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PROJECTIVE VARIETIES WITH CONES AS TANGENTIAL SECTIONS

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Abstract. Let $X \subset \mathbf{P}^n$ be an integral non-degenerate m-dimensional variety defined over an algebraically closed field $\mathbb{K}$. Assume the existence of a non-empty open subset U of X_{reg} such that $T_P X \cap X$ is an $(m-1)$-dimensional cone with vertex containing P. Here we prove that either X is a quadric hypersurface or $\mathrm{char}(\mathbb{K}) = p > 0$, $n = m+1$, $\deg(X) = p^e$ for some $e \geq 1$ and there is a codimension two linear subspace $W \subset \mathbf{P}^n$ such that $W \subset T_P X$ for every $P \in X_{reg}$. We also give an " explicit " description (in terms of polynomial equations) of all examples arising in the latter case; $\dim(\mathrm{Sing}(X)) = m-1$ for every such X.

1. Varieties With Cones As Tangential Sections

Let $X \subset \mathbf{P}^n$ be an integral non-degenerate m-dimensional variety defined over an algebraically closed field $\mathbb{K}$. For any $P \in X_{reg}$ let $T_P X \subset \mathbf{P}^n$ denote the Zariski tangent space of X at P. Here we prove the following result.

Theorem 1.1. *Let $X \subset \mathbf{P}^n$, $n \geq 3$, be an integral non-degenerate m-dimensional variety such that there is a non-empty open subset U of X_{reg} such that $T_P X \cap X$ is an $(m-1)$-dimensional cone with vertex containing P. Then either X is a quadric hypersurface or $char(\mathbb{K}) = p > 0$, $n = m+1$ and there is an integer $e \geq 1$ such that $deg(X) = p^e$. In the latter case all examples are described in Example 1.2 and any such example (except if $p = 2$, $e = 1$, i.e. in the hyperquadric case) has $(m-1)$-dimensional non-empty singular locus.*

Example 1.2. Fix integers $n \geq 2$, a prime integer p and an integer $e \geq 1$. Set $q := p^e$. Assume $\mathrm{char}(\mathbb{K}) = p > 0$. Let $W \subset \mathbf{P}^n$ be a codimension two linear subspace. Notice that all two such linear subspaces are projectively equivalent and hence (up to a projective transformation) the family of examples we will give will not depent from the choice of W. Here we describe all degree p^e hypersurfaces $Y \subset \mathbf{P}^n$ such that $W \subset T_P Y$, $(T_P Y \cap Y)_{red} = \langle W \cup \{P\} \rangle$ for every $P \in Y_{reg} \backslash W$ and p^e is the inseparable degree of the rational map on X induced by the linear projection from W. This is the only case concerning Theorem 1.1; for a more general set-up in which $\deg(Y) = sp^e$ for some $e \geq 1$, instead of $\deg(Y) = p^e$, see [**2**], §4, for the case $n = 2$ or use the set-up of [**1**]. Let $u : M \to \mathbf{P}^n$ be the blowing - up of W. The rational map $\mathbf{P}^n \backslash W \to \mathbf{P}^1$ induced by the linear projection from W induces a morphism $v : M \to \mathbf{P}^1$. Furthermore, v is a $\mathbf{P}^{n-1}$-bundle and (with respect to the projection v) we have $M \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}^{n-1} \otimes \mathcal{O}_{\mathbf{P}^1}(1))$. Let $E := u^{-1}(W)$ denote the exceptional

