^{-\frac{3}{2}}\|t^{\frac{1}{2}}k(t)\|_2 + O(m^{-\frac{3}{2}})$  as  $m \rightarrow \infty$  and hence  $\sigma_i(K_h) = O(i^{-\frac{3}{2}})$  as  $i \rightarrow \infty$ .

**Proof** Since  $t^{-\frac{1}{2}}h(t) = t^{\frac{\alpha+\beta+1}{2}}h(t) \in L^2$ , by theorem 2.5, it follows that the Hilbert-Schmidt norm of  $K_k$  is equal to  $c\|t^{\frac{1}{2}}k(t)\|_2$  where  $c = \sqrt{\frac{\Gamma(\beta+2)\Gamma(\alpha+2)}{\Gamma(\alpha+\beta+4)}}$  and  $\|K_k\|_{HS}$  is greater than or equal to

$$(\sigma_1^2(K_k) + \cdots + \sigma_m^2(K_k))^{\frac{1}{2}} \geq m^{\frac{1}{2}}\sigma_m(K_k).$$

Hence  $\sigma_{2m}(K_h) \leq (\frac{A}{(\alpha+1)\pi m} + o(m^{-1}))\sigma_m(K_k) \leq \frac{A}{(\alpha+1)\pi}cm^{-\frac{3}{2}}\|t^{\frac{1}{2}}k(t)\|_2 + O(m^{-\frac{3}{2}})$  as required.  $\square$

Having obtained an estimate for the singular values  $(\sigma_i)$  of the Hankel integral operator  $K_h$  in the case that  $h$  has compact support, we are now in a position to extend these results to general  $h \in L^2(0, \infty)$  with sufficiently fast decrease to zero at infinity.

**Theorem 4.6** Suppose  $h \in L^2$  be such that  $\dot{h}$  exists and  $\dot{h}, t^{-\frac{1}{2}}h(t)$  belong to  $L^2$ . Suppose further that there exist constants  $B > 0$  and  $\lambda > 1$  such that if  $t$  is sufficiently large (say  $t \geq t_0$ ), then  $\max(|h(t)|) \leq Bt^{-\lambda}$ . Then  $\sigma_n(K_h) = O(n^{\frac{3(1-\lambda)}{2\lambda}})$  as  $n \rightarrow \infty$ . Hence if  $\lambda > 3$ , the operator  $K_h$  is nuclear.

**Proof** Suppose  $\nu = \frac{3}{2\lambda}$ , and let  $n$  be sufficiently large that  $A = n^\nu \geq t_0$  and  $\delta = BA^{-\lambda} < 1$ . We may find a function  $j(t)$  such that  $j(t) = h(t)$  if  $0 \leq t \leq A$ ,  $j(t) = 0$  if  $t \geq A + 1$ , and  $j(t)$  has a square integrable derivative in  $[A, A + 1]$ : for example,  $j(t) = c(t - A - 1)^2$ ,  $A \leq t \leq A + 1$ , for some constant  $c$  depending linearly on  $h(A)$ . Let  $L$  be the constant independent of  $n$  such that  $\sup(|j(t)|) \leq L\delta$  for  $A \leq t \leq A + 1$ . Now  $(\int_A^\infty |h(t)|dt) \leq \frac{BA^{1-\lambda}}{\lambda-1}$ , so that  $\|K_h - K_j\| \leq \|h - j\|_{L^1} = O(A\delta)$ . We have that  $\sigma_n(K_j) = O(An^{-\frac{3}{2}})$ , by corollary 4.5, since  $\|t^{\frac{\alpha+\beta+3}{2}} \frac{d}{dt} \left( \frac{j(t)}{t^{\frac{\alpha+\beta+2}{2}}} \right)\|$  is bounded above by some absolute constant independent of  $A$  and  $n$  as  $t^{-\frac{1}{2}}h(t) \in L^2$ . Hence  $\sigma_n(K_h) = O(An^{-\frac{3}{2}}) + O(A^{1-\lambda}) = O(n^{\frac{3(1-\lambda)}{2\lambda}})$  since  $\nu - \frac{3}{2} = \nu(1 - \lambda)$ .  $\square$

From theorem 4.6 it follows that if  $p > \frac{2}{3}$  and  $\lambda > \frac{3p}{3p-2}$  then since  $\frac{3(1-\lambda)}{2\lambda} < -\frac{1}{p}$  the integral operator  $K_h$  will lie in the Schatten class  $S_p$ .