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divisor of u. Hence $E \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^{n-2}} \otimes \mathcal{O}_{\mathbf{P}^{n-2}}(1))$ and $u_{|E}$ is the projection of this $\mathbf{P}^1$-bundle structure over W. We have $\mathrm{Pic}(M) \cong \mathbb{Z}^{\oplus 2}$ and we may take as a basis of $\mathrm{Pic}(M)$ a fiber F of the ruling v and the exceptional divisor E. The normal bundle of F in M is trivial and hence F^2 is the zero codimension two class in the Chow ring of M. The line bundle $\mathcal{O}_M(F|E)$ induces the degree one line bundle on every fiber of $v_{|E}$. We have $u^*(\mathcal{O}_{\mathbf{P}^n}(1)) \cong \mathcal{O}_M(E+F)$. Hence $\mathcal{O}_E(E+F) \cong u^*(\mathcal{O}_W(1))$ is a spanned line bundle whose global sections contract the ruling of E and then induce the identity map $W \to W$. Hence $h^0(M, \mathcal{O}_M(xE+xF)) = \binom{n+x}{n}$ for all integers $x \geq 0$. Fix integers $y \geq x \geq 0$. Inspired from [**2**] and [**1**] we use the following polynomial expression of the elements of $H^0(M, \mathcal{O}_M(xE+yF))$. Fix $n+1$ variables $w_1, \dots, w_{n-1}, x_0, x_1$. We see $w_1, \dots, w_{n-1}$ as affine coordinates on each fiber of the ruling u in which we take as hyperplane at infinity the intersection of that fiber with E. We see x_0, x_1 as homogeneous coordinate on the base of the ruling v. There is an isomorphism between $H^0(M, \mathcal{O}_M(xE+yF))$ and the vector space of all polynomials in the variable $w_1, \dots, w_{n-1}, x_0, x_1$ which are homogeneous of degree y, which are homogeneous in the variables x_0, x_1 and with order at most x in the variables $w_1, \dots, w_{n-1}$. Now we take $x = q$. Let $A(q,y)$ be the subset of $H^0(M, \mathcal{O}_M(xE+yF))$ corresponding to the polynomials $f(w_1, \dots, w_{n-1}, x_0, x_1)$ in which each w_i appears to the power q. Since q is a p-power, $A(q,y)$ is a $\mathbb{K}$-vector space. Take $f \in A(q,y) \backslash \{0\}$ and $P = (\bar{x_0}, \bar{x_1}) \in \mathbf{P}^1$. Set $F_P := u^{-1}(P)$. Notice that $f(w_1, \dots, w_{n-1}, \bar{x_0}, \bar{x_1})$ either vanishes identically on F_P or it vanishes exactly on an $(n-2)$-dimensional linear space and with multiplicity q. The former case cannot occur if $\{f = 0\}$ is irreducible. If $y = x \geq 3$, then we see that v maps the divisor $\{f = 0\}$ of M isomorphically onto a degree q hypersurface which, if integral, satisfies all the assumptions of Theorem 1.1. The divisor $\{f = 0\}$ is integral for a general f by a dimension count; indeed, it is sufficient to show that for general f the divisor $\{f = 0\}$ has neither E nor any fiber of the ruling v as a component; for the latter check use that the linear system $|\mathcal{O}_M(xE + (x-1)F)|$ has E as a base component.

Remark 1.3. Notice that all examples Y given by Example 1.2 and with $\deg(Y) \geq 3$ are singular in codimension one. This is an easy generalition of Luiss' theorem stating that all strange curves are singular, except the plane conic in characteristic two. The case used in Theorem 1.1 follows from Luiss' theorem taking a general codimension $m-1$ linear section.

We stress that Theorem 1.1 is very easy (and certainly well-known) in characteristic zero (see Remark 1.6).