The rich involvement of Hankel operators with best approximation is well known. In fact the singular values of Hankel operators play an important role in approximation problems. The most important result in this connection is the Adamjan, Arov and Krein theorem which characterizes the singular values of a Hankel operator on the Hardy space in terms of rational approximation of the symbol function. The asymptotic results obtained in this section on the singular values indicate the growth of the singular values  $\sigma_n(K_h)$  of the Hankel integral operator when  $n \rightarrow \infty$ .

One can obtain further estimates for the singular values for smooth kernels with rapid decay at infinity using techniques similar to those of [5] and [6]. Although this is beyond the scope of the present work, one can relate these to achievable errors in rational approximation. Consider the case  $\alpha = \beta = 0$ . It should also be noted that the  $L^2((0, \infty), \frac{dt}{t})$ -norm of the kernel  $h$  of the integral operator  $K_h$  does not correspond to a unitarily equivalent norm of the little Hankel operator: however a scaling of the Hankel integral operator that does not change its rank can be used to make the Hilbert-Schmidt norm of the scaled Hankel operator equal to the  $L^2((0, \infty), \frac{dt}{t})$ -norm of the kernel  $h$  of the integral operator  $K_h$ . For example one can consider the Hankel integral operator  $(\Theta u)(t) = \int_0^\infty \frac{h(t+\tau)}{t+\tau} u(\tau) d\tau$  whose Hilbert-Schmidt norm is equal to  $\|h(t)\|_{L^2((0, \infty), \frac{dt}{t})} = \frac{1}{\sqrt{\pi}} \|\mathcal{L}h\|_{L_a^2(\mathbb{C}_+)}$ . Such considerations play an important role in rational approximation in the Hardy space case [6] where lower bounds on the achievable error in rational approximation are derived from the singular values of a scaled Hankel operator.

**Remark 4.7** Thus we see that if the reproducing kernel of a weighted Bergman space  $L_a^2(\mathbb{D}, \asymp(F)\mathbb{A}(F))$  is some power of  $K(z, w)$ , the reproducing kernel for the Bergman space  $L_a^2(\mathbb{D})$  then we can relate this particular type of integral operator to little Hankel operators on the weighted Bergman space and we can explicitly calculate the symbol correspondence. So the natural question that arises at this point is for which positive weight function  $v(z)$  does the reproducing kernel have the following form  $K_v(z, w) = [K(z, w)]^\beta$  where  $K(z, w) = \frac{1}{(1-z\bar{w})^2}$  and  $K_v(z, w)$  is the reproducing kernel for the space  $L_a^2(\mathbb{D}, \asymp(F)\mathbb{A}(F))$ .

The answer is only for  $v(z) = (1 - |z|^2)^\alpha$ ,  $\alpha > -1$ ,  $\alpha = 2(\beta - 1)$ . The argument is as follows:

It is well known that the reproducing kernel determines the function space in a standard fashion: Let  $K_0(z, w)$  be a reproducing kernel. Let

$$A = \text{linear span of } \{K_0(., w) : w \in \mathbb{D}\}$$

with norm

$$\left\| \sum c_j K_0(., w_j) \right\|^2 = \sum c_i \bar{c}_j K_0(w_j, w_i).$$

Then the completion of  $A$  is the Hilbert space with reproducing kernel  $K_0$ . If  $K_v = K^\beta$  then (up to a constant)

$$\int |f|^2 v dA = \int |f|^2 (1 - |z|^2)^\alpha dA$$

for all polynomials  $f$  and thus

$$\int g v dA = \int g (1 - |z|^2)^\alpha dA$$

for all  $g$  of the form  $\sum a_j |f_j|^2$ . By the Stone-Weierstrass theorem, such  $g$  are dense in  $C_{\mathbb{R}}(\overline{\mathbb{D}})$ . So  $v(z) = (1 - |z|^2)^\alpha$ .