Lemma 1.4. *Let $X \subset \mathbf{P}^n$, $n \geq 3$, be an integral degree $d \geq 3$ hypersurface such that for a general $P \in T_PX$ the set $(X \cap T_PX)_{red}$ is an $(n-2)$-dimensional linear space, i.e. the scheme-theoretic intersection $X \cap T_PX$ is a hyperplane of T_PX counted with multiplicity d. Then char$(\mathbb{K}) = p > 0$, $d = p^e$ for some integer $e \geq 1$ and there is a codimension two linear subspace $W \subset \mathbf{P}^n$ such that $W \subset T_QX$ for every $Q \in X_{reg}$.*

Proof. Let $M \subset \mathbf{P}^n$ be a general plane. By Bertini's theorem ([**3**], parts 3) and 4) of Th. 6.3) $C := M \cap X$ is an integral curve. By assumption for a general $P \in C_{reg}$ the tangent line T_PC to C at P intersects C only at P and hence it has contact order d with C at P. By [**4**], Th. 1, char$(\mathbb{K}) = p > 0$, $d = p^e$ for some integer $e \geq 1$ and there is $P_M \in M \backslash C$ such that all the tangent lines to C at its smooth points contain P_M. Since a general hyperplane tangent to X is tangent along an $(n-2)$-dimensional subscheme, the dual variety $X^* \subset \mathbf{P}^{n*}$ is a curve D. The definition of the Gauss mapping for hypersurfaces implies that the codimension two linear subspace W with the claimed properties exists if and only D is a line. To check $\deg(D) = 1$ it is sufficient to prove that $D \cap \Gamma$ is a unique point for a general hyperplane Γ of $\mathbf{P}^{n*}$, i.e. that for a general $A \in \mathbf{P}^n$ there is a unique hyperplane $H \subset \mathbf{P}^n$ tangent to X and with $A \in H$.

Claim: Fix a general line $R \subset \mathbf{P}^n$ and set $\{P_1, \ldots, P_d\} := X \cap R$. There is a codimension two linear space $W \subset \mathbf{P}^n$ such that $W \subset T_{P_i}X$ for all i.

Proof of the Claim: Set $W := T_{P_1}X \cap T_{P_2}X$. By the generality of R the pair (P_1, P_2) may be considered as a general element of $X \times X$ and hence $T_{P_1}X \neq T_{P_2}X$, i.e. W has codimension two. We want to check $W \subset T_{P_i}X$ for every $i \geq 3$. Fix the index i such that $3 \leq i \leq d$. It is sufficient to prove that for a general plane M containing R the line $T_{P_i}(X \cap M)$ meets the line $T_{P_1}(X \cap M)$ at the point $W \cap M$. By the generality of R the plane M may be considered as a general plane of $\mathbf{P}^n$ and hence the curve $C := X \cap M$ is integral. We proved that C is strange and hence $T_{P_i}C$ meets $T_{P_1}C$ at the strange point of C, i.e. at the point $T_{P_1}C \cap T_{P_2}C$, proving the Claim.

Assume that for a general $A \in \mathbf{P}^n$ there are at least two hyperplanes passing through A and tangent to X (say to P_1 and P_2 with $P_1 \neq P_2$). Fixing P_1 and move A inside $T_{P_1}X$ the point P_2 describes an open subset of X. Hence we may assume that the line $R := \langle\{P_1, P_2\}\rangle$ is a general secant line of X and hence $R \cap X$ consists of d distinct smooth points $P_1, \ldots, P_d$ of X. By a very trivial byproduct of the Claim we have $A \notin R$. and hence $N := \langle\{P_1, P_2, A\}\rangle$ is a plane. A priori N is not general, but at least $N \nsubseteq X$ and the degree d curve $X \cap N$ contains d smooth collinear points. Hence the effective divisor $X \cap N$ has no multiple component. Even the reduced curve $X \cap N$ must be strange. Since $A \in T_{P_1}X \cap T_{P_2}X \cap N$, A is the strange point of $X \cap N$. Hence infinitely many tangent tangent lines of $X \cap N$ contain A. Hence infinitely many tangent hyperplanes of X contain A, contradiction. $\square$

We singled out the Claim in the proof of Lemma 1.4 because it would be nice to have a classification of all hypersurfaces with that property.

Remark 1.5. Let $\{D_t\}_{t \in T}$ be a family of lines of $\mathbf{P}^a$, $a \geq 3$, parametrized by an integral quasi-projective variety T. Assume $D_t \cap D_s \neq \emptyset$ for all t, s. Take $t_1, t_2 \in T$ such that $D_{t_1} \neq D_{t_2}$. By assumption $D_{t_1} \cap D_{t_2}$ is a point and hence $\langle D_{t_1} \cup D_{t_2}\rangle$ is a plane. Since T is integral, it is easy to check that either $D_t \subset \langle D_{t_1} \cup D_{t_2}\rangle$ for every $t \in T$ or $(D_{t_1} \cap D_{t_2}) \in D_t$ for every $t \in T$.

Proof of Theorem 1.1. The “ if ” part in the hyperquadric case is obvious, while in case (ii) it is checked in Example 1.2. We divide the proof of the “ only if ” part into four parts.

(a) Let $\{D_t\}_{t \in T}$ be an algebraic family of lines of $\mathbf{P}^r$, $r \geq 3$, parametrized by an integral quasi-projective variety. Assume $D_t \cap D_s \neq \emptyset$ for a general $(t, s) \in T \times T$. By Remark 1.5 either there is $P \in \mathbf{P}^r$ such that $P \in D_t$ for every $t \in T$ or there is a plane $M \subset \mathbf{P}^r$ such that $D_t \subset M$ for every $t \in T$. Let $A \subset \mathbf{P}^r$, $r \geq 2$, be an integral closed two-dimensional variety containing infinitely many lines. Let $G(1,r)$ denote the Grassmannian of all lines of $\mathbf{P}^r$. Set $L(A) := \{D \in G(1,r) : D \subset A\}$. Since $G(1,r)$ is complete, $L(A)$ is the disjoint union of finitely many integral projective variety. By assumption at least one of these varieties has dimension at least one. It is easy to check that A is a plane if and only if it contains a two-dimensional family. A is a cone if and only if there is $Q \in A$ such that infinitely many elements of $L(A)$ contain Q.

From now on in part (a) we assume that A is not a cone. The first part of the proof implies the existence of a one-dimensional irreducible family $\{D_t\}_{t\in T}$ of $L(A)$ such that $D_t \cap D_s = \emptyset$ for a general $(t,s) \in T \times T$. Let $f : B \to A$ be the normalization map. The total transform of the lines $\{D_t\}_{t\in T}$ gives a family $\{R_t\}_{t\in T}$ of integral curves of B. Since $D_t \cap D_s = \emptyset$ for a general $(t,s) \in T \times T$ we obtain $R_t \cdot R_t = 0$ for any t (intersection product in the normal surface B). Since A is not a cone and B has only finitely many singular points, we also get $D_t \subset B_{reg}$ for a general $t \in T$. Now assume the existence of another one-dimensional irreducible family $\{L_s\}_{s\in S}$ of $L(A)$ and call $\{C_s\}_{s\in S}$ the corresponding family of total transforms in B. The rational number $R_t \cdot C_s$ does not depend from the choice of the pair $(t,s) \in T \times S$. Notice that if $R_t \cdot C_s > 0$, then $D_a \cap L_b \neq \emptyset$ for all $(a,b) \in T \times S$. Now assume $r = 3$ and $R_t \cdot C_s > 0$. Fix general $D_{t_1}, D_{t_2}, D_{t_3}$. There is a unique quadric surface E containing the 3 disjoint lines $D_{t_1}, D_{t_2}, D_{t_3}$. Since the family of lines $\{L_s\}_{s\in S}$ covers A and $\sharp(L_s \cap (D_{t_1} \cup D_{t_2} \cup D_{t_3})) \geq 3$ for all s, we obtain $A = E$. Now assume $R_t \cdot C_s = 0$. For a general $Q \in A$ there are $T_Q \in T$ and $s_Q \in S$ such that $Q \in D_{t_Q} \cap L_{s_Q}$. Since D_{t_Q} and L_{s_Q} are irreducible, we get $D_{t_Q} = L_{s_Q}$ and hence the two families of lines are the same.