**Remark 4.8** Since  $L^2(\mathbb{D}, \lesssim(F)\mathbb{A}(F))$  and  $L^2(\mathbb{C}_+, \lesssim(\sim)\tilde{\mathbb{A}}(\sim))$  are Hilbert spaces, there is always a large number of unitary maps from one to the other. But to calculate the symbol correspondence we prefer a unitary map of the form

$$(Wg)(s) = h(s)g(Ms)$$

where  $h$  is holomorphic in  $\mathbb{C}_+$  and  $M$  is a conformal map from  $\mathbb{C}_+$  to  $\mathbb{D}$ . Substituting this into the integrals for

$$\langle f, g \rangle_{L^2(v)} = \langle Wf, Wg \rangle_{L^2(\tilde{v})}$$

we have the following (by changing variables):

$$\begin{aligned} \int f \bar{g} v(z) dA(z) &= \int (Wf) \overline{(Wg)} \tilde{v}(s) d\tilde{A}(s) \\ &= \int h(s) \overline{f(Ms)} \overline{h(s)} \overline{g(Ms)} \tilde{v}(s) d\tilde{A}(s) \\ &= \int |h(s)|^2 f(Ms) \overline{g(Ms)} \tilde{v}(s) d\tilde{A}(s) \\ &= \int |h(\Psi(z))|^2 f(z) \overline{g(z)} \tilde{v}(\Psi(z)) |\Psi'(z)|^2 dA(z) \end{aligned}$$

where  $\Psi$  is a conformal map of  $\mathbb{D}$  onto  $\mathbb{C}_+$ . Thus a necessary and sufficient condition for such a unitary map to exist is that there exists a conformal map of  $\mathbb{D}$  onto  $\mathbb{C}_+$  such that  $\frac{v(z)}{\tilde{v}(\Psi(z))}$  is the modulus squared of an analytic function on  $\mathbb{D}$  (i.e.,  $\frac{v(z)}{\tilde{v}(\Psi(z))} = |p(s)|^2$ ).

If  $v$  and  $\tilde{v}$  correspond in the sense above and  $v(z) = (1 - |z|^2)^\alpha$ ,  $z \in \mathbb{D}$  then

$$(\operatorname{Re} s)^\alpha = \frac{|s+1|^{2\alpha}}{4^\alpha} |p(s)|^2 \tilde{v}(s)$$

where  $p$  is analytic in  $\mathbb{C}_+$  and  $s = \Psi(z)$ . Thus  $\tilde{v}(s) = x^\alpha$  is only the simplest case (occurring when  $p(s) = \frac{2^\alpha}{(1+s)^\alpha}$ ). Nevertheless, all other cases are easily connected to the simplest, via a multiplication operator.

The Möbius invariance of  $(1 - |z|^2)^\alpha$  (i.e.,  $(1 - |Mz|^2)^\alpha = |M'(z)|^\alpha (1 - |z|^2)^\alpha$  for all Möbius map  $M : \mathbb{D} \rightarrow \mathbb{D}$ ) is the important property of the measure  $(1 - |z|^2)^\alpha dA(z)$ .

This property is studied on symmetric spaces; if  $S$  is a symmetric space and  $dV$  is an invariant measure then the weights  $w$  with the property that

$$w \circ M = (\det M')^\alpha w$$

are important tools in the geometry, topology and analysis on  $S$  [9].

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Namita Das  
P. G. Dept. of Mathematics  
Utkal University  
Vanivihar, Bhubaneswar  
751004, Orissa  
INDIA  
namitadas440@yahoo.co.in