(b) Here we assume $n = m+1 = 3$. Hence for a general $P \in X$ the plane curve $X \cap T_P X$ is a degree $d := \deg(X)$ plane curve which is a union of lines through P (perhaps counted with certain multiplicities). Let $x \geq 1$, be the " general number of these lines ", i.e. the integer $\deg((X \cap T_P X)_{reg})$ for a general $P \in X$. First assume $x \geq 2$ and that X is not a cone. By part (a) there is a unique one-dimensional family $\{D_t\}_{t\in T}$ of lines contained in X. Call $\{R_t\}_{t\in T}$ the associated family of curves in the normalization B of X. Since $x \geq 2$ and $P \in X_{reg}$, we get $D_t \cdot D_t > 0$. By Remark 1.5 we obtain that X is a cone. Now assume $x \geq 2$ and that X is a cone. Taking the tangential section at any smooth point of X we see that $x = 1$, contradiction. Now assume $x = 1$. We are in the set-up of Lemma 1.4 for $n = 3$ and hence we are in case (ii) of Theorem 1.1.

(c) Here we assume $n = m+1 \geq 4$. By part (b) and induction on the integer m we may assume that the result is true for the integers $n' = m' + 1 = m$. Let $H \subset \mathbf{P}^n$ be a general hyperplane. By Bertini's theorem ([**3**], parts 3) and 4) of Th. 6.3) the scheme $X \cap H$ is integral. Fix a general $Q \in U \cap H$. Hence $Q \in (X \cap H)_{reg}$. Since $T_Q X \cap X$ is an $(m-1)$-dimensional cone with vertex containing Q and $X \cap H$ is not a linear space, $T_Q(X \cap H) \cap (X \cap H) = T_Q X \cap X \cap H$ is an $(m-2)$-dimensional cone with vertex containg Q. By the inductive assumption $X \cap H$ is either in case (i) or in case (ii) of Theorem 1.1. If $X \cap H$ is a quadric hypersurface of H, then X is a quadric hypersurface. If $X \cap H$ is in case (ii), then X is in case (ii) by Lemma 1.4.

(d) Here we assume $n \geq m+2$. Let $X' \subset \mathbf{P}^{m+1}$ be a general projection of X from a general $(n-m-2)$-dimensional linear subspace of $\mathbf{P}^n$. A sufficiently general linear projection of a cone is a cone. Hence we see that X' satisfies all the assumptions of Theorem 1.1. By parts (b) and (c) we obtain that X' is either a quadric hypersurface or belongs to case (ii). If X' is a quadric hypersurface, then $\deg(X) = \deg(X') = 2$. Since X is non-degenerate, we obtain $n = m + 1$ in this case, contradiction. Assume that X' is in case (ii). A general codimension $(m - 1)$ linear section, $C \subset \mathbf{P}^{n-m+1}$, of X satisfies all the assumptions of [**4**], Th. 1. By [**4**], Th. 1, we get $n - m + 1 = 2$, contradiction.

Remark 1.6. Assume $\mathrm{char}(\mathbb{K}) = 0$. Here we give one of the uncountably many reasons which show that Theorem 1.1 is a very easy exercise in this case. If $\mathrm{char}(\mathbb{K}) = 0$ for a general $P \in X$ and a general hyperplane $M \subset \mathbf{P}^n$ containing $T_P X$ the hyperplane M is tangent to X exactly along a linear subspace (perhaps reduced to P) and the scheme $H \cap X$ has a quadratic singularity at P. As in Step (d) of the proof of Theorem 1.1 we reduce to the case $n = m + 1$. In this case $M = T_P X$. Since a cone with vertex containing P has a quadratic singularity if and only if the cone is a quadric hypersurface of its linear span, we get $\deg(X) = 2$ and hence X is a quadric hypersurface.

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